# Detection of a concentrated damage in a parabolic arch by measured static displacements 

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#### Abstract

The present paper deals with the identification of a concentrated damage in an elastic parabolic arch through the minimization of an objective function which measures the differences between numerical and experimental values of static displacements. The damage consists in a notch that reduces the height of the cross section at a given abscissa and therefore causes a variation in the flexural stiffness of the structure. The analytical values of static displacements due to applied loads are calculated by means of the principle of virtual work for both the undamaged and damaged arch. First, pseudo-experimental data are used to study the inverse problem and investigate whether a unique solution can occur or not. Various damage intensities are considered to assess the reliability of the identification procedure. Then, the identification procedure is applied to an experimental case, where displacements are measured on a prototype arch. The identified values of damage parameters, i.e., location and intensity, are compared to those obtained by means of a dynamic identification technique performed on the same structure.


Keywords: damage identification; inverse problems; arches; experimental static displacements

## 1. Introduction

The analysis of the integrity of structures and their health monitoring has been worthy of great interest in the last few decades and has produced many studies. The presence of damage can be detected by means of conventional methods such as simple visual inspection, radiography, thermal analysis, ultrasonic testing, which in any case require knowledge of the damaged zone location and its direct access. These methods are impractical when the damage search involves the whole structure. Other techniques allow to detect the presence of damage and, in some cases, detect its location and intensity by means of non destructive tests. These techniques are based on the analysis of both static or dynamic variations of responses of damaged structures with respect to the correspondent undamaged ones. In fact, damage implies a loss of structural stiffness and therefore may strongly modify the behaviour of the structure. Different techniques have been presented in the literature; these are based on the variation of dynamic characteristics, such as natural frequencies, mode shapes, dynamic flexibilities, or static quantities, such as displacement induced by applied

[^0]loads. Two recent review papers (Farrar and Worden 2007, Friswell 2007) contain a rich and up-to date bibliography.

Response measurements obtained by performing non destructive load tests, hence, represent crucial data for damage identification. The use of experimental data for damage detection leads to a class of inverse problems which are often ill-conditioned and sometimes undetermined. Dynamic load tests provide, in general, a large amount of information and a wide range of research, reported in the literature, has been devoted to the study of dynamic identification procedures. Although less numerous, in the literature there are studies proposing identification procedures based on measurements by static tests which are easily executable and provide additional information to the dynamic identification without any introduction of uncertainties due to masses and damping ratios. (Sanayei and Onipede 1991, Banan et al. 1994, Sanayei and Saletnik 1996, Sanayei et al. 1997, Hjelmstadt and Shin 1997, Oh and Jung 1998, Tseng 2000, Wang et al. 2001, Di Paola and Bilello 2004, Caddemi and Morassi 2005, Bakhtiari-Nejad et al. 2005, Caddemi and Greco 2006, Shenton and Hu 2006, Buda and Caddemi 2007, Caddemi and Morassi 2007, Rucka and Wilde 2006).

The way in which damage influences the behaviour of structural members has been mainly studied with reference to straight beams rather than curved bars. To the authors' knowledge, research literature regarding experimental studies on damaged parabolic arches is quite modest. Some contributions are those by Cerri and Ruta (2004), Viola et al. (2005), Pau et al. (2011).

A very sensitive aspect in damage identification techniques is the difficulty of damage modelling. In many studies damage is represented by one or more fully open cracks along the axis of the beam. In this instance, the concentrated damage may be modelled by a reduction in the rigidity of the beam at the correspondent abscissae described by means of a rotational spring (Ostachowicz and Krawczuk 1991). Elsewhere a weaker element in a FEM code is introduced while others present one dimensional continuum theories (Christides and Barr 1984, Chondros and Dimarogonas 1998).

In this paper, a parabolic arch, in which a notch reduces the height of the cross section at a given abscissa, has been studied in its undamaged and damaged states. The damage has been modelled by means of a rotational spring of suitable rigidity (Cerri and Vestroni 2004). No crack closure phenomenon is considered, hence linear behaviour of the damaged beam is assumed; however, no restriction concerning the damage intensity is introduced. In the study of the direct problem analytical vertical displacements of the undamaged and the damaged arch under concentrated loads have been obtained by means of the principle of virtual work.

The inverse problem has been studied for both the undamaged and damaged arch. In the former, the Young's modulus of the elastic material is identified, thus allowing a reliable updating of the analytical model. For the damaged arch, the parameters assumed as characteristics of the damage, i.e. location and intensity, are identified. Both inverse problems are solved by minimizing an objective function which measures the differences between analytical and measured vertical displacements at given abscissae. In order to test the reliability of the proposed procedure, the analyses have been firstly carried out using pseudo-experimental data. The same inverse problems are then solved by using experimental data obtained from static tests on a prototype model of the considered arch.

The performance of the proposed procedure for the identification of damage parameters using experimental data is compared to an analogous procedure based on the variation of natural frequencies of the undamaged and damaged arch already applied to a similar arch by the authors themselves in a previous paper (Pau et al. 2011).

## 2. Model of the undamaged and damaged arch

A plane linear elastic double hinged parabolic arch with the following centreline equation is considered

$$
\begin{equation*}
y(x)=0.004 f x \frac{L-x}{L} \tag{1}
\end{equation*}
$$

where $f=200 \mathrm{~mm}$ is the mid-span height and $L=1000 \mathrm{~mm}$ the horizontal length of the arch, as shown in Fig. 1. The cross section is rectangular having width $b=40 \mathrm{~mm}$ and height $h=8 \mathrm{~mm}$. The values of the Young's modulus $E$, Poisson ratio $v$ and mass density $m$ are listed in the same figure, which also reports the Cartesian reference system used in the following analytical developments.
As it is well known, in structures with curved centre line, the radius of curvature $r(x)=$ $\sqrt{\left(1+y^{\prime}(x)^{2}\right)^{3}} / y^{\prime \prime}(x)$ modifies the distribution of axial stresses in each cross section with respect to the case of the straight beam and therefore the bending curvature under the hypothesis of linear elastic behaviour is (Baldacci 1970)

$$
\begin{equation*}
\chi(x)=\frac{M(x)}{E J_{r}}-\frac{1}{r(x) E A}\left(N(x)-\frac{M(x)}{r(x)}\right) \cong \frac{M(x)}{E J_{r}} \tag{3}
\end{equation*}
$$

where $M(x)$ and $N(x)$ are respectively the bending moment and the axial force in the arch. For a generic shape of the cross section of area $A$ and principal axis $\xi$ and $\eta$, the reduced moment of inertia of the cross section with respect to $\xi$ axis, $J_{r}$ is given by

$$
\begin{equation*}
J_{r}=\int_{A} \eta^{2} \frac{r(x)}{r(x)-\eta} d A \tag{4}
\end{equation*}
$$

As it can be seen from Eq. (4), the reduced moment of inertia $J_{r}$ for $r \rightarrow \infty$, coincides with the ordinary moment of inertia with respect to $\xi$ axis (Baldacci 1970). For the considered arch, the reduced moment of inertia $J_{r}$ results almost constant in the range $0 \leq x \leq L$ and equals the ordinary moment of inertia $I_{u}=b h^{3} / 12=1706.67 \mathrm{~mm}^{4}$; therefore, $J_{r}$ and $I_{u}$ are assumed to be coincident in the following calculations.

With regard to the shear deformability (Baldacci 1970), an approximate treatment of the problem shows that in a rectangular section, the shear deformability of a curved beam equals the shear deformability of a straight beam when $h / r(x)<0.2$. In the present case, the mentioned ratio assumes the maximum value in the middle of the arch and is $h / r(L / 2)=0.013 \ll 0.2$, then the shear deformability of the arch can be considered equal to that of a straight beam. In a simply supported slender arch, the case considered here, the shear deformability provides an increment in transverse displacements of $0.01 \%$, that is hence considered negligible. However, in general, in thick curved


$$
\begin{aligned}
& E=199950 \mathrm{~N} / \mathrm{mm}^{2} \\
& v=0.3 \\
& m=7.849 * 10^{-9} \mathrm{~N} / \mathrm{mm}^{3}
\end{aligned}
$$

Fig. 1 Geometrical and mechanical characteristics of the arch
beams the shear contribution cannot be neglected. A full treatment of the problem for non-circular arches is presented, for instance, in (Lee et al. 2004).

Besides the undamaged model, the case in which a concentrated notch reduces the height of the cross section has been considered. It has been assumed that the width of the notch is such that it is possible to neglect the reduction in the total mass of the structure. The damage determines, at a given abscissa, a reduction in flexural and axial rigidities. However, the authors have shown in a previous paper (Greco and Pau 2008) that the reduction in axial rigidity for the examined structure is negligible. Consequently, only a rotational spring equivalent to the notch is introduced to model the damage. In the literature, many different models of the stiffness of the equivalent spring are reported. Here, this stiffness is evaluated by means of the procedure proposed by (Cerri and Vestroni 2004), briefly described hereafter. The notch causes a perturbation in the tension state in a damaged zone, where length $L_{d}$ is greater than the effective width of the notch itself. The relative rotation $\phi_{d}$ between the sections delimiting the damaged zone can be written as $\phi_{d}=\phi_{u}+\Delta \phi$, where $\phi_{u}$ is the rotation between the two limit sections in the undamaged case and $\Delta \phi$ is the increase in rotation due to damage. Then $\Delta \phi=M L_{d} \beta / E I_{u}(1-\beta), \beta=\left(E I_{u}-E I_{d}\right) / E I_{u}$, and $E I_{u}$ and $E I_{d}$ are respectively the flexural rigidities of the undamaged and damaged cross sections. Assuming that the notch causes an exponential decay of the stiffness and requiring that the deformability of this beam equals the deformability of a beam with a step variation of stiffness with length $L_{d}$, resulting in $L_{d}=$ $h / 2$. In the case of localized damage, $\Delta \phi=M / K$, therefore, the nondimensional stiffness of an equivalent spring $k=K /\left(E I_{u} / L\right)$ can be written as

$$
\begin{equation*}
k_{d}=\frac{2 L}{h} \frac{1-\beta}{\beta} \tag{2}
\end{equation*}
$$

The values of this nondimensional stiffness can vary from zero, i.e., $E I_{d}=0$, which means that the transverse section is fully damaged, to infinity, i.e., $E I_{u}=E I_{d}$, which happens when no damage occurs.

## 3. Direct problem

The static response of the arch under a concentrated load has been analytically calculated by means of the principle of virtual work and also measured in experimental tests. In particular, the vertical displacements of the centre line of both the undamaged and damaged arch have been evaluated in order to provide the data used in the solution of the inverse problem.

### 3.1 Vertical displacements of the undamaged arch under a concentrated load

When the arch is subjected to a vertical concentrated load $Q$ at the abscissa $x_{Q}$, the horizontal reactions of the hinges $R_{x}^{A}=R_{x}^{B}$ of the present statically undetermined structure can be analytically evaluated using the principle of virtual work and result

$$
\begin{equation*}
R_{x}^{A}=R_{x}^{B}=\frac{5 Q}{8 f\left(1+\frac{15 J}{8 f^{2} b h}\right)}\left(\frac{x_{Q}}{L}-\frac{2 x_{Q}^{3}}{L^{3}}+\frac{x_{Q}^{4}}{L^{4}}\right) \tag{5}
\end{equation*}
$$



Fig. 2 Vertical displacements of the undamaged arch under a concentrated load

Vertical displacements of the arch centre line at the generic abscissa $x^{*}$, whose origin coincides with the origin of the Cartesian coordinate system illustrated in Fig. 1, are evaluated again by means of the principle of virtual work. Denoting with $v_{1}\left(x^{*}\right)$ the displacements for $x^{*} \leq x_{Q}$ and with $v_{2}\left(x^{*}\right)$ the displacements for $x^{*} \geq x_{Q}$ these turn out to be

$$
\begin{align*}
& v_{1}\left(x^{*}\right)=\int_{0}^{x^{*}} M_{1}^{f}(x) \frac{M_{1}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x^{*}}^{x_{Q}} M_{2}^{f}(x) \frac{M_{1}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x_{Q}}^{L} M_{2}^{f}(x) \frac{M_{2}^{s}(x)}{\cos \theta(x) E J} d x \\
& v_{2}\left(x^{*}\right)=\int_{0}^{x_{Q}} M_{1}^{f}(x) \frac{M_{1}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x_{Q}}^{x^{*}} M_{1}^{f}(x) \frac{M_{2}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x^{*}}^{L} M_{2}^{f}(x) \frac{M_{2}^{s}(x)}{\cos \theta(x) E J} d x \tag{6}
\end{align*}
$$

where the bending moments, in the fictitiously balanced system are indicated by the superscript $f$ and those of the actual kinematically compatible system are denoted by the superscript $s$

$$
\begin{gathered}
M_{1}^{f}(x)=\frac{\left(L-x^{*}\right)}{L} x \quad 0 \leq x \leq x^{*} ; \quad M_{2}^{f}(x)=\frac{\left(L-x^{*}\right)}{L} x-\left(x-x^{*}\right) x^{*} \leq x \leq L \\
M_{1}^{s}(x)=\frac{Q\left(L-x_{Q}\right)}{L} x+R_{x}^{B} y(x) \quad 0 \leq x \leq x_{Q} ; \quad M_{2}^{s}(x)=\frac{Q\left(L-x_{Q}\right)}{L} x-Q\left(x-x_{Q}\right)+R_{x}^{B} y(x) \quad x_{Q} \leq x \leq L
\end{gathered}
$$

and $\theta(x)$ is the trigonometric tangent of the angle between the geometric tangent to the centre line and $x$ axis. Fig. 2 shows vertical displacements for $Q=100 \mathrm{~N}, x_{Q}=362 \mathrm{~mm}$ and $E=2^{*} 10^{5} \mathrm{~N} / \mathrm{mm}^{2}$.

### 3.2 Vertical displacements of the damaged arch under concentrated loads

In order to evaluate the vertical displacements of the cracked arch under a concentrated load, it is necessary to distinguish the case in which the damage is located at an abscissa $x_{d} \leq x_{Q}$ or $x_{d} \geq x_{Q}$, respectively denoted as case (a) and (b) in Fig. 3. In each case, three different displacement functions must be defined over regular domains that can be closed since congruence of vertical displacements is required, as the shear deformation is neglected.
Case a) Denoting with $v_{a 1}$ vertical displacements for $0 \leq x^{*} \leq x_{d}$, $v_{a 2}$ vertical displacements for $x_{d} \leq x^{*} \leq x_{Q}$ and $v_{a 3}$ vertical displacements for $x_{Q} \leq x^{*} \leq L$ these turn out to be


Fig. 3 Damaged arches

$$
\begin{align*}
& v_{a 1}\left(x^{*} ; x_{d}, k_{d}\right)=\int_{0}^{x^{*}} M_{1}^{f}(x) \frac{M_{1}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x^{*}}^{x_{Q}} M_{2}^{f}(x) \frac{M_{1}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x_{Q}}^{L} M_{2}^{f}(x) \frac{M_{2}^{s}(x)}{\cos \theta(x) E J} d x+M_{2}^{f}\left(x_{d}\right)\left(\frac{M_{1}^{s}\left(x_{d}\right)}{k_{d}}\right) \\
& v_{a 2}\left(x^{*} ; x_{d}, k_{d}\right)=\int_{0}^{x^{*}} M_{1}^{f}(x) \frac{M_{1}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x^{*}}^{x_{Q}} M_{2}^{f}(x) \frac{M_{1}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x_{Q}}^{L} M_{2}^{f}(x) \frac{M_{2}^{s}(x)}{\cos \theta(x) E J} d x+M_{1}^{f}\left(x_{d}\right)\left(\frac{M_{1}^{s}\left(x_{d}\right)}{k_{d}}\right) \\
& v_{a 3}\left(x^{*} ; x_{d}, k_{d}\right)=\int_{0}^{x_{Q}} M_{1}^{f}(x) \frac{M_{1}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x_{Q}}^{x^{*}} M_{1}^{f}(x) \frac{M_{2}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x^{*}}^{L} M_{2}^{f}(x) \frac{M_{2}^{s}(x)}{\cos \theta(x) E J} d x+M_{1}^{f}\left(x_{d}\right)\left(\frac{M_{1}^{s}\left(x_{d}\right)}{k_{d}}\right) \tag{7}
\end{align*}
$$

Case b) Denoting with $v_{b 1}$ vertical displacements for $0 \leq x^{*} \leq x_{Q}$, $v_{b 2}$ vertical displacements for $x_{Q} \leq x^{*} \leq x_{d}$ and $v_{b 3}$ vertical displacements for $x_{d} \leq x^{*} \leq L$ these turn out to be

$$
\begin{align*}
& v_{b 1}\left(x^{*} ; x_{d}, k_{d}\right)=\int_{0}^{x^{*}} M_{1}^{f}(x) \frac{M_{1}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x^{*}}^{x_{Q}} M_{2}^{f}(x) \frac{M_{1}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x_{Q}}^{L} M_{2}^{f}(x) \frac{M_{2}^{s}(x)}{\cos \theta(x) E J} d x+M_{2}^{f}\left(x_{d}\right)\left(\frac{M_{2}^{s}\left(x_{d}\right)}{k_{d}}\right) \\
& v_{b 2}\left(x^{*} ; x_{d}, k_{d}\right)=\int_{0}^{x_{Q}} M_{1}^{f}(x) \frac{M_{1}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x_{Q}}^{x^{*}} M_{1}^{f}(x) \frac{M_{2}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x^{*}}^{L} M_{2}^{f}(x) \frac{M_{2}^{s}(x)}{\cos \theta(x) E J} d x+M_{2}^{f}\left(x_{d}\right)\left(\frac{M_{2}^{s}\left(x_{d}\right)}{k_{d}}\right) \\
& v_{b 3}\left(x^{*} ; x_{d}, k_{d}\right)=\int_{0}^{x_{Q}} M_{1}^{f}(x) \frac{M_{1}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x_{Q}}^{x^{*}} M_{1}^{f}(x) \frac{M_{2}^{s}(x)}{\cos \theta(x) E J} d x+\int_{x^{*}}^{L} M_{2}^{f}(x) \frac{M_{2}^{s}(x)}{\cos \theta(x) E J} d x+M_{1}^{f}\left(x_{d}\right)\left(\frac{M_{2}^{s}\left(x_{d}\right)}{k_{d}}\right) \tag{8}
\end{align*}
$$



Fig. 4 Vertical displacements of the damaged arch

As expected, Eqs. (7) and (8) depend on both the damage parameters position $x_{d}$ and intensity $k_{d}$. Fig. 4 shows vertical displacements of the centre line of the arch for $Q=100 N, x_{Q}=362 \mathrm{~mm}, E=$ $2^{*} 10^{5} \mathrm{~N} / \mathrm{mm}^{2}, k_{d}=30$ and $k_{d}=300$.

## 4. Inverse problem

Some properties of the undamaged and damaged structure are identified using the vertical displacements. In particular, the Young's modulus of the elastic material is identified for the undamaged arch, while for the damaged arch the damage parameters, i.e., position and intensity, are evaluated. These inverse problems are both solved by minimizing a proper objective function. In the present section, in order to test the reliability of the proposed procedure, the analyses have been firstly carried out using pseudo-experimental data. The same inverse problems are then solved by using experimental data obtained from static tests on a prototype model of the considered arch.

### 4.1 Identification of the Young's modulus in the undamaged model

With reference to the undamaged arch, the identification of the Young's modulus of the material is obtained minimizing the following objective function, which measures the differences between analytical and measured vertical displacements at $n$ chosen abscissae

$$
\begin{equation*}
G(E)=\sqrt{\sum_{i=1}^{n}\left(\frac{v_{i}^{a}(E)-v_{i}^{e}}{v_{i}^{e}}\right)^{2}} \tag{9}
\end{equation*}
$$

In Eq. (9), $v_{i}^{a}(E)$ are discrete values of the deformed shape, which is a linear function of $E$. Hence, no matter where the displacements are measured and even in the presence of errors, it can be analytically shown that the minimum of this function is unique.
In the following analyses, four pseudo-experimental values of the vertical displacements $v_{i}^{e}$ due to a load of 100 N applied at the abscissa $x_{Q}=362 \mathrm{~mm}$ are calculated for a nominal value of the elasticity modulus $E=2 * 10^{5} \mathrm{~N} / \mathrm{mm}^{2}$ by means of Eq. (6). Choosing four evenly scattered abscissae $x_{1}=128 \mathrm{~mm}, x_{2}=376 \mathrm{~mm}, x_{3}=594 \mathrm{~mm}, x_{4}=847 \mathrm{~mm}$, the values of the displacements turn out to be $v_{1}^{e}=-0.15 \mathrm{~mm}, v_{2}^{e}=-0.26 \mathrm{~mm}, v_{3}^{e}=0.16 \mathrm{~mm}, v_{4}^{e}=0.28 \mathrm{~mm}$. Fig. 5 shows the corresponding


Fig. 5 Objective function $G(E)$ with experimental and pseudo-experimental data


Fig. 6 Experimental setup


Fig. 7 Static test setup
objective function, defined in Eq. (9), with a bold line. $G(E)$ has a unique global minimum which is equal to zero, in the absence of errors, and confirms the reliability of the proposed algorithm. The abscissa of this global minimum provides the value of $E$ which corresponds to the exact solution to the inverse problem.

An experimental investigation was performed on a prototype arch in the Laboratory of the Department of Structural and Geotechnical Engineering of La Sapienza University of Rome on the structure of Fig. 6. The static tests were carried out according to the setup represented in Fig. 7. The vertical displacements due to a load applied at the abscissa $x_{Q}=362 \mathrm{~mm}$ were measured at four points, whose abscissae are the mentioned $x_{1}, x_{2}, x_{3}, x_{4}$. The load was applied in steps and different cycles were made increasing and decreasing the load; each cycle was repeated 10 times in order to have statistical significance of the measurements. The values of the Young's modulus of the material identified using different values of the applied load showed an irrelevant coefficient of variation, then in the following results, for the sake of brevity, only the data related to the concentrated load of 100 N are reported.

The mean values of the vertical displacements recorded in the four measurement points are reported in Table 1 and provide the objective function reported in Fig. 5 with the thin line. The shape of the objective function is close to the analytical one, however, the minimum is less sharp because of the presence of errors and the value of the function at the minimum is greater than zero. The value of the Young's modulus which corresponds to the minimum of the objective function is $E$

Table 1 Mean values of experimental vertical displacements

| $Q[N]$ | $x_{1}=128 \mathrm{~mm}$ | $x_{2}=376 \mathrm{~mm}$ | $x_{3}=594 \mathrm{~mm}$ | $x_{4}=847 \mathrm{~mm}$ |
| :---: | :---: | :---: | :---: | :---: |
| 100 | -0.1561 | -0.2830 | 0.1534 | 0.2807 |

$=196363 \mathrm{~N} / \mathrm{mm}^{2}$. The identified value of Young's modulus allows to update the model of the arch and therefore will be used when detecting damage parameters.

### 4.2 Identification of damage parameters by means of pseudo-experimental data

Given a damaged configuration, the vertical displacement $v_{i}$ at a point $x_{i}$ is a function of both damage position $x_{d}$, and stiffness $k_{d}$ of the spring equivalent to the notch. Inversely, it is possible to evaluate a continuous function $k_{i d}\left(x_{d}\right)$ that provides the stiffness that a spring located at $x_{d}$ should have in order to provide a given displacement $v_{i}$ at $x_{i}$. For each given value $v_{i}$ at a point $x_{i}$, a continuous curve $k_{d}\left(x_{d}\right)$ can be plotted. The inverse problem of determining $x_{d}$ and $k_{d}$, based on the measurement of $i$ displacements $v_{i}$, can then be solved finding the intersection between at least two curves. In the absence of errors, the coordinates of the intersection identify the correct values of $x_{d}$ and $k_{d}$.

The minimum number $i$ of curves $k_{i d}\left(x_{d}\right)$ necessary to identify two parameters is 2 , however it can be greater since these curves can show multiple intersections This statement is proved by means of the following example, where an arbitrary number of displacements greater than 2 is taken. It is assumed to measure displacements at the same four abscissae in the arch with a concentrated damage at $x_{d}=430 \mathrm{~mm}$ with depth 2 mm height. The stiffness of the equivalent spring, calculated by means of Eq. (2), is $k_{d}=182.43$; the concentrated load $Q=100 \mathrm{~N}$, applied at $x_{Q}=362 \mathrm{~mm}$, provides at the abscissae $x_{1}^{*}=128 \mathrm{~mm}, x_{2}^{*}=376 \mathrm{~mm}, x_{3}^{*}=594 \mathrm{~mm}, x_{4}^{*}=847 \mathrm{~mm}$ the following pseudo-experimental displacements: $v_{1}=-0.157 \mathrm{~mm}, v_{2}=-0.281 \mathrm{~mm}, v_{3}=0.146 \mathrm{~mm}, v_{4}=$ 0.277 mm for $E=196363 \mathrm{~N} / \mathrm{mm}^{2}$, obtained by the formulae (7)-(8).


Fig. 8 Curves $k_{d}\left(x_{d}\right)$

Fig. 8(a) reports the four curves $k_{d}\left(x_{d}\right)$, which show multiple intersections. However, by using three values of the displacements, the exact damage parameters can be determined univocally, as it can be seen in Fig. 8(b), which represents an enlargement of the intersection area. It must be noted that in the diagrams there are points where different couples of curves intersect, which indicate critical situations. In fact, in experimental cases, where errors occur, these points can provide solutions not corresponding to the actual values of damage parameters.
Therefore, to obtain an overdetermined system, in the following analyses, four values of static displacements are considered as pseudo-experimental data in order to minimize the effect of experimental and modelling errors. Damage parameters can be identified by a procedure analogous to that used to evaluate the Young's modulus in the undamaged arch. In the case of the damaged arch, the objective function is defined as the sum of the squares of the differences between the analytical $\Delta v_{i}^{a}\left(x_{d}, k_{d}\right)$ and experimental $\Delta v_{i}^{e}$ values of the variation of displacements between the undamaged and the damaged state, and normalized with respect to the experimental displacements of the undamaged arch. Assuming as pseudo-experimental data the variations of static displacements measured in the four abovementioned points, the objective function is

$$
\begin{equation*}
G\left(x_{d}, k_{d}\right)=\sqrt{\left(\frac{\Delta v_{1}^{a}\left(x_{d}, k_{d}\right)-\Delta v_{1}^{e}}{v_{1}^{e}}\right)^{2}+\left(\frac{\Delta v_{2}^{a}\left(x_{d}, k_{d}\right)-\Delta v_{2}^{e}}{v_{2}^{e}}\right)^{2}+\left(\frac{\Delta v_{3}^{a}\left(x_{d}, k_{d}\right)-\Delta v_{3}^{e}}{v_{3}^{e}}\right)^{2}+\left(\frac{\Delta v_{4}^{a}\left(x_{d}, k_{d}\right)-\Delta v_{4}^{e}}{v_{4}^{e}}\right)^{2}} \tag{10}
\end{equation*}
$$

The measurement points and load position define 6 intervals along the arch span, as shown in Fig. 7. Within each interval, a different objective function is defined according to the analytical values of displacements at the measurement points, easily recognizable between the previous cases a) or b).

The damage parameters can be determined by successively seeking two distinct minima. For each objective function and for each possible $x_{d}$ within the interval in which the functions is defined, the minimum with respect to the spring stiffness is determined

$$
\begin{equation*}
\tilde{G}\left(x_{d}\right)=\min _{k} G\left(k, x_{d}\right) \tag{11}
\end{equation*}
$$

then, the global minimum of the functions is sought with respect to $x_{d}$.


Fig. 9 Objective function $\tilde{G}\left(x_{d}\right)$ with pseudo-experimental data

Assuming the same data used in Fig. 8, the objective function $\tilde{G}\left(x_{d}\right)$ for the pseudo-experimental case is shown in Fig. 9. As it can be seen, the global minimum is well defined and gives a zero value of the objective function at the exact damage abscissa ( $x_{d}=430 \mathrm{~mm}$ ), nevertheless another local minimum appears. This can cause difficulties in the solution to the inverse problem in presence of experimental errors.

### 4.3 Identification of damage parameters by means of experimental data

The static test setup for the damaged arch is the same as that used for the identification of Young's modulus of the undamaged structure. An asymmetric notch 1 mm in width was made with a cutting device composed by an inclinable rotating disk mounted on a system of two adjustable slides, as shown in Fig. 6. Four increasing intensities of damage at the abscissa $x_{d}=430 \mathrm{~mm}$ were considered, as reported in Table 2, each one related to a different height of the damaged section $h_{d}$.
Table 3 reports the analytical displacements calculated in the measurements points by means of the principle of virtual work in each damage configuration. A comparison between the values of the displacements in the undamaged and damaged configurations shows that, apart from the values calculated in $x_{1}=128 \mathrm{~mm}$, the static displacements increase with the depth of the notch. The reason of the different trend of behaviour observed in $x_{1}$ is due to the fact that the local decrease of flexural stiffness caused by damage, modifies the deformed shape of the arch and, depending on the damage location, may result in a reduction in the rotation near the hinges.
Moreover in the experimental test on the damaged arch, in each damage configuration, many load cycles have been performed but in the following analyses only the results for $Q=100 \mathrm{~N}$ are reported for the sake of brevity. Table 4 reports the mean values of the experimental static displacements measured in the four considered abscissae for each damage configuration. As it can be noticed, due to experimental errors, the illustrated increase of the displacements with the intensity of the damage does not always occur.

The damage parameters for each damage configuration are again determined by seeking first the minimum with respect to the spring stiffness for each possible damage location and then evaluating

Table 2 Mechanical characteristics of the notch

|  | $h-h_{d}[\mathrm{~mm}]$ | $\delta=\left(h-h_{d}\right) / h$ | $k_{d}$ |
| :---: | :---: | :---: | :---: |
| D1 | 1 | 0.125 | 507.40 |
| D2 | 2 | 0.250 | 182.43 |
| D3 | 3 | 0.375 | 80.75 |
| D4 | 4 | 0.500 | 35.71 |

Table 3 Analytical values of the static displacements (in mm) for $Q=100 \mathrm{~N}$

|  | $x_{1}=128 \mathrm{~mm}$ | $x_{2}=376 \mathrm{~mm}$ | $x_{3}=594 \mathrm{~mm}$ | $x_{4}=847 \mathrm{~mm}$ |
| :---: | :---: | :---: | :---: | :---: |
| U | -0.1575 | -0.2791 | 0.1456 | 0.2759 |
| D1 | -0.1574 | -0.2798 | 0.1457 | 0.2763 |
| D2 | -0.1573 | -0.2811 | 0.1460 | 0.2772 |
| D3 | -0.1570 | -0.2835 | 0.1465 | 0.2787 |
| D4 | -0.1565 | -0.2887 | 0.1476 | 0.2822 |

Table 4 Experimental values of the mean static displacements (in mm) for $Q=100 \mathrm{~N}$

|  | $x_{1}=128 \mathrm{~mm}$ | $x_{2}=376 \mathrm{~mm}$ | $x_{3}=594 \mathrm{~mm}$ | $x_{4}=847 \mathrm{~mm}$ |
| :---: | :---: | :---: | :---: | :---: |
| U | -0.1561 | -0.2830 | 0.1534 | 0.2807 |
| D1 | -01552 | -0.2746 | 0.1596 | 0.2841 |
| D2 | -0.1586 | -0.2776 | 0.1329 | 0.2813 |
| D3 | -0.1513 | -0.2843 | 0.1557 | 0.2914 |
| D4 | -0.1487 | -0.2875 | 0.1628 | 0.2943 |



Fig. 10 Experimental objective functions for damage configurations D1 (a), D2 (b), D3 (c), D4 (d)

Table 5 Identified damage parameters by means of static tests and related errors

|  | D1 |  | D2 |  | D3 |  | D4 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Ident. val | err | Ident. val | err | Ident. val | err | Ident. val | err |
| $k_{d}$ | 93.12 | 81.6\% | 48.53 | 73.4\% | 39.41 | 51.2\% | 24.14 | 32.4\% |
| $x_{d}$ | 590 | 37.2\% | 500 | 16.3\% | 460 | 7\% | 430 | 0\% |

the global minimum with respect to $k_{d}$. The four objective functions $\tilde{G}\left(x_{d}\right)$ related to the damage configurations considered are reported in Fig. 10. These functions show one global minimum together with other local minima. The values of the identified parameters, reported in Table 5, show that the errors may be in general large, but the position is detected with a smaller error than
stiffness. Furthermore the errors decrease with the intensity of damage.

## 5. Comparison with the experimental results of dynamic tests

The performance of the proposed procedure for the identification of damage parameters using experimental data is compared to an analogous procedure based on the variation of natural frequencies of the undamaged and damaged arch. This procedure is experimentally applied to the same arch and damage configurations considered in the present paper. The full details of the method and experimental tests are reported in a recent research paper (Pau et al. 2011).
Analogously to the static case, it is important to know the minimum number of frequencies necessary to provide a unique solution. In a damaged configuration characterized by the parameters $x_{d}, k_{d}$ the values of the natural frequencies $\omega_{i}$ can be calculated, for instance, by a finite element code. On the contrary, when $\omega_{i}$ is known for the damaged arch, for each possible damage position $x_{d}$, a stiffness $k_{i d}\left(x_{d}\right)$ exists, which corresponds to a value of the $i$-th natural frequency equal to $\omega_{i}$. Therefore, by taking into account all the possible positions of the damage, a curve $k_{i d}\left(x_{d}\right)$ can be obtained for each frequency $i$. Analogously to what is found in the static tests regarding the minimum $i$, it can be shown that, for the geometric ratios of the arch considered, a unique solution to the inverse problem is found using at least three frequencies.
The optimal estimate of $x_{d}, k_{d}$ is obtained by minimizing an objective function which measures the difference between numerical (FE model) $\Delta \omega_{i}\left(x_{d}, k_{d}\right)$ and experimental variations $\Delta \omega_{e i}$ of natural frequencies in the undamaged and damaged states, normalized to their undamaged counterparts $\omega_{i}^{U}$ and $\omega_{e i}^{U}$

$$
\begin{equation*}
G\left(x_{d}, k_{d}\right)=\sum_{i=1}^{n}\left(\frac{\Delta \omega_{i}\left(x_{d}, k_{d}\right)}{\omega_{i}^{U}}-\frac{\Delta \omega_{e i}}{\omega_{e i}^{U}}\right)^{2} \tag{12}
\end{equation*}
$$

In order to evaluate the experimental natural frequencies, the structure has been excited by an instrumented hammer and its response measured by means of accelerometers at seven locations. Each test was repeated 10 times. Table 6 reports the values of the equivalent stiffness and damage location, obtained by means of the procedure based on the frequency comparison and for the damage configurations described. Satisfactory results are obtained with a mean error of $15.9 \%$ on the equivalent spring stiffness, and $2.2 \%$ on the location. The errors decrease with increasing damage, where the systematic errors are less important.
The comparison between the two identification techniques performed on this structure allows to conclude that the errors on the identified damage parameters using static identification are much higher than the corresponding errors obtained through dynamic identification even for the strongest damage scenario which is the one with the best performance.

Table 6 Identified, damage parameters by means of dynamic tests and related errors

|  | D1 |  | D2 |  | D3 |  | D4 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Ident. val | err | Ident. val | err | Ident. val | err | Ident. val | err |
| $k_{d}$ | 393.0 | 29.1\% | 158.2 | 15.3\% | 72.4 | 11.6\% | 33.2 | 7.5\% |
| $x_{d}$ | 396 | 8.5\% | 429 | 2.1\% | 430 | 0.0\% | 430 | 0.0\% |

## 6. Conclusions

In this paper, the static response of a parabolic arch, in which a notch reduces the height of the cross section at a given abscissa, has been studied in its undamaged and damaged states. The damage has been modelled by means of a rotational spring of suitable rigidity. The damage parameters are hence location and stiffness of the equivalent spring. The direct problem of static response and inverse problem of damage identification have been studied. The damage parameters are identified by minimizing an objective function which measures the differences between analytical vertical displacements and measured ones. In order to test the reliability of the proposed procedure, the analyses have been firstly carried out using pseudo-experimental data and then by experimental data. The objective functions obtained using pseudo-experimental data provide as expected a unique global minimum correspondent to the exact solution, while the experimental ones provide errors in the identified damage parameters. The errors decrease with the intensity of the damage. In the objective functions, together with the global minimum, other local minima occur, sometimes close to the global one, causing difficulties in the solution to the inverse problem. The performance of the proposed procedure for the identification of damage parameters using experimental data is compared to an analogous procedure based on the measurements of natural frequencies of the undamaged and damaged arch. The comparison allows to conclude that, for the considered case, the errors on the identified damage parameters using static identification is much higher than the corresponding error obtained through dynamic identification even for the strongest damage scenario which is the one with the best performance.

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