Structural Engineering and Mechanics, Vol. 39, No. 5 (2011) 669-682 669
DOI: http://dx.doi.org/10.12989/sem.2011.39.5.669

Matrix-based Chebyshev spectral approach to dynamic analysis of non-uniform Timoshenko beams

W.Y. Wang[1], J.Y. Liao[2] and L.W. Hourng*[1]

[1]*Department of Mechanical Engineering, National Central University, Taoyuan, Taiwan*
[2]*System Development Center, Chung-Sun Institute of Science and Technology, Taoyuan, Taiwan*

(Received May 17, 2010, Accepted June 20, 2011)

Abstract. A Chebyshev spectral method (CSM) for the dynamic analysis of non-uniform Timoshenko beams under various boundary conditions and concentrated masses at their ends is proposed. The matrix-based Chebyshev spectral approach was used to construct the spectral differentiation matrix of the governing differential operator and its boundary conditions. A matrix condensation approach is crucially presented to impose boundary conditions involving the homogeneous Cauchy conditions and boundary conditions containing eigenvalues. By taking advantage of the standard powerful algorithms for solving matrix eigenvalue and generalized eigenvalue problems that are embodied in the MATLAB commands, *chebfun* and *eigs*, the modal parameters of non-uniform Timoshenko beams under various boundary conditions can be obtained from the eigensolutions of the corresponding linear differential operators. Some numerical examples are presented to compare the results herein with those obtained elsewhere , and to illustrate the accuracy and effectiveness of this method.

Keywords: Chebyshev spectral method; modal analysis; spectral differentiation matrix; chebfun; Timoshenko beam

1. Introduction

Members with variable cross-sections have been extensively used in many industrial fields, including the mechanical, civil, aerospace and rocket engineering fields, to optimize the distribution of weight and strength, and sometimes to satisfy some special requirements. The cost of fabricating such members is relatively high and offsets their advantages. However, when weight and performance are the most important considerations, members with a variable cross-section are preferred.

Approximate solutions to such problems can be found by variational methods. Based on practical considerations, energy methods are felt not to be straightforward since they require *a priori* selection of displacement functions that satisfy at least, the geometric boundary conditions. Satisfying these conditions is difficult, especially in cases of mixed boundaries. Also, the necessary application of variational calculus commonly requires a knowledge of the principles of mechanics that frequently exceeds that of many engineers. Including the shear deformation of beam analysis

*Corresponding author, Professor, E-mail: lwhong@cc.ncu.edu.tw

which significantly affects the dynamic characteristics of short beams, especially at higher modes of vibrations would complicate the matter. Exact solutions for the behavior of Timoshenko beams with arbitrary variable coefficients governing equations do not exist, so related problems must be studied using approximate numerical methods, such as the finite element method (Rossi *et al.* 1990, Cleghorn and Tabarrok 1992, Gutierrez *et al.* 1991, Rossi and Laura 1993), the transfer matrix method (Irie *et al.* 1980), and the Rayleigh-Ritz method (Gutierrez 1991). In special cases in which the coefficients in the governing differential equations are of polynomial form, the method of Frobenius has sometimes been used (Lee and Lin 1992, 1995, Leung and Zhou 1995). Recently, Posiadala (1997) examined the free vibrations of uniform Timoshenko beams with attachments using the Lagrange multiplier formalism. Ho and Chen (1998) analyzed general elastically restrained non-uniform beams using a differential transform approach. Karami and Malekzadeh (2003) developed a differential quadrature element method for determining the vibration of shear deformable beams under general boundary conditions. Hsu *et al*. (2009) solved the free vibration problem of uniform Timoshenko beams using the Adomian modified decomposition method.

The formulation of the free vibration of a Timoshenko beam using the Chebyshev spectral method is straightforward and sufficiently powerful to produce approximate solutions that are close to exact solutions. This method has been highly successful in such areas as turbulence modeling, weather prediction and nonlinear waves. Lee and Schultz (2004) presented an eigenvalue analysis of Timoshenko beams and axisymmetric Mindlin plates using the pseudo spectral method. They used Chebyshev series expansion to generate a recurrence formula of expansion coefficients. Ruta (2006) applied Chebyshev series approximation to solve the vibration problem of a non-prismatic Timoshenko beam resting a two-parameter elastic foundation. Salarieh and Ghorashi (2006) analyzed the free vibration of a cantilever Timoshenko beam with a rigid tip mass. Ferreira and Fasshuer (2006) explored the free vibration of Timoshenko beams and Mindlin plates using the RBF-pseudospectral method.

In this study, a novel matrix-based Chebyshev spectral method (Don and Solomonoff 1995, Costa and Don 2000, Trefethen 2000) is implemented via Matlab matrix eigenvalue commands to analyze the modal parameters of non-uniform Timoshenko beams under various boundary conditions.

The system that is introduced herein is built on the chebfun system. In the chebfun system, vectors are replaced by functions that are defined on an interval [a, b], and commands like these are overloaded by their continuous analogues such as integral, derivative, or L^2-norm. The functions are represented by interpolants in suitably rescaled Chebyshev points $\cos(j\pi/n)$, $0 \le j \le n$ or interpolants in suitably rescaled Chebyshev polynomials, either globally or piecewise. The process is terminated when the Chebyshev coefficients fall to a relative magnitude of about 10^{-16}. Thus the central principle of the chebfun system is to evaluate functions in sufficiently many Chebyshev points for a polynomial interpolant to be accurate to machine precision. This study proposes a method of so doing based on collocation in the Chebyshev points and lazy evaluation of the associated spectral discretization matrices, all implemented in object-oriented *Matlab* on top of the chebfun system.

2. Chebyshev spectral method and differentiation matrix

According to Don and Solomonoff (1995), in the domain $x \in [-1, 1]$, the Nth-order Chebyshev polynomial $T_N(x)$ and Chebyshev-Gauss-Lobatto (CGL) collocation points x_j can be expressed respectively as

$$T_N(x) = \cos(N\cos^{-1}x) \tag{1}$$

$$x_j = \cos\left(\frac{j\pi}{N}\right), \quad j = 0, \ldots, N \tag{2}$$

Notably, the CGL points, which are numbered from right to left for convenience, are clustered near ±1.

Let $p(x)$ be a smooth function $p(x)$ in the domain $x \in [-1, 1]$. $p(x)$ is interpolated by constructing the N-order interpolation polynomial $g(x)$, where $g(x)$ is the polynomial of degree N, $g_j = g(x_j) = p(x_j)$, and $j = 0,\ldots, N$. The function $g(x)$ can be written as

$$g(x) = \sum_{j=0}^{N} \frac{(-1)^{j+1}(1-x^2)T'_N(x) \cdot g_j}{c_j N^2 (x-x_j)}, \quad j = 0, \ldots, N$$

$$c_j = \begin{cases} 1, & j = 1, \ldots, N-1 \\ 2, & j = 0 \text{ or } N \end{cases} \tag{3}$$

The derivative of $g(x)$ at the CGL points x_j can then be computed via matrix-vector multiplication, which can be formally represented as

$$\begin{bmatrix} g'(x_0) \\ g'(x_1) \\ \vdots \\ g'(x_N) \end{bmatrix} = [\mathbf{D}_N] \begin{bmatrix} g'(x_0) \\ g'(x_1) \\ \vdots \\ g'(x_N) \end{bmatrix} \tag{4}$$

where $[\mathbf{D}_N]$ is an $(N+1) \times (N+1)$ matrix. The elements of the matrix in (Don and Solomonoff 1995) are

$$(\mathbf{D}_N)_{00} = \frac{2N^2+1}{6}$$

$$(\mathbf{D}_N)_{NN} = -(\mathbf{D}_N)_{00}$$

$$(\mathbf{D}_N)_{jj} = \frac{-x_j}{2\sin^2\left(\frac{j\pi}{N}\right)}, \quad j = 1, \ldots, N-1$$

$$(\mathbf{D}_N)_{jj} = \frac{-c_i}{2c_j} \frac{(-1)^{i+j}}{\sin\frac{(i+j)2\pi}{N} \sin\frac{(i-j)2\pi}{N}}, \quad i \neq j, \quad i,j = 0, \ldots, N-1 \tag{5}$$

Concerning higher derivatives, we remark that often the second- and higher-derivative matrices are equal to the first-derivative matrix raised to the appropriate power. However, computing higher derivative matrices by computing the powers of the first-derivative matrix is not recommended. The computation of powers of a full matrix requires $O(N^3)$ flops, compared to the $O(N^2)$ flops for the recursive algorithm (Costa and Don 2000), which is described as follows. Not only is this recursion faster, but also introduces less roundoff error compared to that when computing matrix powers.

$$(\mathbf{D}_N^m)_{ij} = m[(\mathbf{D}_N^{m-1})_{ii}(\mathbf{D}_N)_{ij} - (x_i - x_j)^{-1}(\mathbf{D}_N^{m-1})_{ij}] \quad i \neq j$$
$$(\mathbf{D}_N^m)_{ii} = -\sum_{j=0, j\neq i}^{N} (\mathbf{D}_N^m)_{ij} \tag{6}$$

Eqs. (5) and (6) play significant roles in this study. By using the Chebyshev spectral differentiation matrix, a linear differential operator may be transformed into a matrix operator; that is, the modal solutions of Timoshenko beams can be eigensolutions of the transformed matrix.

3. Free vibration analysis of a non-uniform Timoshenko beam by the Chebyshev spectral method

The material and geometric parameters E, G, A, I, ν, and ρ of a non-uniform Timoshenko beam (Fig. 1), are functions of the longitudinal coordinate x. Applying Hamilton's Principle and carrying out integration by parts yield the governing equations and external boundary conditions for free vibration with frequency ω of the non-uniform Timoshenko beam

$$\frac{d}{dx}\left[\kappa GA\left(\frac{dw(x)}{dx} - \phi(x)\right)\right] + \omega^2 \rho A w(x) = 0 \tag{7a}$$

$$\kappa GA\left(\frac{dw(x)}{dx} - \phi(x)\right) + \frac{d}{dx}\left(EI\frac{d\phi(x)}{dx}\right) + \omega^2 \rho I \phi(x) = 0 \tag{7b}$$

in which a shear factor κ is applied to all terms that involve G and A, such that κGA accounts for the non-uniform distribution of shear stress across the cross-sectional area.

The corresponding external boundary conditions are

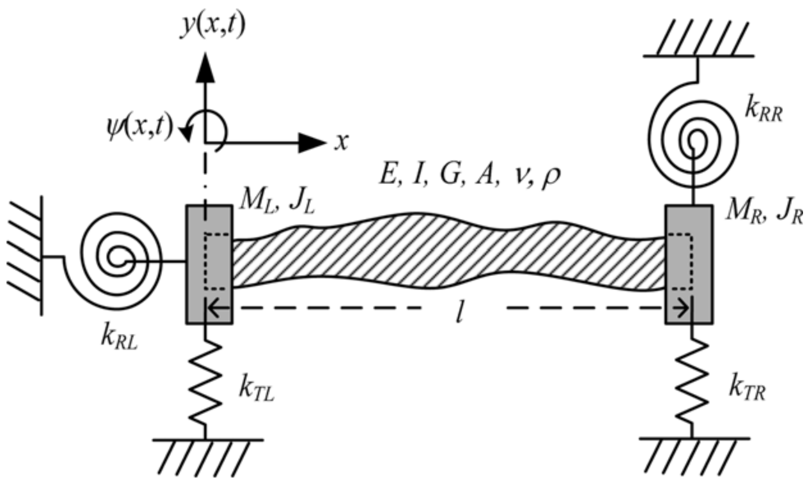

Fig. 1 The non-uniform Timoshenko beams with various boundary conditions

$$x = 0$$

$$\kappa GA\left(\frac{dw}{dx}-\phi\right)-(k_{TL}-M_L\omega^2)w = 0$$

$$EI\frac{d\phi}{dx}-(k_{RL}-J_L\omega^2)\phi = 0 \tag{8a}$$

$$x = l$$

$$\kappa GA\left(\frac{dw}{dx}-\phi\right)+(k_{TR}-M_R\omega^2)w = 0$$

$$EI\frac{d\phi}{dx}+(k_{RR}-J_R\omega^2)\phi = 0 \tag{8b}$$

where k_{TL} and k_{RL} are the translational and rotational spring constants, respectively, at the left end; k_{TR} and k_{RR} are the translational and rotational spring constants, respectively, at the right end; M_L and J_L are the concentrated mass and moment of inertia, respectively, of the mass attached to left end of the beam, and M_R and J_R are the concentrated mass and moment of inertia, respectively, of the mass attached to right end of the beam.

The following dimensionless quantities are defined.

$$u = \frac{x}{l},\quad w(u) = \frac{w(x)}{l},\quad \phi(u) = \phi(x)$$

$$p(u) = \frac{\kappa G(u)A(u)}{\kappa G(0)A(0)},\quad q(u) = \frac{E(u)I(u)}{E(0)I(0)},\quad m(u) = \frac{\rho(u)A(u)}{\rho(0)A(0)}$$

$$r(u) = \frac{\rho(u)I(u)}{\rho(0)I(0)},\quad \xi = \frac{\kappa G(0)A(0)l^2}{E(0)I(0)},\quad \eta = \frac{I(0)}{A(0)l^2},\quad \lambda^2 = \frac{\omega^2\rho(0)A(0)l^4}{E(0)I(0)}$$

$$K_{TL} = \frac{k_{TL}l^3}{E(0)I(0)},\quad K_{TR} = \frac{k_{TR}l^3}{E(0)I(0)},\quad K_{RL} = \frac{k_{RL}l}{E(0)I(0)},\quad K_{RR} = \frac{k_{RR}l}{E(0)I(0)}$$

$$\mu_R = \frac{M_R}{\rho(0)A(0)l},\quad \mu_L = \frac{M_L}{\rho(0)A(0)l},\quad \gamma_R = \frac{J_R}{\rho(0)A(0)l^3},\quad \gamma_L = \frac{J_L}{\rho(0)A(0)l^3} \tag{9}$$

Notably, the range of the independent variable is $u \in [0, 1]$; however, in the Chebyshev spectral method, the domain prefers to normalize in [−1, 1], with the following transformation

$$z = 2u-1,\qquad z\in[-1,1] \tag{10}$$

Substituting Eqs. (9) and (10) into Eq. (7) allows the governing equations may be rewritten in the following dimensionless forms.

$$\frac{\xi}{m(z)}\left\{2\frac{d}{dz}\left[p(z)\left(2\frac{dw(z)}{dz}-\phi(z)\right)\right]\right\}+\lambda^2 w(z) = 0$$

$$\frac{\xi p(z)}{\eta r(z)}\left(2\frac{dw(z)}{dz}-\phi(z)\right)+\frac{2}{\eta r(z)}\frac{d}{dz}\left(2q(z)\frac{d\phi(z)}{dz}\right)+\lambda^2\phi(z) = 0 \tag{11}$$

Let $\mathfrak{D}$ denotes the differentiation operator, with $\mathfrak{D}^k h = d^k h/dz^k$ then Eq. (11) is rearranged in a matrix form as

$$\begin{bmatrix} \frac{4\xi}{m(z)}(p(z)\mathfrak{D}^2 + p'(z)\mathfrak{D}) & \frac{-2\xi}{m(z)}(p(z)\mathfrak{D} + p'(z)) \\ \frac{2\xi p(z)}{\eta r(z)}\mathfrak{D} & \frac{1}{\eta r(z)}(4q(z)\mathfrak{D}^2 + 4q'(z)\mathfrak{D} - \xi p(z)) \end{bmatrix} \begin{Bmatrix} w \\ \phi \end{Bmatrix} = -\lambda^2 \begin{Bmatrix} w \\ \phi \end{Bmatrix} \tag{12}$$

Let $\{W\}$ and $\{\Phi\}$ denote the vectors of w_i and ϕ_i, respectively, evaluated at CGL collocation points z_i, and be expressed as

$$\{W\} = \begin{Bmatrix} w_0 \\ w_1 \\ \vdots \\ w_N \end{Bmatrix}; \quad \{\Phi\} = \begin{Bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_N \end{Bmatrix} \tag{13}$$

where $z_i = \cos\left(\frac{i\pi}{N}\right)$, $w_i = w(z_i)$, $\phi_i = \phi(z_i)$, $i = 0, \ldots, N$.

In terms of the Chebyshev spectral differentiation matrix, Eqs. (4)-(6), Eq. (12) can then be reduced to

$$[\mathbf{L}]\{\mathbf{Z}\} = \lambda^2[\mathbf{Z}] \tag{14}$$

where

$$[\mathbf{L}] = \begin{bmatrix} -\frac{4\xi}{m}\mathbf{P}\mathbf{D}_N^2 - \frac{4\xi}{m}\mathbf{P}'\mathbf{D}_N & \frac{2\xi}{m}\mathbf{P}\mathbf{D}_N + \frac{2\xi}{m}\mathbf{P}' \\ \frac{-2\xi}{\eta r}\mathbf{P}\mathbf{D}_N & -\frac{4q}{\eta r}\mathbf{Q}\mathbf{D}_N^2 - \frac{4}{\eta r}\mathbf{Q}'\mathbf{D}_N + \frac{\xi}{\eta r}\mathbf{P} \end{bmatrix}$$

$$[\mathbf{Z}] = \begin{Bmatrix} W \\ \Phi \end{Bmatrix}$$

$$\begin{aligned} \mathbf{P} &= \mathrm{diag}\{p(x_0), p(x_1), \ldots, p(x_N)\}_{(N+1)\times(N+1)} \\ \mathbf{P}' &= \mathrm{diag}\{p'(x_0), p'(x_1), \ldots, p'(x_N)\}_{(N+1)\times(N+1)} \\ \mathbf{Q} &= \mathrm{diag}\{q(x_0), q(x_1), \ldots, q(x_N)\}_{(N+1)\times(N+1)} \\ \mathbf{Q} &= \mathrm{diag}\{q'(x_0), q'(x_1), \ldots, q'(x_N)\}_{(N+1)\times(N+1)} \end{aligned} \tag{15}$$

where diag{…} is the diagonal matrix.

The boundary conditions Eq. (8a) and (8b) can be reduced in terms of the Chebyshev spectral differentiation matrix to yield

$$\begin{Bmatrix} B_{L,0} \\ B_{L,N+1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} B_{R,N} \\ B_{R,2N+1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{16}$$

where vectors $\{B_L\}$ and $\{B_R\}$ are defined as

$$\{B_L\} = \begin{Bmatrix} B_{L,0} \\ B_{L,1} \\ \vdots \\ B_{L,2N+1} \end{Bmatrix} = \left(\begin{bmatrix} 2\xi \mathbf{P}\mathbf{D}_N & -\xi \mathbf{P} \\ \mathbf{0}_{N+1} & 2\mathbf{Q}\mathbf{D}_N \end{bmatrix} - [A_L(\lambda)] \right) \{\mathbf{Z}\} \tag{17a}$$

where $[A_L(\lambda)] = \begin{bmatrix} (K_{TL} + \mu_L \lambda^2)\mathbf{I}_{N+1} & \mathbf{0}_{N+1} \\ \mathbf{0}_{N+1} & (K_{RL} + \gamma_L \lambda^2)\mathbf{I}_{N+1} \end{bmatrix}$, and

$$\{B_R\} = \begin{Bmatrix} B_{R,0} \\ B_{R,1} \\ \vdots \\ B_{R,2N+1} \end{Bmatrix} = \left(\begin{bmatrix} 2\xi \mathbf{P}\mathbf{D}_N & -\xi \mathbf{P} \\ \mathbf{0}_{N+1} & 2\mathbf{Q}\mathbf{D}_N \end{bmatrix} + [A_R(\lambda)] \right) \{\mathbf{Z}\} \tag{17b}$$

where $[A_R(\lambda)] = \begin{bmatrix} (K_{TR} - \mu_R \lambda^2)\mathbf{I}_{N+1} & \mathbf{0}_{N+1} \\ \mathbf{0}_{N+1} & (K_{RR} - \gamma_R \lambda^2)\mathbf{I}_{N+1} \end{bmatrix}$.

Consequently, various external classical and non-classical boundary conditions can be modeled by assembling different conditions in Eq. (17). For illustrative purposes, under classical boundary conditions, such as free, pinned and clamped, Eqs. (17a) and (17b) can further be reduced as follows.

Free at end X (either left or right)

$$K_{TX} = K_{RX} = \mu_X = \gamma_x = 0$$

$$\{B_X\} = \begin{bmatrix} 2\mathbf{P}\mathbf{D}_N & -\mathbf{P} \\ \mathbf{0}_{N+1} & 2\mathbf{Q}\mathbf{D}_N \end{bmatrix} \{\mathbf{Z}\} \tag{18a}$$

Simple supported at end X

$$K_{TX} \to \infty, \quad K_{RX} = \mu_X = \gamma_x = 0$$

$$\{B_X\} = \begin{bmatrix} \mathbf{I}_{N+1} & -\mathbf{0}_{N+1} \\ \mathbf{0}_{N+1} & 2\mathbf{P}\mathbf{D}_N \end{bmatrix} \{\mathbf{Z}\} \tag{18b}$$

Clamped at end X

$$K_{TX}, \ K_{RX} \to \infty, \ \mu_X = \gamma_x = 0$$

$$\{B_X\} = \begin{bmatrix} \mathbf{I}_{N+1} & \mathbf{0}_{N+1} \\ \mathbf{0}_{N+1} & \mathbf{I}_{N+1} \end{bmatrix} \{\mathbf{Z}\} \tag{18c}$$

Furthermore, for the case of a mass attached at one end, the boundary condition of a Timoshenko beam with the left end clamped and mass attached at the free right end is illustrated. Under the specified boundary conditions, with $K_{TL} \to \infty, K_{RL} \to \infty, K_{TR} = 0, K_{RR} = 0$ Eqs. (17a) and (17b) can be reduced to

$$\{B_L\} = \begin{bmatrix} \mathbf{I}_{N+1} & -\mathbf{0}_{N+1} \\ \mathbf{0}_{N+1} & \mathbf{I}_{N+1} \end{bmatrix} \{\mathbf{Z}\} \tag{19a}$$

and

$$\{B_R\} = \begin{bmatrix} 2\xi \mathbf{P}\mathbf{D}_N - \lambda^2 \mu_R \mathbf{I}_{N+1} & -\xi \mathbf{P} \\ \mathbf{0}_{N+1} & 2\mathbf{Q}\mathbf{D}_N - \lambda^2 \mu_R \mathbf{I}_{N+1} \end{bmatrix} \{\mathbf{Z}\} \tag{19b}$$

Substituting Eq. (16) into Eq. (19b) allows the right end condition to be further reduced to

$$\begin{Bmatrix} B_{\lambda,N} \\ B_{\lambda,2N+1} \end{Bmatrix} = \lambda^2 \begin{Bmatrix} w_N \\ \phi_N \end{Bmatrix} \tag{20}$$

where vector $\{B_\lambda\}$ is defined as

$$\{B_\lambda\} = \begin{Bmatrix} B_{\lambda,0} \\ B_{\lambda,1} \\ \vdots \\ B_{\lambda,2N+1} \end{Bmatrix} = \frac{1}{\mu_R} \begin{bmatrix} 2\xi \mathbf{P}\mathbf{D}_N & -\xi \mathbf{P} \\ \mathbf{0}_{N+1} & \dfrac{2\mu_R \mathbf{Q}\mathbf{D}_N}{\gamma_R} \end{bmatrix} \begin{Bmatrix} W \\ \Phi \end{Bmatrix} \tag{21}$$

From Eq. (20), the right end boundary condition contains an eigenvalue due to the attached mass. For a specific beam vibrating mode, the attached mass will vibrate with the corresponding natural frequency.

4. Imposing boundary conditions on the Chebyshev spectral matrices

In summary of Eqs. (18a)-(18c) and (20), two boundary conditions must be handled, i.e., a homogeneous Cauchy condition, and boundary condition varying with the eigenvalues of the dynamic system.

Based on the properties of the boundary conditions, matrix [**L**] and vector [**Z**] in Eq. (14) can be rearranged and partitioned as follows.

$$\begin{bmatrix} \mathbf{L}_{rr} & \mathbf{L}_{rc} & \mathbf{L}_{r\lambda} \\ \mathbf{L}_{cr} & \mathbf{L}_{cc} & \mathbf{L}_{c\lambda} \\ \mathbf{L}_{\lambda r} & \mathbf{L}_{\lambda c} & \mathbf{L}_{\lambda\lambda} \end{bmatrix} \begin{Bmatrix} \mathbf{Z}_r \\ \mathbf{Z}_c \\ \mathbf{Z}_\lambda \end{Bmatrix} = \lambda^2 \begin{Bmatrix} \mathbf{Z}_r \\ \mathbf{Z}_c \\ \mathbf{Z}_\lambda \end{Bmatrix} \tag{22}$$

where $\{\mathbf{Z}_c\}$ and $\{\mathbf{Z}_\lambda\}$ are the vectors obtained by homogeneous Robin conditions and by boundary conditions containing eigenvalues, respectively. Additionally, $\{\mathbf{Z}_r\}$ is the vector of generalized coordinates in the interior CGL points.

In the case of homogeneous Robin conditions, such as given by Eqs. (18a)-(18c), the equations can be rewritten as

$$[\mathbf{C}_{cr}\ \ \mathbf{C}_{cc}\ \ \mathbf{C}_{c\lambda}]\{\mathbf{Z}\} = 0\{\mathbf{Z}_c\} \tag{23}$$

For the case of boundary conditions containing eigenvalues, such as in Eq. (20), the equation can be rewritten as

$$[\mathbf{C}_{\lambda r}\ \ \mathbf{C}_{\lambda c}\ \ \mathbf{C}_{\lambda\lambda}]\{\mathbf{Z}\} = \lambda^2\{\mathbf{Z}_\lambda\} \tag{24}$$

Substituting Eq. (23) for the second row in Eq. (22), and substituting Eq. (24) from the third row in Eq. (22), then Eq. (22) can be reduced to

$$\begin{bmatrix} \mathbf{L}_{rr} & \mathbf{L}_{rc} & \mathbf{L}_{r\lambda} \\ \mathbf{C}_{cr} & \mathbf{C}_{cc} & \mathbf{C}_{c\lambda} \\ \mathbf{L}_{\lambda r}-\mathbf{C}_{\lambda r} & \mathbf{L}_{\lambda c}-\mathbf{C}_{\lambda c} & \mathbf{L}_{\lambda\lambda}-\mathbf{C}_{\lambda\lambda} \end{bmatrix} \begin{Bmatrix} \mathbf{Z}_r \\ \mathbf{Z}_c \\ \mathbf{Z}_\lambda \end{Bmatrix} = \lambda^2 \begin{Bmatrix} \mathbf{Z}_r \\ \mathbf{0}_c \\ \mathbf{0}_\lambda \end{Bmatrix} \tag{25}$$

where $\{\mathbf{0}_c\}$ and $\{\mathbf{0}_\lambda\}$ denote zero vectors with the same size as vectors $\{\mathbf{Z}_c\}$ and $\{\mathbf{Z}_\lambda\}$, respectively.

Using the definitions,

$$\{\mathbf{Z}_B\} = \begin{Bmatrix} \mathbf{Z}_c \\ \mathbf{Z}_\lambda \end{Bmatrix}, \quad \{\mathbf{0}_B\} = 0\{\mathbf{Z}_B\}, \quad [\mathbf{R}_{rc}] = [\mathbf{L}_{rc}\ \ \mathbf{L}_{c\lambda}]$$

$$[\mathbf{R}_{cc}] = \begin{bmatrix} \mathbf{C}_{cc} & \mathbf{C}_{c\lambda} \\ \mathbf{L}_{\lambda c}-\mathbf{C}_{\lambda c} & \mathbf{L}_{\lambda\lambda}-\mathbf{C}_{\lambda\lambda} \end{bmatrix}, \quad [\mathbf{R}_{cr}] = \begin{bmatrix} \mathbf{C}_{cr} \\ \mathbf{L}_{\lambda r}-\mathbf{C}_{\lambda r} \end{bmatrix} \tag{26}$$

Eq. (25) can be rewritten as

$$\begin{bmatrix} \mathbf{L}_{rr} & \mathbf{R}_{rc} \\ \mathbf{R}_{cr} & \mathbf{R}_{cc} \end{bmatrix} \begin{Bmatrix} \mathbf{Z}_r \\ \mathbf{Z}_B \end{Bmatrix} = \lambda^2 \begin{Bmatrix} \mathbf{Z}_r \\ \mathbf{0}_B \end{Bmatrix} \tag{27}$$

Consequently, Eq. (27) can further be reduced to the following condensed form

$$[\mathbf{L}_{rr}-\mathbf{R}_{rc}(\mathbf{R}_{cc})^{-1}\mathbf{R}_{cr}]\{\mathbf{Z}_r\} = [\mathbf{L}_{cond}]\{\mathbf{Z}_r\} = \lambda^2\{\mathbf{Z}_r\} \tag{28}$$

Notably, the eigensolutions of the condensed matrix of $[\mathbf{L}_{cond}]$ in Eq. (28) satisfy the imposed boundary conditions, and the modal parameters of Timoshenko beams can then be obtained. Let $\{\hat{\mathbf{Z}}_r\}$ be the eigensolution of Eq. (28); the solution of vector $[\hat{\mathbf{Z}}_B]$ can be obtained by

$$[\hat{\mathbf{Z}}_B] = -[(\mathbf{R}_{cc})^{-1}\mathbf{R}_{cr}]\{\hat{\mathbf{Z}}_r\} \tag{29}$$

5. Numerical results

In the following examples, the material properties of the beam are assumed to be constant, while the cross-sectional properties vary with length. Case 1 is a Timoshenko beam with constant width and linearly varying thickness. The dimensionless parameters of the cross-section are

$$p(u) = m(u) = 1 + \alpha u$$
$$q(u) = r(u) = (1 + \alpha u)^3 \tag{30}$$

Case 2 is a Timoshenko beam which width and depth both vary linearly with the taper ratio α. The dimensionless parameters of the cross-section are

$$p(u) = m(u) = (1 + \alpha u)^2$$
$$q(u) = r(u) = (1 + \alpha u)^4 \tag{31}$$

A preliminary run of the check for convergence of the eigenvalues of the non-uniform Timoshenko beam under various boundaries. Tables 1 to 3 present the results for case 1 under C-F, C-S and C-C boundary conditions. The number of collocation points determines the size of the problem. The rapid convergence of the Chebyshev spectral method is evidenced by $N < 20$ for the convergence of the first five eigenvalues to five significant digits. Tables 1 to 3 also show the first five dimensionless natural frequencies. Excellent agreement is achieved between the results of the

Table 1 Convergence results of the first five dimensionless frequencies for $N = 30$ in Case 1. (C-F) ($\alpha = -0.2$, $\gamma_R = 0$, $\mu_R = 0$, $\eta = 0.01$, $\xi = 1/3.12\eta$, $\upsilon = 0.3$, $\kappa = 5/6$)

N	$\lambda_1^{(N)}$	$\lambda_2^{(N)}$	$\lambda_3^{(N)}$	$\lambda_4^{(N)}$	$\lambda_5^{(N)}$
8	3.3306	14.2974	30.7985	47.8769	62.0107
10	3.3306	14.2891	30.7081	47.8338	65.3291
13	3.3307	14.2892	30.7108	47.7510	64.9867
15	3.3307	14.2892	30.7108	47.7502	64.9978
18	3.3307	14.2892	30.7108	47.7502	64.9770
20	3.3307	14.2892	30.7108	47.7502	64.9770
25	3.3307	14.2892	30.7108	47.7502	64.9770
30	3.3307	14.2892	30.7108	47.7502	64.9770
*Leung and Zhou (2001)	3.33	14.29	30.71	47.70	-------

Table 2 Convergence results of the first five dimensionless frequencies for N=30 in Case 1. (C-S) ($\alpha = -0.2$, $\gamma_R = 0$, $\mu_R = 0$, $\eta = 0.01$, $\xi = 1/3.12\eta$, $\upsilon = 0.3$, $\kappa = 5/6$)

N	$\lambda_1^{(N)}$	$\lambda_2^{(N)}$	$\lambda_3^{(N)}$	$\lambda_4^{(N)}$	$\lambda_5^{(N)}$
8	10.6875	26.1159	43.8044	60.2425	68.2347
10	10.6869	26.1070	43.6079	61.6402	68.4820
12	10.6869	26.1072	43.5000	61.6486	68.4135
15	10.6869	26.1072	43.5907	61.6560	68.4210
17	10.6869	26.1072	43.5907	61.6559	68.4207
20	10.6869	26.1072	43.5907	61.6560	68.4207
25	10.6869	26.1072	43.5907	61.6560	68.4207
30	10.6869	26.1072	43.5907	61.6560	68.4207
*Leung and Zhou (2001)	10.69	26.11	43.60	60.04	-------

Table 3 Convergence results of the first five dimensionless frequencies for N = 30 in Case 1. (C-C) ($\alpha = -0.2$, $\gamma_R = 0$, $\mu_R = 0$, $\eta = 0.01$, $\xi = 1/3.12\eta$, $\upsilon = 0.3$, $\kappa = 5/6$)

N	$\lambda_1^{(N)}$	$\lambda_2^{(N)}$	$\lambda_3^{(N)}$	$\lambda_4^{(N)}$	$\lambda_5^{(N)}$
8	13.2223	27.7757	45.0600	60.6723	72.4833
10	13.2223	27.7782	44.7048	61.7450	75.5222
13	13.2223	27.7782	44.6971	61.8062	72.5549
15	13.2223	27.7782	44.6971	61.8066	72.5547
18	13.2223	27.7782	44.6971	61.8066	72.5547
20	13.2223	27.7782	44.6971	61.8066	72.5547
25	13.2223	27.7782	44.6971	61.8066	72.5547
30	13.2223	27.7782	44.6971	61.8066	64.5547
*Leung and Zhou (2001)	13.32	27.78	44.72	60.16	-------

Table 4 First six dimensionless frequencies of a cantilever tapered beam with attached mass at right end : Case 1 $\upsilon = 0.3$, $\kappa = 5/6$, $\alpha = -0.2$, $\gamma_R = 0.0$, $\mu_R = \mu(1 + \alpha/2)$, $\xi = 1/3.12\eta$

	$\mu = 0.2$, $\eta = 0.0016$					$\mu = 0.2$, $\eta = 0.01$				
	Present			Leung and Zhou (2001)	Rossi *et al.* (1990)	Present			Leung and Zhou (2001)	Rossi *et al.* (1990)
	N = 10	15	20			N = 10	15	20		
λ_1	2.5888	2.5888	2.5888	2.59	2.59	2.4618	2.4618	2.4618	2.46	2.46
λ_2	15.6704	15.6708	15.6708	15.67	15.67	12.2684	12.2687	12.2687	2.27	12.27
λ_3	41.7452	41.5309	41.5309	41.53	41.56	27.7448	27.7252	27.7252	27.73	27.78
λ_4	78.3230	75.6669	75.6632	75.67	75.84	45.4958	44.8895	44.8893	44.89	45.15
λ_5	114.8122	115.0394	114.9829	-	-	65.4581	62.9629	62.9477	-	-
λ_6	140.5508	157.2472	157.4484	-	-	71.9424	68.9766	68.9785	-	-

Table 5 First six dimensionless frequencies of a cantilever tapered beam with attached mass at right end : Case 2, $\upsilon = 0.3$, $\kappa = 5/6$, $\alpha = -0.1$, $\eta = 0.0008$, $\xi = 400$

	$\gamma_R = \mu_R = 0$					$\gamma_R = \mu_R = 0.2$				
	Present			Leung and Zhou (2001)	Lee and Lin (1992)	Present			Leung and Zhou (2001)	Lee and Lin (1992)
	N = 10	15	20			N = 12	15	20		
λ_1	3.6464	3.6464	3.6464	3.65	3.65	1.6656	1.6656	1.6656	1.67 1.67	
λ_2	20.5725	20.5742	20.5742	20.57	20.57	5.1390	5.1391	5.1391	5.14	5.14
λ_3	53.4353	53.4251	53.4251	53.45	53.43	23.9839	24.0107	24.0107	24.01	24.01
λ_4	98.1446	97.1232	97.1230	96.91	97.12	56.3211	56.5033	56.4986	56.51	56.50
λ_5	159.9709	148.5359	148.5191	-	-	105.0615	99.3709	99.3207	-	-
λ_6	221.2256	205.1447	205.0865	-	-	153.1226	148.4395	149.2887	-	-

analysis herein and those of Leung and Zhou (2001).

To demonstrate the efficiency of the present algorithm for beams with non-classical boundary conditions, especially those with heavy masses and those with rotary inertias, various examples are

Table 6 First four dimensionless frequencies of cantilever tapered beam restrainted and carrying a tip mass at the free end : Case 1 $\upsilon = 0.3$, $\kappa = 5/6$, $\alpha = -0.2$, $\eta = 0.01$, $\xi = 1/3.12\eta$

	$K_{TR} = 0$, $\mu_R = 0$	$K_{TR} = 0$, $\mu_R = 1$	$K_{TR} = 1$, $\mu_R = 1$	$K_{TR} = 1$, $\mu_R = 10$	$K_{TR} = 1$, $\mu_R = 100$	$K_{TR} = 1$, $\mu_R = \infty$	$K_{TR} = 0$, $\mu_R = \infty$
	(C-F)						(C-S)
λ_1	3.3307	1.3946	1.6659	0.5720	0.1825	0.0002	0.0000
λ_2	14.2892	11.1193	11.1224	10.7349	10.6917	10.6869	10.6869
λ_3	30.7108	26.5009	26.5014	26.1493	26.1114	26.1072	26.1072
λ_4	47.7502	43.8924	43.8926	43.6226	43.5939	43.5907	43.5907
λ_5	64.9970	61.9689	61.9689	61.6893	61.6593	61.6560	61.6560
λ_6	70.5880	68.5382	68.5382	68.4328	68.4220	68.4207	68.4207

Table 7 First four dimensionless frequencies of cantilever tapered beam restrainted and carrying a tip mass at the free end : Case 1 $\upsilon = 0.3$, $\kappa = 5/6$, $\alpha = -0.2$, $\eta = 0.01$, $\xi = 1/3.12\eta$

	$K_{TR} = 0$, $\mu_R = 0$	$K_{TR} = 0$, $\mu_R = 0$	$K_{TR} = 1$, $\mu_R = 1$	$K_{TR} = 10$, $\mu_R = 1$	$K_{TR} = 100$, $\mu_R = 1$	$K_{TR} = \infty$, $\mu_R = 1$	$K_{TR} = \infty$, $\mu_R = 0$
	(C-F)						(C-S)
λ_1	3.3307	3.9355	1.6659	3.1934	8.6266	10.6869	10.6869
λ_2	14.2892	14.4087	11.1224	11.1526	11.8588	26.1072	26.1072
λ_3	30.7108	30.7561	26.5014	26.5061	26.5604	43.5907	43.5907
λ_4	47.7502	47.7700	43.8926	43.8939	43.9079	61.6560	61.6560
λ_5	64.9970	65.0023	61.9689	61.9696	61.9766	68.4207	68.4207
λ_6	70.5880	70.5934	68.5382	68.5384	68.5407	79.6605	79.6605

considered here. In most cases, the results are presented at different collocation points to reveal the high rate of convergence of the method. Tables 4 and 5 list the first six dimensionless natural frequencies of cantilever non-uniform beams with an attached mass at their right ends in cases 1 and 2, respectively. The rapid convergence of the results is obvious. The results of the present method are compared with the results of the dynamic stiffness method of Leung and Zhou (2001) and other convergent solutions provided by Lee and Lin (1992), which are believed to be very near to the exact solutions. Comparing the results indicates an excellent rate of convergence and high accuracy.

In other applications, one non-uniform cantilever beam carries a heavy tip mass and is restrained at the free end with elastic restraints of various strengths. Tables 6 and 7 present dimensionless natural frequencies with different tip masses and elastic restraints. From Table 6, when the dimensionless elastic coefficients of support K_{TR} are held constant, the natural frequency decreases as the dimensionless concentrated attached mass increases. In Table 7, the dimensionless concentrated attached mass is constant and the natural frequency increases with the dimensionless elastic coefficient of the support increases. Tables 6 and 7 also reveal that the simple-support boundary condition applies when either the elastic coefficient K_{TR} or the concentrated attached mass is sufficiently large.

6. Conclusions

This study presented a differentiation matrix to determine the modal parameters of Timoshenko beams using the Chebyshev spectral approach. A simple and efficient matrix-based method is used to integrate complicated boundary conditions into a condensation matrix. This method allows complete natural frequencies and mode shapes to be calculated simultaneously using the standard matrix eigenvalue algorithm in Matlab software. Numerical results are compared with those obtained using other methods. Nevertheless, the rapid convergence and high accuracy of the proposed method were demonstrated. The method is a straightforward and efficient approach for computing eigensolutions of linear differential operators in engineering problems.

References

Cleghorn, W.L. and Tabarrok, B. (1992), "Finite element formulation of tapered Timoshenko beam for free lateral vibration analysis", *J. Sound Vib.,* **152**, 461-470.

Costa, B. and Don, W.S. (2000), "On the computation of high order pseudospectral derivatives", *Appl. Numer. Math*., **33**, 151-159.

Don, W.S. and Solomonoff, A. (1995), "Accuracy and speed in computing the Chebyshev collocation derivative", *SIAM J. Sci. Comput*., **16**, 1253-1268.

Ferreira, A.J.M. and Fasshauer, G.E. (2006), "Computation of natural frequencies of shear deformable beams and plates by an RBF-pseudospectral method", *Comput. Meth. Appl. Mech. Eng*., **196**, 124-146.

Gutierrez, R.H., Laura, P.A.A. and Rossi, R.E. (1991), "Fundamental frequency of vibration of a Timoshenko beam of non-uniform thickness", *J. Sound Vib.,* **145**, 241-245.

Ho, S.H. and Chen, C.K. (1998), "Analysis of general elastically restrained non-uniform beams using differential transform", *Appl. Math. Mode*., **22**, 219-234.

Hsu, J.C., Lai, H.Y. and Chen, C.K. (2009), "An innovative eigenvalue problem solver for free vibration of uniform Timoshenko beams by using the Adomian modified decomposition method", *J. Sound Vib.,* **325**, 451-470.

Irie, T., Yamada, G. and Takahashi, I. (1980), "Vibration and stability of a non-uniform Timoshenko beam subjected to a flower force", *J. Sound Vib*., **70**, 503-512.

Karami, G., Malekzadeh, P. and Shahpari, S.A. (2003), "A DQEM for vibration of shear deformable non-uniform beams with general boundary conditions", *Eng. Struct*., **25**, 1169-1178.

Lee, J. and Schultz, W.W. (2004), "Eigenvalue analysis of Timoshenko beams and axisymmetric Mindlin plates by pseudospectral method", *J. Sound Vib*., **269**, 609-621.

Lee, S.Y. and Lin, S.M. (1992), "Exact vibration solutions for non-uniform Timoshenko beams with attachments", *AIAA J*., **30**, 2930-2934.

Lee, S.Y. and Lin, S.M. (1995), "Vibration of elastically restrained non-uniform Timoshenko beams", *J. Sound Vib*., **181**, 403-415.

Leung, A.Y.T. and Zhou, W.E. (1995), "Dynamic stiffness analysis of non-uniform Timoshenko beams", *J. Sound Vib*., **181**, 447-456.

Leung, A.Y.T., Zhou, W.E., Lim, C.W., Yuen, R.K.K. and Lee, U. (2001) , "Dynamic stiffness for piecewise non-uniform Timoshenko column by power series-part I: conservative axial force", *Inter. J. Numer. Meth. Eng*., **51**, 505-529.

Posiadala, B. (1997), "Free vibration of uniform Timoshenko beams with attachments", *J. Sound Vib*., **204**(2), 359-369.

Rossi, R.E., Laura, P.A.A. and Gutierrez, R.H. (1990), "A note on transverse vibrations of Timoshenko beam of non-uniform thickness clamped at one end and carrying a concentrated mass at the other", *J. Sound Vib*., **143**, 491-502.

Rossi, R.E. and Laura, P.A.A. (1993), "Numerical experiments on vibrating linearly tapered Timoshenko beam",

J. Sound Vib., **168**, 179-183.
Ruta, P. (2006), "The application of a Chebyshev polynomials to the solution of the non-prismatic Timoshenko beam vibration problem", *J. Sound Vib*., **296**, 243-263.
Salarieh, H. and Ghorashi, M. (2006), "Free vibration of Timoshenko beam with finite mass rigid tip load and flexural-torsional coupling", *Inter. J. Mech. Sci*., **48**, 763-779.
Trefethen, L.N. (2000), *Spectral Methods in Matlab*, SIAM, Phila.