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A boundary element method based on time-stepping approximation for transient heat conduction in anisotropic solids

Masa. Tanaka† and T. Matsumoto‡

Department of Mechanical Engineering, Faculty of Engineering, Shinshu University Nagano, Japan

Q.F. Yang‡†

Graduate School of Shinshu University, Nagano, Japan

Abstract. The time-stepping boundary element method has been so far applied by the authors to transient heat conduction in isotropic solids as well as in orthotropic solids. In this paper, attempt is made to extend the method to 2-D transient heat conduction in arbitrarily anisotropic solids. The resulting boundary integral equation is discretized by means of the boundary element with quadratic interpolation. The final system of equations thus obtained is solved by advancing the time step from the given initial state to the final state. Through numerical compuation of a few examples the potential usefulness of the proposed method is demonstrated.

Key words: boundary element method; integral equation formulatiuon; numerical analysis; time-stepping approximation; time-dependent problem; heat conduction; anisotropic solid

1. Introduction

A wide class of linear problems can be formulated into boundary integral equations, and hence numerical implementation of their solution requires the discretization of only the boundary surface of the domain by using boundary elements. Because of this inherent advantage, the boundary element method has been recognized as a powerful alternative to the domain-type numerical methods of solution such as the finite difference and finite element methods. The boundary element method has been successfully applied to linear problems in static and steady states, such as elastostatics, elastodynamics, thermoelasticity, acoustics, electromagnetcs and so on. However, there seems to be still some difficulty for the linear problems in dynamic and nonsteady states, i.e., the time-dependent problems. Several boundary element formulations have been so far presented for the time-dependent problems. They can be classified into the following three kinds: *Transform Method* (Rizzo and Shippy 1970), *Direct Method* (Liggett and Liu 1979, Wrobel and Brebbia 1979, Pina and Fernandes 1984, Taigbenu and Liggett 1985), and *Time-*

† Professor
‡ Associate Professor
‡† Graduate Student

Stepping Method (Curran, Cross and Lewis 1980, Roures and Alarcon 1983, Ingber 1987, Tanaka and Matsumoto 1990, Matsomoto, Tanaka and Fujii 1990, Ingber and Phan-Thien 1992, Tanaka, Matsumoto and Yang 1993 and 1994). The first one makes use of the Laplace transform, while the second one does the time-dependent fundamental solution and derives a set of boundary integral equations in space and time. The third one first approximates time derivatives involved in the governing differential euqations, and then transforms the reduced governing differential equations thus-obtained into boundary integral equations. Although each method of solution has advantages and disadvantages, at present *Direct Method* has been popularly used in practical analysis (Brebbia, Telles and Wrobel, 1984), mainly because this method treats directly the time-dependent variables in the boundary-only formulation and requires no inverse transformation as in *Transform Method*. In this respect, *Time-Stepping Method* may have almost the same advantage as *Direct Method.*

The authors previously reported on the time-stepping boundary element methods for transient heat conduction in isotropic and also orthotropic solids (Tanaka, Matsumoto and Yang, 1993 and 1994). This paper aims at its extension to the solution of transient heat conduction in arbitrarily anisotropic solids. Several papers have discussed this method for the solution of parabolic differential equations. Curran et al. (1980), first discussed the method and investigated its properties for one-dimensional problems, introducing linear and quadratic finite difference approximations. A simple finite difference scheme was applied to approximation of time derivative by Roures and Alarcon in (1983), Ingber (1987), and Ingber and Phan-Thien (1992), dealing with two-dimensional problems of the parabolic differential equation. It is interesting to note that the time-stepping boundary element method has been also successfully applied to transient elastodynamic problems including first- and second-order time derivatives in the governing differential equations (see Tanaka and Matsumoto 1990, Matsumoto, Tanaka and Fujii 1990).

In this paper, time derivative in the governing differential equation of transient heat conduction is approximated by a generalized time-stepping scheme introducing the so-called time-scheme parameter. Then, the reduced governing differential equation is transformed into a boundary integral equation in the usual manner using the fundamental solution of the modified Helmholtz equation. A numerical implementation of the formulation is then discussed in detail, and a few examples of the two-dimensional problems are computed by a new computer code developed in this study. To obtain accurate numerical solutions, the so-called regularization technique is applied to the resulting boundary integral equation so that no Cauchy principal-value integrals are included in the formulation. The usefulness of the proposed method is demonstrated through numerical experiment for a couple of problems. It should be mentioned that the proposed method is easily applicable to 3-D problems if the 3-D fundamental solution is employed for the formulation.

2. Formulation and implementation

Let us consider two-dimensional problems of transient heat conduction in arbitrarily anisotropic solids as shown in Fig. 1.

The material constants are assumed to be constant in the temperetature range under consideration. The governing differential equation of the problem can be expressed as follows (Carslaw and Jaeger 1978):

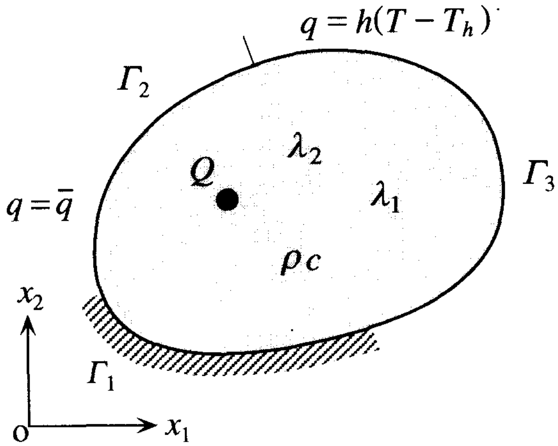

Fig. 1 Transient heat conduction problem

$$\kappa_{11}\frac{\partial^2 T}{\partial x_1^2}+(\kappa_{12}+\kappa_{21})\frac{\partial^2 T}{\partial x_1 \partial x_2}+\kappa_{22}\frac{\partial^2 T}{\partial x_2^2}+Q=\rho c\frac{\partial T}{\partial t} \tag{1}$$

where

$$\left.\begin{array}{ll}\kappa_{11}=n_1^2\lambda_1+m_1^2\lambda_2, & \kappa_{12}=n_1 n_2\lambda_1+m_1 m_2\lambda_2 \\ \kappa_{21}=n_2 n_1\lambda_1+m_2 m_1\lambda_2, & \kappa_{22}=n_2^2\lambda_1+m_2^2\lambda_2\end{array}\right\} \tag{2}$$

The notation used in the above equations is summarized in the following:

T: temperature,
Q: internal heat source,
t : time,
ρ: mass density,
c : specific heat,
λ_1 and λ_2: principal values of heat conduction coefficients,
n_i and m_i: direction cosines defined by

$$\left.\begin{array}{ll}n_1=\cos(\lambda_1, x_1) & m_1=\cos(\lambda_2, x_1) \\ n_2=\cos(\lambda_1, x_2) & m_2=\cos(\lambda_2, x_2)\end{array}\right\} \tag{3}$$

The boundary conditions can be such that
Dirichlet type:

$$T=\bar{T} \quad \text{on } \Gamma_1 \tag{4}$$

Neumann type:

$$q=-\alpha\frac{\partial T}{\partial x_1}n_{x_1}-\beta\frac{\partial T}{\partial x_2}n_{x_2}=\bar{q} \quad \text{on } \Gamma_2 \tag{5}$$

Robin type:

$$q=h(T-T_h) \quad \text{on } \Gamma_3 \tag{6}$$

In the above equations, α and β denote the heat conduction coefficients in the coordinates

x_1 and x_2, respectively; The direction cosines of the outward normal n to the boundary are denoted by n_{x_1} and n_{x_2}, that is,

$$n_{x_1}=\cos(n,\ x_1),\qquad n_{x_2}=\cos(n,\ x_2) \tag{7}$$

Furthermore, the given initial condition is as follows:

$$T|_{t=0}=T_0 \tag{8}$$

We now divide the time axis into small time steps, introduce the time-scheme parameter θ, and approximate the temperature T as well as the internal heat source Q between time steps $t=t_n$ and $t=t_{n+1}$ as

$$T=\theta T_{n+1}+(1-\theta)T_n,\qquad Q=\theta Q_{n+1}+(1-\theta)Q_n \tag{9}$$

Time derivative is approximated by

$$\frac{\partial T}{\partial t}=\frac{T_{n+1}-T_n}{\Delta t} \tag{10}$$

where $\Delta t=t_{n+1}-t_n$.

Using Eqs. (9) an (10) for the governing differential Eq. (1), we obtain

$$\begin{aligned}&\kappa_{11}\frac{\partial^2 T_{n+1}}{\partial x_1^2}+(\kappa_{12}+\kappa_{21})\frac{\partial^2 T_{n+1}}{\partial x_1\partial x_2}+\kappa_{22}\frac{\partial^2 T_{n+1}}{\partial x_2^2}\\&\quad+\frac{(1-\theta)}{\theta}\left\{\kappa_{11}\frac{\partial^2 T_n}{\partial x_1^2}+(\kappa_{12}+\kappa_{21})\frac{\partial^2 T_n}{\partial x_1\partial x_2}+\kappa_{22}\frac{\partial^2 T_n}{\partial x_2^2}\right\}\\&\quad=\rho c\frac{T_{n+1}-T_n}{\theta\Delta t}-\left\{Q_{n+1}+\frac{(1-\theta)}{\theta}Q_n\right\}\end{aligned} \tag{11}$$

Depending on the value of θ, Eq. (11) can lead us to the following solution schemes:

$\theta=1/2$ corrsponds to the Crank-Nicolson scheme,
$\theta=2/3$ to the Galerkin scheme, and
$\theta=1$ to the backward finite difference scheme.

Now we rewrite the reduced differential Eq. (11) into the following form:

$$\begin{aligned}&\nabla_x^{\bar{t}}\Lambda\nabla_x T_{n+1}+\frac{(1-\theta)}{\theta}\nabla_x^{\bar{t}}\Lambda\nabla_x T_n\\&\quad=\rho c\frac{T_{n+1}-T_n}{\theta\Delta t}-\left\{Q_{n+1}+\frac{(1-\theta)}{\theta}Q_n\right\}\end{aligned} \tag{12}$$

where ∇_x denotes the Hamiltonian operator, and $\bar{t}$ the transpose of a matrix. Matrx A in Eq. (12) can be given by

$$A=U^{\bar{t}}\Lambda U=\begin{bmatrix}\kappa_{11} & \kappa_{12}\\ \kappa_{21} & \kappa_{22}\end{bmatrix} \tag{13}$$

Denoting by α and β the eigenvalues of matrix A, and by U the transform matrix corresponding to eigenvectors, we can express Λ and U as follows:

$$\Lambda=\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix},\quad U=\begin{bmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{bmatrix} \tag{14}$$

Using the transform matrix U, we can relate the coordinate axes x_1 and x_2 to the orthogonal principal axes X_1 and X_2 through

$$\begin{Bmatrix}X_1\\ X_2\end{Bmatrix}=\begin{bmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} \tag{15}$$

Using Eqs. (12) to (14) for Eq.(11), and arranging it leaving the variable T_{n+1} on the left hand side, we obtain

$$\begin{aligned}&\alpha\frac{\partial^2 T_{n+1}}{\partial X_1^2}+\beta\frac{\partial^2 T_{n+1}}{\partial X_2^2}-\rho c\frac{T_{n+1}}{\theta\Delta t}\\ &=-\frac{(1-\theta)}{\theta}\left\{\alpha\frac{\partial^2 T_n}{\partial x_1^2}+\beta\frac{\partial^2 T_n}{\partial x_2^2}\right\}\\ &\quad-\rho c\frac{T_n}{\theta\Delta t}-\left\{Q_{n+1}+\frac{(1-\theta)}{\theta}Q_n\right\}\end{aligned} \tag{16}$$

In the above equation, T_{n+1} is the unknown to be determined through analysis at time $t=t_{n+1}$, whereas all the terms on the right hand side have been known from previous computations up to $t=t_n$.

For the solution of Eq. (16), we shall apply the standard boundary element method. To this end, we have to introduce the fundamental solution to the differential operator of Eq. (16), the so-called modified Helmholtz equation. The fundamental solution T^* is defined for infinite domain by

$$\alpha\frac{\partial^2 T^*}{\partial X_1^2}+\beta\frac{\partial^2 T^*}{\partial X_2^2}-\rho c\frac{T^*}{\theta\Delta t}=\delta(X-Y) \tag{17}$$

where $\delta(X-Y)$ denotes Dirac's delta function. The fundamental solution can be given by (Tanaka, Matsumoto and Yang, 1993 and 1994)

$$T^*=\frac{-1}{2\pi\sqrt{\alpha\beta}}K_0\left\{\frac{r_0(X,\ Y)}{\sqrt{S}}\right\} \tag{18}$$

in which $S=\theta\Delta t/\rho c$, K_0 denotes the zeroth order modified Bessel function of second kind, and r_0 is the distance between two points X and Y defied by

$$r_0(X,\ Y)=\sqrt{\frac{(X_1-Y_1)^2}{\alpha}+\frac{(X_2-Y_2)^2}{\beta}} \tag{19}$$

The flux q^* derived from the fundamental solution T^* can be given by

$$\begin{aligned}q^*&=-\alpha\frac{\partial T}{\partial X_1}\frac{\partial X_1}{\partial n}-\beta\frac{\partial T}{\partial X_2}\frac{\partial X_2}{\partial n}\\ &\quad-\alpha\frac{\partial T}{\partial X_1}n_{X_1}-\beta\frac{\partial T}{\partial X_2}n_{X_2}\end{aligned} \tag{20}$$

Multiplying Eq. (16) with the fundamental solution T^*, and applying the Gaussin divergence theorem to the resulting identity, we finally obtain the following integral equation:

$$\mu T_{n+1}(Y)-\int_\Gamma T^*(X,\ Y)q_{n+1}(X)d\Gamma+\int_\Gamma q^*(X,\ Y)T_{n+1}(X)d\Gamma$$
$$=\frac{(1-\theta)}{\theta}\left\{-\mu T_n(Y)+\int_\Gamma T^*(X,\ Y)q_n(X)d\Gamma-\int_\Gamma q^*(X,\ Y)T_n(X)d\Gamma\right\}$$
$$+\frac{1}{S}\int_\Omega T^*(X,\ Y)\left[\frac{T_n(X)}{\theta}-\frac{\Delta t}{\rho c}\{\theta Q_{n+1}(X)+(1-\theta)Q_n(X)\}\right]d\Omega \tag{21}$$

Note that the third terms on both the sides of Eq. (21) should be evaluated in the sense of Cauchy's principal value integral. To circumvent this difficulty, we shall here regularize Eq.(21). To this end, we introduce the flux $\tilde{q}^*$ of the fundamental solution to the Laplace equation. Considering the field of uniform temperature governed by the Laplace equation, we have (Tanaka, Matsumoto and Yang, 1993 and 1994)

$$\mu+\int_\Gamma \tilde{q}^*(X,Y)\,d\Gamma=0 \tag{22}$$

where

$$\tilde{q}^*(X,Y)=\frac{-1}{2\pi r_0\sqrt{\alpha\beta}}\left\{\alpha\frac{\partial r_0}{\partial X_1}n_{X_1}+\beta\frac{\partial r_0}{\partial X_2}n_{X_2}\right\} \tag{23}$$

Making use of Eq.(22), we can regularize Eq.(21) as follows:

$$\left[\int_\Gamma \{q^*(X,Y)-\tilde{q}^*(X,Y)\}d\Gamma\right]T_{n+1}(Y)-\int_\Gamma T^*(X,Y)q_{n+1}(X)d\Gamma$$
$$+\int_\Gamma q^*(X,Y)\{T_{n+1}(X)-T_{n+1}(Y)\}d\Gamma$$
$$=\frac{(1-\theta)}{\theta}\left\{-\left[\int_\Gamma \{q^*(X,Y)-\tilde{q}^*(X,Y)\}\right]T_n(Y)\right.$$
$$\left.+\int_\Gamma T^*(X,Y)q_n(X)d\Gamma+\int_\Gamma q^*(X,Y)\{T_n(X)-T_n(Y)\}d\Gamma\right\}$$
$$+\frac{1}{S}\int_\Omega T^*(X,Y)\left[\frac{T_n(X)}{\theta}+\frac{\Delta t}{\rho c}\{\theta Q_{n+1}(X)+(1-\theta)Q_n(X)\}\right]d\Omega \tag{24}$$

Eq. (24) holds for point Y in the inner domain as well as on the boundary if T_n and T_{n+1} satisfy the Hölder continuity, and this equation is so regularized that it includes no Cauchy principal value integrals. It should be here mentioned that for the solution of boundary integral Eq. (24) we have to subdivide the inner domain into small cells to evaluate the domain integral resulting from the "pseudo initial condition" at each time step, even if the given initial condition is homogeneous. This seems to be an inherent disadvantage of the time-stepping boundary element method proposed in this paper.

We now summarize the main flow of the proposed time-stepping boundary element method. At $t=t_1$, advancing Δt from the initial state, we first calculate the right hand side of Eq. (24) from the given initial condition. Then, we can determine all the unknowns on the boundary and then compute the necessary values in the inner domain using the discrtized version of

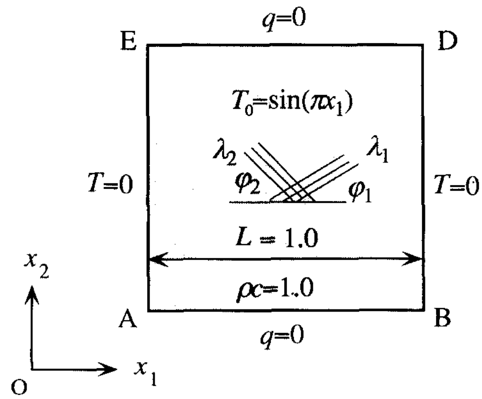

Fig. 2 Analysis model

Eq. (24). From the next steps, the inner temperatures thus calculated are to be used as the "pseudo" initial condition. If we iterate this procedure in the step-wise manner until the final time of analysis, we can solve the problem under investigation.

3. Numerical results and discussion

In order to demonstrate the validity fo the proposed time-stepping BEM, we shall analyze two typical problems of transient heat conduction. As in the previous papers (Tanaka, Mastumoto and Yang, 1993 and 1994), we introcude the diffusion number κ defined by

$$\kappa=\frac{a\Delta t}{\Delta L^2} \tag{25}$$

where $a=(\lambda_1 cos^2\varphi_1+\lambda_2\cos^2\varphi_2)/\rho c$, Δt is the time-setp width, and ΔL is the averaged distance between the two adjacent nodal points. Moreover, we introduce the averaged percentage error defined by

$$E_r(\%)=\frac{100}{K}\sum_{i=1}^{k}\left(\frac{100}{N}\sum_{j=1}^{N}\frac{|T^{ij}{}_{be}-T^{ij}{}_{ex}|}{|T^{ij}{}_{ex}|}\right) \tag{26}$$

where K is the total mumber of time steps and N is that of nodal points, whereas subscripts *ex* and *be* denote the exact solution and the present boundary element solution, respectively.

The domain of the two problems to be analyzed is a square, and the notation and coordinate system are shown in Fig. 2. The boundary conditions, initial conditions and material constants are such that

Problem 1:

$T=0$ at $x_1=0$ and 1; $q=0$ at $x_2=0$ and 1; $T_0=\sin(\pi x_1)$; $\lambda_1=1.0$, $\lambda_2=0.5$, $\varphi_1=15[\text{deg}]$, $\varphi_2=45[\text{deg}]$ (27)

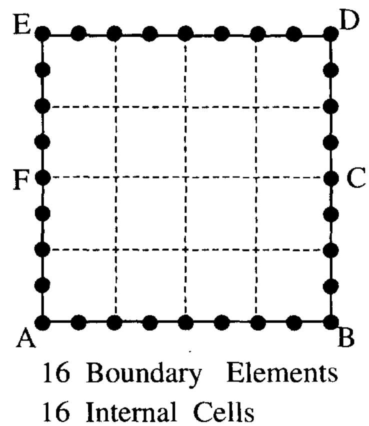

Fig. 3 Boundary element mesh and internal cells

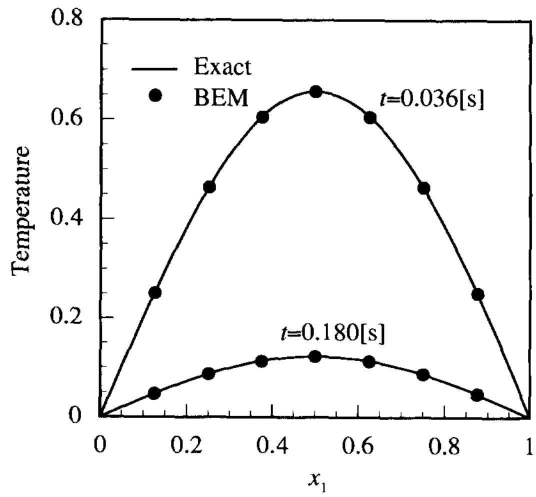

Fig. 4 Results for *Problem 1*

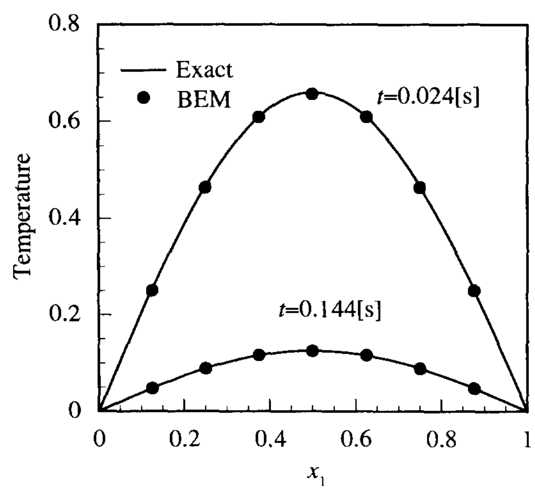

Fig. 5 Results for *Problem 2*

Problem 2:

$T=0$ at $x_1=0$ and 1; $q=0$ at $x_2=0$ and 1; $T_0=\sin(\pi x_1)\sin(\pi x_2)$; $\lambda_1=1.0$, $\lambda_2=0.5$,
$\varphi_1=15[\text{deg}]$, $\varphi_2=45[\text{deg}]$ (28)

The boundary element discretization and the internal cell subdivision are shown in Fig. 3. It is noted that the whole boundary is uniformly divided into quadratic boundary elements and the inner domain also uniformaly into eight-node quadratic internal cells.

Tanaka, Matsumoto and Yang (1993) previously reported on the numerical properties of the time-stepping BEM for transient heat conuction in orthotropic solids. It is revealed there that there is an optimal value of the diffusion number k to minimize the averaged error if the uniform mesh is employed. A chart has been available for the relationship between the diffusion number and the number of boundary elements in uniform mesh (Tanaka, Matsumoto and Yang 1993). This chart be helpful to determine an appropriate time-step width. In the present computation, we shall make use of this chart to estimate an optimal time step. For *Problem 1*, as $a=1.18$

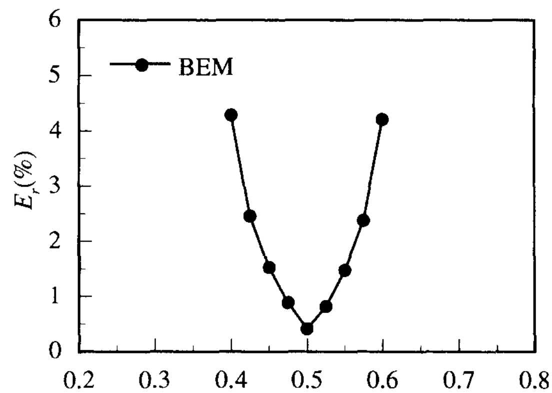

Fig. 6 Relation between κ anhd minimal error Er

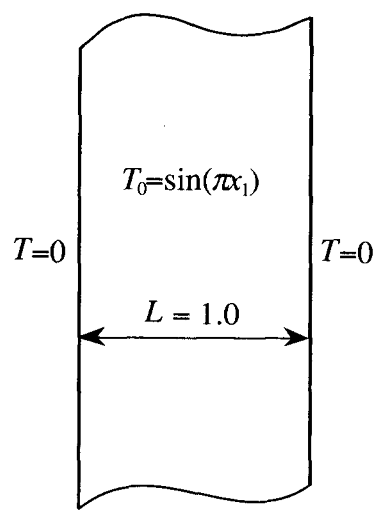

Fig. 7 Transient heat conduction in thick-walled plate

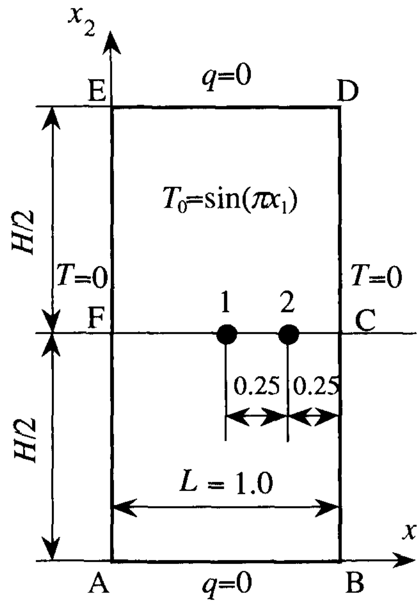

Fig. 8 Analysis model of thick-walled plate

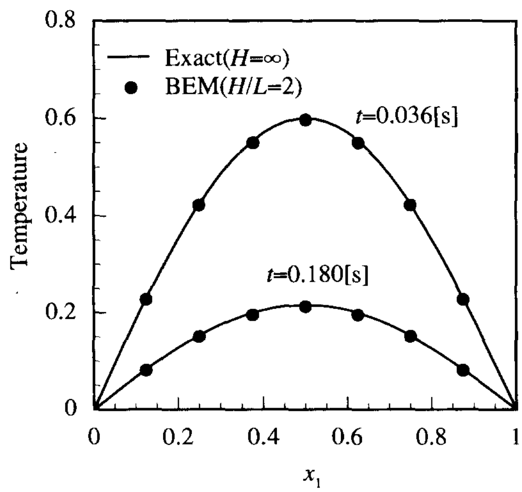

Fig. 9 Results for temperature of *FC* ($\Delta t = 0.036$)

and $\Delta L = 0.125$, we have from the chart for one-dimensional heat flow $\kappa = 2.55$ and hence from Eq. (25) we have $\Delta L = 0.036$. Assuming $\theta = 1/2$, we can obtain the results as sown in Fig. 4, where the computational results are compared with the exact ones (Carslaw and Jaeger 1987). Good agreement can be recognized between the results, which may imply partially the usefulness of the chart. Furthermore, for *Problem 2*, as $a = 1.18$ and $\Delta L = 0.125$, we have from the chart for two-dimensional heat flow $\kappa = 1.70$ and hence from Eq. (25) we have $\Delta t = 0.024$. Assuming also $\theta = 1/2$, we can obtain the results as shown in Fig. 5, where good agreement is again recognized between the computational results and exact ones (Carslaw and Jaeger 1987).

Now, we check which value can be recommended for the time-scheme parameter θ. In Fig.

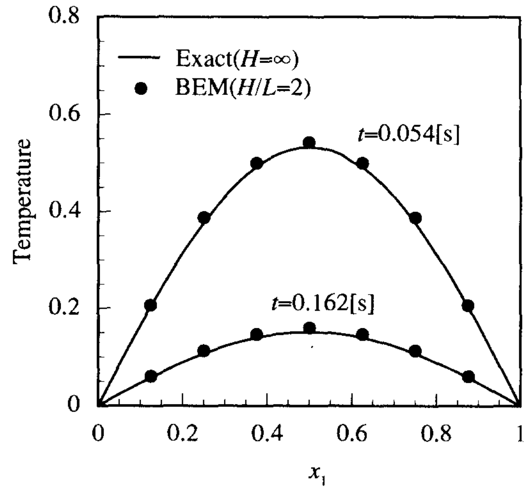

Fig. 10 Results for temperature on *FC* (Δt=0.054)

6. are shown the results obtained for *Problem 2* when we change θ from 0.4 to 0.6 by the increment 0.05. To minimize the averaged error *Er*, we should choose $\theta = 1/2$ which corresponds to the Crank-Nicholson scheme.

Finally, we shall apply the method to transient heat conduction in a thick-walled plate infinitely long in axis x_2 as shown in Fig. 7. The material constants are as follows:

$$\lambda_1 = 1.0,\ \lambda_1 = 0.5,\ \varphi_1 = 30[\text{deg}],\ \text{and}\ \varphi_1 = 30[\text{deg}] \tag{29}$$

The other information required for analysis is presented in Fig. 7. We consider a two-dimensional model of this problem as shown in Fig. 8 in which the Neumann condition is assumed on the sides AB and ED. The results obtained for the two cases $H/L=1$ and $H/L=2$ were compared, but better results were obtained when $H/L=2$.

Assuming $\rho c = 1.0$ also in this problem, we have $a = 1.125$. If we divide either of the sides AB and ED such that $\Delta L = 0.125$, we can obtain from the chart for one-dimensional heat flow the recommended diffusion number as $\kappa = 2.55$, from which we may choose by Eq. (25) $\Delta t = 0.036$. The computational results obtained in this case are compared with the exact ones for infinitely long plate (Carslaw and Jaeger 1987). Good agreement can be recognized between both the results. In Fig. 10 are shown the results obtained when a larger time step $\Delta t = 0.054$ is selected. In this case, less satifactory agreement is appreciated.

The above investigation would imply broader applicability of the chart for the relationship between the number of boundary elements and the diffusion number to other transient heat conduction problems.

4. Conclusions

The time-stepping boundary element method has been discussed in detail for the solution of two-dimensional transient heat conduction problems. This approach is an extension of the solution procedure previously proposed by the authors in 1993 and 1994 to transient heat conduction in arbitrarily anisotropic solids. As reported on in the previous papers, the time-scheme parameter θ should be chosen as $\theta = 1/2$, which corresponds to the Crank-Nicolson scheme in the finite difference method. It is also shown that the diffusion number defined as in the isotropic cases is an important factor for numerical accuracy of the solution.

In this paper, investigation is limited to the two-dimensional problems, but it is an easy matter to extend the proposed method of solution to three-dimensional problems, if the corresponding fundamental solution is used for the formulation. Such extension and also application to more complicated problems with sharp concentrations even in the two-dimensional case could be recommended for future research work in this direction.

References

Brebbia, C.A. Telles, J.C.F and Wrobel, L.C. (1984), *Boundary Element Techniques-Theory and Applications in Engineering,* Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 141-176.

Carslaw, H.S. and Jaeger, J.C. (1978), *Conduction of Heat in Solids,* 2nd ed., clarendon Press, Oxford.

Curran, D., Cross, M. and Lewis, B.A. (1980), "Solution of parabolic differential equations using discretization in time", *Appl. Math. modelling,* **4**, 389-400.

Ingber, M.S. (1978), "An efficient boundary element method for a class of parabolic differential equations using discretization in time", *Numerical Methods in Partial Differential Equations,* **3**, 187-197.

Ingber, M.S. and Phan-Thien, N. (1992), "A boundary element approach for parabolic differential equations using a classs of particular solutions", *Appl. Math. Modelling,* **16**, 124-132.

Liggett, J.A. and Liu, P.L.F. (1979), "Unsteady flow in confined aquifers: A comparison of two boundary integral methods", *Water Resour. Res.,* **15**, 861-866.

Matsumoto, T., Tanaka, M. and Fujii, H. (1990). "Transient analysis of scalar wave equation using the boundary element method based kon a time-stepping scheme", *Advances in BEM in Japan and USA.,* Computational Mechanics Publications, Southampton-Boston, 203-218.

Matsumoto, T. and Tanaka, M. (1993), "Boundary stress calculation using regularized boundary integral equation for displacement gradients", *Int. J. Numerical Methods in Engineering,* **36**, 783-797.

Matsumoto, T., Tamaka, M. and Hondoh, K. (1993), "Some nonsingular direct formulations of boundary integral equations for thin elastic palte bending", *Appl. Math. Modelling,* **17**, 586-594.

Pina, H.L.G. and Fernandes, J.L.M. (1984), "Applications in transient heat conduction", *Topics in Boundary Element Research,* **1**, ed. by C.A. Brebbia, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 41-58.

Rizzo, F.J. and Shippy, D.J. (1970), "A method of solution for certain problems of transient heat conduction", *AIAA J.,* **8**, 2004-2009.

Roures, V. and Alarcon, E. (1983), "Transient heat conduction problems using B.I.E.M.", *Computers and Structures,* **16**(6), 717-730.

Sladek, V., Sladek, J. and Tanaka, M. (1993), "Regularization of hypersingular and nearly singular integrals in the potential theory and elasticity", *Int. J. Numerical Methods in Engineering,* **36**, 1609-1628.

Taigbenu, A.E. and Liggett, J.A. (1985), "Boundary element calculations for the diffusion equation", *J. Eng. Mech.,* **111**(3), 311-328.

Tanaka, M. and Matsumoto, T. (1990), "Transient elastodynamic boundary element formulations based on the time-stepping scheme", *Int. J. Pressure Vessels & Piping,* **42**, 371-332.

Tanaka, M., Matsumoto, T. and Yang, Q.F. (1993), "A time-stepping boundary element method for transient heat conduction in orthotropic bodies", *Eng. Anal. with Boundary Elements,* **12**, 85-91.

Tanaka, M., Matsumoto, T. and Yang, Q.F. (1994), "Time-stepping boundary element method applied to 2-D transient heat conduction problems", *Appl. Math. Modelling,* **18**, 569-576.

Wrobel, L.C. and Brebbia, C.A. (1979), "The boundary element method for steady-state and transient heat conduction", *Numerical Methods in Thermal Problems,* **1**, ed. by R.W. Lewis and K. Morgan, Pineridge Press, Swansea, 58-73.