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Curved laminate analysis

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Abstract. This paper is devoted to the development of the equations which describe the elastic response of a curved laminate subjected to in-plane loads and bending moments. Similar to the classic 6×6 ABD matrix constitutive relation of a flat laminate, a new 6×6 matrix constitutive relation between force resultants, moment resultants, mid-plane strains and deformed curvatures for a curved laminate is formulated. This curved lamination theory will provide the fundamental basis for the analyses of curved laminated structures. The stress predictions by the present curved lamination theory are compared to those by the curved laminate analysis that neglected the nonlinear terms in the derivation of the constitutive relation. The results show that the curved laminate analysis that neglected the nonlinear terms cannot reflect the effect of curvature and can no longer predict the stresses accurately as the curvature becomes noticeable. In this paper, a curved lamination theory that retains the nonlinear terms and, therefore, accounts for the effect of the non-flat geometry of the structure will be developed.

Keywords: curved laminate; constitutive relation; nonlinear

1. Introduction

The classic lamination theory originally presented by Pister and Dong (1959), Reissner and Stavsky (1961) and Dong *et al.* (1962) has been widely applied to analyze the laminated composite structures. Thereafter, many books (Vinson and Chou 1975, Christensen 1979, Vinson and Sierakowski 1987, Herakovich 1998) were devoted to present the theory more completely and applied the theory to perform structure analyses of laminated beams, plates, columns and rods etc. with straight or flat geometries. However, it is frequent that the laminated composite structures are curved rather than straight or flat for many applications (e.g., airplane frame and skin). Curved laminated beams and plates are instances where stresses and displacements must be determined on the basis of a lamination theory that accounts for the non-flat geometry of the structure.

The fundamental work on the thin elastic shell by Koiter (1959) has shown that Love's first approximation is indeed a consistent first approximation in the general theory of thin elastic shells. In Koiter's work and the other thin shell theories (Ambartsumyan 1964, Kraus 1967, Chiang 2006) including thin laminated shell theories (Whitney 1987, Reddy 2004, Bozhevolnaya and Frostig 1997, Nemeth and Smeltzer 2000, Volovoi and Hodges 2002) (i.e., $h \ll R$, as shown in Fig. 1), the assumption of $z/R \ll 1$ (where $-h/2 \leqq z \leqq h/2$) is made to simplify the nonlinear terms into the linear terms. As a result, the derived constitutive relations were of the same form as that of the classic

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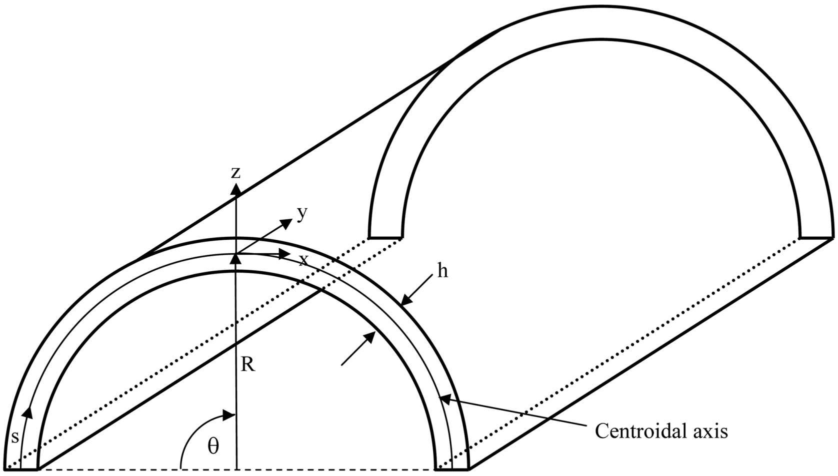
lamination theory. However, from the analyses of curved structures (e.g., curved beams by Bickford (1998), curved laminated beams by Lin and Hsieh (2007), curved plate by Ventsel *et al*. (2001) and curved lamination by Ren *et al*. (2003), Altınok *et al*. (2008)), it is indicated the effect of the non-flat geometry of the structure is reflected by the nonlinear terms which are critical in the analysis of the curved structure. Therefore, for a rigorous curved lamination theory the effect of curvature should be taken into account and the nonlinear teams should not be neglected. In the present analysis, a new curved lamination theory will be developed by retaining the nonlinear terms and, therefore, the effect of the non-flat geometry can be taken into account in the analysis. Actually, the difficulty and challenge on the analyses of curved structures are ascribed to the existence of the nonlinear terms in the analysis.

In the present paper, a curved lamination theory that describes the elastic response of a curved laminate subjected to stretching and bending will be developed. Individual layers in the curved laminate are assumed to be homogeneous, orthotropic and in a state of plane stress. Assumed that the curved laminate deforms according to the Kirchhoff-Love hypothesis, the strains can be expressed in terms of the mid-plane strains and the deformed curvatures in the nonlinear forms. Subsequently, the stresses in the individual layer can be evaluated from the plane stress constitutive relations. By integrating the stresses through the laminate thickness, the force resultants and the moment resultants are obtained in terms of the mid-plane strains and the deformed curvatures in a 6×9 matrix form which is not applicable for mathematical operation. Then, the next effort is devoted to transform the 6×9 matrix form into the 6×6 matrix constitutive formula through finding one extra relation and rearranging the matrix. Similar to the classic ABD matrix constitutive formulation of a flat laminate, this new curved laminate constitutive relation will provide the fundamental basis to the analyses of curved laminated structures. The influence of the laminate lay-up sequences on the computation of the 6×6 matrix will be discussed. The thermal behavior of a curved laminate will also be investigated in the paper.

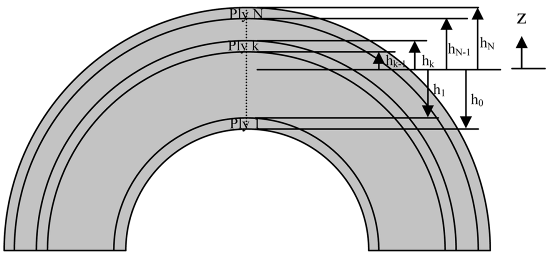
The application of the curved lamination theory is demonstrated by the stress calculations of the curved laminate. The stress predictions by the present curved lamination theory are compared to those by the curved laminate analysis that neglected the nonlinear terms in the derivation of the constitutive relation. In situations where the thickness-radius ratio (h/R) ratio is small, the curved laminate analysis that neglected the nonlinear terms continues to give acceptable accuracy. However, as the h/R ratio is getting larger, the analysis can no longer predict the stresses accurately. The present curved lamination theory that retains the nonlinear terms and, therefore, accounts for the effect of the non-flat geometry of the structure is essentially required for proper structure analyses.

2. Curved lamination theory

Consider a curved laminate of thickness h as depicted in Fig. 1(a). Here, the x-axis is passing everywhere through the centroid of the section and tangent to a circular arc of radius R, that is, $ds = Rd\theta$, where θ is the angular variable associated with a change in location along the curved section. The z-axis lies along the local direction of the radius R with the y-axis such that a right-handed rectangular coordinate system is formed. As depicted in Fig. 1(b), the curved laminate has N layers numbered from bottom lamina to top lamina. Coordinates h_k are the vertical distances from the mid-plane to the interfaces and they have the sign conventions of the z coordinates.



(a)



(b)

Fig. 1 (a) Geometry for a curved laminate, (b) notation for location of ply interface

2.1 Strain-displacement relationships

The curved laminate consists of perfectly bonded layers and its individual layer is assumed to be homogeneous, orthotropic and in a state of plane stress. Furthermore, the curved laminate deforms according to the Kirchhoff-Love hypothesis for stretching and bending of plates:

(1) A lineal element of the curved laminate extending through the laminate thickness is normal to the mid-plane (instantaneous *xy* plane). Upon application of load, the lineal element remains straight and normal to the deformed mid-plane.

(2) The lineal element does not change length.

Based upon the foregoing assumptions, the most general form for the displacements in the *x* and *y* directions is

$$u(x, y, z) = u_0(x, y) + z\alpha(x, y) \tag{1}$$

$$v(x, y, z) = v_0(x, y) + z\beta(x, y) \tag{2}$$

where u_0 and v_0 denote the mid-plane displacements in the x and y directions and α and β are notations which will be defined later. From the assumption (2), requires that $\varepsilon_z = 0$ and in turn means that the displacement in the z direction can be expressed as

$$w(x, y, z) = w_0(x, y) = w \tag{3}$$

where w_0 denotes the mid-plane displacement in the z direction.

By specializing the cylindrical coordinate strain-displacement relations to the present situation, the strain-displacement relations become

$$\varepsilon_x = \frac{1}{1+\kappa z}\left(\frac{\partial u}{\partial s} + \kappa w\right) = \frac{1}{1+\kappa z}\left(\frac{\partial u_0}{\partial s} + z\frac{\partial \alpha}{\partial s} + \kappa w\right) \tag{4}$$

$$\varepsilon_y = \frac{\partial v_0}{\partial y} + z\frac{\partial \beta}{\partial y} \tag{5}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0 \tag{6}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{1}{1+\kappa z}\frac{\partial v}{\partial s} = \frac{\partial u_0}{\partial y} + z\frac{\partial \alpha}{\partial y} + \frac{1}{1+\kappa z}\left(\frac{\partial v_0}{\partial s} + z\frac{\partial \beta}{\partial s}\right) \tag{7}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} - \frac{\kappa}{1+\kappa z}u + \frac{1}{1+\kappa z}\frac{\partial w}{\partial s} = \frac{\partial u_0}{\partial z} + \alpha - \frac{\kappa}{1+\kappa z}(u_0 + z\alpha) + \frac{\kappa}{1+\kappa z}\frac{\partial w}{\partial s} \tag{8}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{\partial v_0}{\partial z} + \beta + \frac{\partial w}{\partial y} \tag{9}$$

where $\kappa = 1/R$ is the geometrical curvature of the curved laminate. The assumption (1) requires that the shear strains of γ_{xz} and γ_{yz} are zero with $du_0/dz = 0$ and $dv_0/dz = 0$ under the expression of the displacements given by Eqs. (1), (2) leads to

$$\alpha = \kappa u_0 - \frac{\partial w}{\partial s} \tag{10}$$

$$\beta = -\frac{\partial w}{\partial y} \tag{11}$$

Substituting Eqs. (10), (11) into Eqs. (4), (5) and (7), the strain-displacement relations can be expressed in matrix form as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z\begin{Bmatrix} 0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{Bmatrix} + \frac{z}{1+\kappa z}\begin{Bmatrix} \kappa_x^1 \\ 0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{12}$$

where the mid-plane strains $\{\varepsilon^0\}$ and deformed curvatures $\{\kappa^0\}$ and $\{\kappa^1\}$ are defined as

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial s} + \kappa w \tag{13a}$$

$$\kappa_x^1 = -\left(\frac{\partial^2 w}{\partial s^2} + \kappa^2 w\right) \tag{13b}$$

$$\varepsilon_y^0 = \frac{\partial v_0}{\partial y} \tag{13c}$$

$$\kappa_y^0 = -\frac{\partial^2 w}{\partial y^2} \tag{13d}$$

$$\gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial s} \tag{13e}$$

$$\kappa_{xy}^0 = -\left(\frac{\partial^2 w}{\partial s \partial y} - \kappa \frac{\partial u_0}{\partial y}\right) \tag{13f}$$

$$\kappa_{xy}^1 = -\left(\frac{\partial^2 w}{\partial s \partial y} + \kappa \frac{\partial v_0}{\partial s}\right) \tag{13g}$$

or mosre simply

$$\{\varepsilon\} = \{\varepsilon^0\} + z\{\kappa^0\} + \frac{z}{1+\kappa z}\{\kappa^1\} \tag{14}$$

Eq. (14) indicates that the strains $\{\varepsilon\}$ at any z-location in the curved laminate are the sum of the mid-plane strains $\{\varepsilon^0\}$ and the strains associated with deformed curvatures $\{\kappa^0\}$ and $\{\kappa^1\}$. It is noted that Eq. (14) is the result derived from Kirchhoff-Love assumptions on deformation and it is independent of material considerations. It means that the result of Eq. (14) is applicable to either isotropic materials or anisotropic materials.

In the analysis of Whitney (1987), the nonlinear term of $z/(1 + \kappa z)$ in the strain-displacement relations of Eq. (12) is simplified to z by the assumption of shallow curvature (i.e., $h<<R$, as shown in Fig. 1(a) of the paper). In addition, the terms of $\kappa^2 w$, $\kappa(\partial u_0/\partial y)$ and $\kappa(\partial v_0/\partial s)$, respectively, in κ_x^1, κ_{xy}^0 and κ_{xy}^1 in Eq. (13) are neglected in the strain-displacement relations. After the simplification of the nonlinear terms to linear terms and the negligence of those terms, the strain-displacement relations given by Whitney are of the same form as those of a flat laminate with the exception of the κw term in the mid-plane tangential strain ε_x^0. This manipulation seems to be illogical. If κz (i.e., z/R where $-h/2 \leqq z \leqq h/2$) in the nonlinear term of $z/(1+\kappa z)$ can be neglected due to the shallow curvature, the term of κw (w/R) should also be neglected because the displacement w should be small compared to the plate thickness h under the Kirchoff-Love deformation assumption. After further negligence of the κw term in the mid-plane tangential strain ε_x^0, the strain-displacement relations of the curved laminate of shallow curvature will be the same as those of the flat laminate and the curved laminate analysis is reduced to that of a flat laminate. If the effect of curvature becomes noticeable, the nonlinear terms and whole terms in Eq. (12) should be retained in the analysis of the curved laminate.

2.2 Stresses

The stresses at any z-location can be determined by substituting the strain equation of (12) into the plane stress constitutive equation, we have

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\bar{Q}]_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{15}$$

where $[\bar{Q}]_k$ is the transformed reduced stiffness of the k^{th} lamina in the x, y coordinate system; and it can be related to the determined stiffness matrix $[Q]_k$ in the fiber direction coordinate system from standard coordinate transformation. Combining Eqs. (12) and (15) gives a general expression of stresses in the k^{th} lamina in terms of position z

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\bar{Q}]_k \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + [\bar{Q}]_k z \begin{Bmatrix} 0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{Bmatrix} + [\bar{Q}]_k \frac{z}{1+\kappa z} \begin{Bmatrix} \kappa_x^1 \\ 0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{16}$$

The first term in Eq. (16) corresponds to the stresses associated with the mid-plane strains, and the second and third terms correspond to the stresses associated with deformed curvatures. It is noted that $\{\varepsilon^0\}$, $\{\kappa^0\}$ and $\{\kappa^1\}$, which are associated with the mid-plane displacements and geometrical curvature κ, are independent of z location.

2.3 Force resultants and moment resultants

The force resultants $\{N\}$ are defined as the through-thickness integrals of the stresses in the curved laminate. A similar interpretation can be given to the moment resultants $\{M\}$. Thus, $\{N\}$ and $\{M\}$ in compact forms are, respectively, expressed as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz \tag{17}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} z\,dz \tag{18}$$

Substituting Eq. (16) into Eqs. (17), (18) and the intervals are taken over the total laminate thickness by summing the intervals over each layer. Then, $\{N\}$ and $\{M\}$ become

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} [\bar{Q}]_k \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} dz + \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} [\bar{Q}]_k \begin{Bmatrix} 0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{Bmatrix} z\,dz + \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} [\bar{Q}]_k \begin{Bmatrix} \kappa_x^1 \\ 0 \\ \kappa_{xy}^1 \end{Bmatrix} \frac{z}{1+\kappa z} dz \tag{19}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} [\bar{Q}]_k \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} z dz + \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} [\bar{Q}]_k \begin{Bmatrix} 0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{Bmatrix} z^2 dz + \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} [\bar{Q}]_k \begin{Bmatrix} \kappa_x^1 \\ 0 \\ \kappa_{xy}^1 \end{Bmatrix} \frac{z^2}{1+\kappa z} dz \tag{20}$$

Remembering $\{\varepsilon^0\}$, $\{\kappa^0\}$ and $\{\kappa^1\}$ are independent of z coordinate and the material properties are constant over each individual layer. Thus, the only variable inside the integrals is z and the integrals are easy to carry out. For example

$$\int_{h_{k-1}}^{h_k} \frac{z}{1+\kappa z} dz = \frac{1}{\kappa}\left(h_k - h_{k-1} - \frac{1}{\kappa}\ln\frac{1+\kappa h_k}{1+\kappa h_{k-1}}\right) \tag{21a}$$

$$\int_{h_{k-1}}^{h_k} \frac{z^2}{1+\kappa z} dz = -\frac{1}{\kappa^2}\left(h_k - h_{k-1} - \frac{1}{\kappa}\ln\frac{1+\kappa h_k}{1+\kappa h_{k-1}}\right) \tag{21b}$$

where ln represents natural logarithms.

2.4 Curved laminate constitutive relations

By carrying out the integrals in Eqs. (19) and (20), the fundamental equation of curved lamination theory can be written in the following form

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B & C \\ B & D & -C/\kappa \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa^0 \\ \kappa^1 \end{Bmatrix} \tag{22}$$

where A, B and D matrices are defined in the classic lamination theory, and the C matrix is defined as

$$C_{ij} = \frac{1}{\kappa}\left(A_{ij} - \frac{1}{\kappa}\sum_{k=1}^{N} (\bar{Q}_{ij})_k \ln\frac{1+\kappa h_k}{1+\kappa h_{k-1}}\right) \tag{23}$$

Eq. (22) can be written in expanded form as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & C_{11} & C_{12} & C_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & C_{12} & C_{22} & C_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & C_{16} & C_{26} & C_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & -C_{11}/\kappa & -C_{12}/\kappa & -C_{16}/\kappa \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & -C_{12}/\kappa & -C_{22}/\kappa & -C_{26}/\kappa \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & -C_{16}/\kappa & -C_{26}/\kappa & -C_{66}/\kappa \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ 0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \\ \kappa_x^1 \\ 0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{24}$$

Eq. (24) contains the seven unknowns of $\varepsilon_x^0, \varepsilon_x^0, \gamma_{xy}^0, \kappa_y^0, \kappa_{xy}^0, \kappa_x^1$ and κ_{xy}^1 but it only contains six relations. One additional relation is needed to bridge the gap. In addition, the 6 × 9 matrix form of Eq. (24) is not suitable for the mathematical matrix operations. Recalling Eq. (13), we can relate the mid-plane shear strain of γ_{xy}^0 to the deformed shear curvatures of κ_{xy}^0 and κ_{xy}^1 as

$$\kappa_{xy}^0 = \kappa\gamma_{xy}^0 + \kappa_{xy}^1 \tag{25}$$

Substituting Eq. (25) into Eq. (24), the fundamental equation of curved lamination theory can be rearranged in the following 6 × 6 matrix form

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & (A_{16}+\kappa B_{16}) & C_{11} & B_{12} & (C_{16}+B_{16}) \\ A_{12} & A_{22} & (A_{26}+\kappa B_{26}) & C_{12} & B_{22} & (C_{26}+B_{26}) \\ A_{16} & A_{26} & (A_{66}+\kappa B_{66}) & C_{16} & B_{26} & (C_{66}+B_{66}) \\ B_{11} & B_{12} & (B_{16}+\kappa D_{16}) & -C_{11}/\kappa & D_{12} & (-C_{16}/\kappa+D_{16}) \\ B_{12} & B_{22} & (B_{26}+\kappa D_{26}) & -C_{12}/\kappa & D_{22} & (-C_{26}/\kappa+D_{26}) \\ B_{16} & B_{26} & (B_{66}+\kappa D_{66}) & -C_{16}/\kappa & D_{26} & (-C_{66}/\kappa+D_{66}) \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x^1 \\ \kappa_y^0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{26}$$

As the radius of the curved laminate R approaches infinite (i.e., $\kappa \to 0$), C and $-C/\kappa$ matrices become B and D matrices, respectively. In addition, the deformed curvatures of κ_x^1, κ_y^0 and κ_{xy}^1 become κ_x, κ_y and $\kappa_{xy}/2$ defined in the classic lamination theory. As a result, Eq. (26) will reduce to the ABD constitutive relation of the classic lamination theory. Similar to the classic 6 × 6 ABD matrix constitutive relation of a flat laminate, this new 6 × 6 ABCD curved lamination theory will provide the fundamental basis to the analyses of curved laminate structures (e.g., curved laminated beams and plates etc.). It is noted that the 6 × 6 ABCD matrix of Eq. (26) is an unsymmetric matrix, which is dissimilar to the 6 × 6 ABD matrix of the classic lamination theory.

3. Thermo-elastic behavior

The development of curved lamination theory including the thermal effects follows essentially the same procedures as used for pure mechanical loading, except that the stress-strain relations of individual layers must now include the thermal strains. With the assumption of the total strains in the curved laminate following the Kirchhoff-love assumptions on displacements, the total strains are the superposition of the mechanical strains and the thermal strains and given as

$$\{\varepsilon\} = \{\varepsilon^{\sigma}\} + \{\varepsilon^{T}\} = \{\varepsilon^0\} + z\{\kappa^0\} + \frac{z}{1+\kappa z}\{\kappa^1\} + \{\varepsilon^{T}\} \tag{27}$$

The thermo-elastic constitutive equation is then given by

$$\{\sigma\} = [\overline{Q}]\left(\{\varepsilon^0\} + z\{\kappa^0\} + \frac{z}{1+\kappa z}\{\kappa^1\} + \{\varepsilon^{T}\}\right) \tag{28}$$

3.1 Thermal forces and moments

From the definitions for $\{N\}$ by Eq. (17) and $\{M\}$ by Eq. (18) and the constitutive equation of

Eq. (26), the thermo-elastic constitutive equation of the curved laminate theory is given as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & (A_{16}+\kappa B_{16}) & C_{11} & B_{12} & (C_{16}+B_{16}) \\ A_{12} & A_{22} & (A_{26}+\kappa B_{26}) & C_{12} & B_{22} & (C_{26}+B_{26}) \\ A_{16} & A_{26} & (A_{66}+\kappa B_{66}) & C_{16} & B_{26} & (C_{66}+B_{66}) \\ B_{11} & B_{12} & (B_{16}+\kappa D_{16}) & -C_{11}/\kappa & D_{12} & (-C_{16}/\kappa+D_{16}) \\ B_{12} & B_{22} & (B_{26}+\kappa D_{26}) & -C_{12}/\kappa & D_{22} & (-C_{26}/\kappa+D_{26}) \\ B_{16} & B_{26} & (B_{66}+\kappa D_{66}) & -C_{16}/\kappa & D_{26} & (-C_{66}/\kappa+D_{66}) \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x^1 \\ \kappa_y^0 \\ \kappa_{xy}^1 \end{Bmatrix} - \begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \\ M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix} \tag{29}$$

where $\{N^T\}$ and $\{M^T\}$ are, respectively, the equivalent thermal forces and moments per unit length. For the case of constant temperature through the laminate thickness, the $\{N^T\}$ and $\{M^T\}$ are given by

$$\{N^T\} = \int_{-h/2}^{h/2} [\overline{Q}]_k\{\varepsilon^T\}dz = \Delta T\sum_{k=1}^{N}\int_{h_{k-1}}^{h_k} [\overline{Q}]_k\{\overline{\alpha}\}_k dz = \Delta T\sum_{k=1}^{N}[\overline{Q}]_k\{\overline{\alpha}\}_k(h_k-h_{k-1}) \tag{30}$$

$$\{M^T\} = \int_{-h/2}^{h/2} [\overline{Q}]_k\{\varepsilon^T\}z dz = \Delta T\sum_{k=1}^{N}\int_{h_{k-1}}^{h_k} [\overline{Q}]_k\{\overline{\alpha}\}_k z dz = \Delta T\sum_{k=1}^{N}[\overline{Q}]_k\{\overline{\alpha}\}_k\left(\frac{h_k^2-h_{k-1}^2}{2}\right) \tag{31}$$

where $\overline{\alpha}$ is the coefficients of thermal expansion in the x, y coordinate, which is assumed to be constant in each layer.

4. Stresses within the layers

Due to different ply orientation and the presence of geometrical curvature, the stresses within the individual layers can be highly non-uniform and nonlinear, even for very simple loadings. However, these stresses can be determined from the equations given in the above sections. The basic scheme is that the strains in the curved laminate can be determined as part of solution process for the particular problem (e.g., 1D beam problem or 2D plate problem). For example, consider a simple statically determined problem in which the force resultants and moment resultants are known, the mid-plane strains and deformed curvatures can be obtained by inverting Eq. (26)

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x^1 \\ \kappa_y^0 \\ \kappa_{xy}^1 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & (A_{16}+\kappa B_{16}) & C_{11} & B_{12} & (C_{16}+B_{16}) \\ A_{21} & A_{22} & (A_{26}+\kappa B_{26}) & C_{21} & B_{22} & (C_{26}+B_{26}) \\ A_{61} & A_{62} & (A_{66}+\kappa B_{66}) & C_{61} & B_{62} & (C_{66}+B_{66}) \\ B_{11} & B_{12} & (B_{16}+\kappa D_{16}) & -C_{11}/\kappa & D_{12} & (-C_{16}/\kappa+D_{16}) \\ B_{21} & B_{22} & (B_{26}+\kappa D_{26}) & -C_{21}/\kappa & D_{22} & (-C_{26}/\kappa+D_{26}) \\ B_{61} & B_{62} & (B_{66}+\kappa D_{66}) & -C_{61}/\kappa & D_{62} & (-C_{66}/\kappa+D_{66}) \end{bmatrix}^{-1} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} \tag{32}$$

The deformed shear curvature of κ_{xy}^0 can then be obtained by Eq. (25). Subsequently, the strain distributions throughout the thickness of the curved laminate can be computed from Eq. (12). Furthermore, the through-thickness stresses in the x, y coordinates can be calculated by Eq. (16).

Finally, the stresses in the fiber directions are easily obtained from a simple coordinate transformation procedure. Generally, the determination of individual layer stresses is the basis for strength design in laminated composites.

5. Discussions and example

5.1 Symmetric laminates

Symmetric laminates, each layer above the mid-plane will be paired with a same lay-up orientation below the mid-plane, are of significant interest. As we well know that the B matrix will vanish as the laminate lay-up is symmetric. The vanishing of B matrix will give an uncoupling of the stretching and bending responses for a flat laminate such that it will much simplify the analysis. However, for a curved symmetric laminate, the vanishing of B matrix results the following 6 × 6 ABCD matrix constitutive relation

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & C_{11} & 0 & C_{16} \\ A_{12} & A_{22} & A_{26} & C_{12} & 0 & C_{26} \\ A_{16} & A_{26} & A_{66} & C_{16} & 0 & C_{66} \\ 0 & 0 & \kappa D_{16} & -C_{11}/\kappa & D_{12} & (-C_{16}/\kappa + D_{16}) \\ 0 & 0 & \kappa D_{26} & -C_{12}/\kappa & D_{22} & (-C_{26}/\kappa + D_{26}) \\ 0 & 0 & \kappa D_{66} & -C_{16}/\kappa & D_{26} & (-C_{66}/\kappa + D_{66}) \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x^1 \\ \kappa_y^0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{33}$$

The stretching and bending responses are still coupled through the parts of [C] and κ[D] matrices for a curved symmetric laminate, as can be seen in Eq. (33). This coupling effect is caused by the non-flat geometry of the structure and it will significantly complicate the analyses for curved symmetric laminates. This coupling effect for a symmetric curved laminate can not be shown by the analysis that neglected the nonlinear terms (Whitney 1987, Reddy 2004).

5.2 Symmetric cross-ply laminates

If the symmetric laminate contains only 0 and 90 plies, it is referred to the "cross-ply laminate" and the constitutive relation further reduces to

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & C_{11} & 0 & 0 \\ A_{12} & A_{22} & 0 & C_{12} & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & C_{66} \\ 0 & 0 & 0 & -C_{11}/\kappa & D_{12} & 0 \\ 0 & 0 & 0 & -C_{12}/\kappa & D_{22} & 0 \\ 0 & 0 & \kappa D_{66} & 0 & 0 & (-C_{66}/\kappa + D_{66}) \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x^1 \\ \kappa_y^0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{34}$$

It is found in Eq. (34) that the normal responses are unrelated to the shear responses. Thus, Eq. (34) can be separated into two equations of

$$\begin{Bmatrix} N_x \\ N_y \\ M_x \\ M_y \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & C_{11} & 0 \\ A_{12} & A_{22} & C_{12} & 0 \\ 0 & 0 & -C_{11}/\kappa & D_{12} \\ 0 & 0 & -C_{12}/\kappa & D_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \kappa_x^1 \\ \kappa_y^0 \end{bmatrix} \tag{35}$$

and

$$\begin{Bmatrix} N_{xy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{66} & C_{66} \\ \kappa D_{66} & (-C_{66}/\kappa + D_{66}) \end{bmatrix} \begin{Bmatrix} \gamma_{xy}^0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{36}$$

The uncoupling between the normal and shear responses will simplify the analysis for the curved symmetric cross-ply laminate.

5.3 Example

The AS4/3501-6 carbon/epoxy laminated composite, of which lamina properties are given in Table 1, is used for case study.

Now consider a curved symmetric cross-ply laminate of $[0_{10}/90_{10}]_s$ with the radius R under the loading of q = 10 kN/m, as shown in Fig. 2. For this simple loading, only the force resultants and the moment resultants in the x direction (i.e., N_x and M_x) exist. Due to the symmetric cross-ply stacking, the normal responses uncouple to the shear responses. Thus, the mid-plane normal strains

Table 1 Lamina properties of AS4/3501-6 carbon/epoxy composite

E_{11} (GPa)	E_{22} (GPa)	ν_{12}	G_{12} (GPa)	t (thickness of lamina, mm)
128	11.1	0.28	6.55	0.132

Data from Swanson (1997)

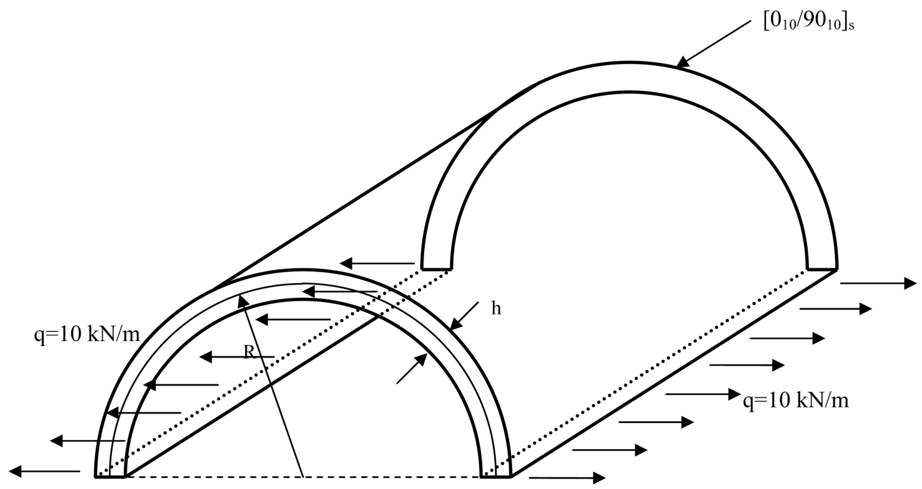


Fig. 2 A curved symmetric cross-ply laminate under q = 10 kN/m at both ends

of ε_x^0 and ε_y^0 and deformed curvatures of κ_x^1 and κ_y^0 can be obtained by inverting Eq. (35)

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \kappa_x^1 \\ \kappa_y^0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & C_{11} & 0 \\ A_{12} & A_{22} & C_{12} & 0 \\ 0 & 0 & -C_{11}/\kappa & D_{12} \\ 0 & 0 & -C_{12}/\kappa & D_{22} \end{bmatrix}^{-1} \begin{Bmatrix} N_x \\ 0 \\ M_x \\ 0 \end{Bmatrix} \tag{37}$$

Subsequently, the normal strains of ε_x and ε_y are given by substituting ε_x^0, ε_y^0, κ_x^1 and κ_y^0 obtained by Eq. (37) into Eq. (12) as

$$\begin{aligned} \varepsilon_x &= \varepsilon_x^0 + \frac{z}{1+\kappa z}\kappa_x^1 \\ &= \frac{1}{A_{11}A_{12}-A_{12}^2}\left[A_{22}N_x + \frac{\kappa D_{22}(A_{22}C_{11}-A_{12}C_{12})}{C_{11}D_{22}-C_{12}D_{12}}M_x\right] - \frac{z}{1+\kappa z}\frac{\kappa D_{22}}{C_{11}D_{22}-C_{12}D_{12}}M_x \end{aligned} \tag{38}$$

$$\begin{aligned} \varepsilon_y &= \varepsilon_y^0 + z\kappa_y^0 \\ &= -\frac{1}{A_{11}A_{22}-A_{12}^2}\left[A_{12}N_x + \frac{\kappa A_{12}D_{22}(A_{22}C_{11}-A_{12}C_{12})}{A_{22}(C_{11}D_{22}-C_{12}D_{12})}M_x\right] + \frac{\kappa C_{12}D_{22}}{A_{22}(C_{11}D_{22}-C_{12}D_{12})}M_x \\ &\quad - z\frac{C_{12}}{C_{11}D_{22}-C_{12}D_{12}}M_x \end{aligned} \tag{39}$$

The through-thickness stresses in the x, y coordinates can be calculated by Eq. (16). Accordingly, the stresses in the fiber directions are easily obtained from a simple coordinate transformation procedure. For example, the stresses in the 0 ply are given by

$$\begin{aligned} \sigma_1 &= Q_{11}\varepsilon_x + Q_{12}\varepsilon_y \\ &= \frac{A_{22}Q_{11}-A_{12}Q_{12}}{A_{11}A_{22}-A_{12}^2}\left[N_x + \frac{\kappa D_{22}(A_{22}C_{11}-A_{12}C_{12})}{A_{22}(C_{11}D_{22}-C_{12}D_{12})}M_x\right] + \frac{\kappa C_{12}D_{22}Q_{12}}{A_{22}(C_{11}D_{22}-C_{12}D_{12})}M_x \\ &\quad - z\frac{C_{12}Q_{12}}{C_{11}D_{22}-C_{12}D_{12}}M_x - \frac{z}{1+\kappa z}\frac{\kappa D_{22}Q_{11}}{C_{11}D_{22}-C_{12}D_{12}}M_x \end{aligned} \tag{40}$$

and

$$\begin{aligned} \sigma_2 &= Q_{12}\varepsilon_x + Q_{22}\varepsilon_y \\ &= \frac{A_{22}Q_{12}-A_{12}Q_{22}}{A_{11}A_{22}-A_{12}^2}\left[N_x + \frac{\kappa D_{22}(A_{22}C_{11}-A_{12}C_{12})}{A_{22}(C_{11}D_{22}-C_{12}D_{12})}M_x\right] + \frac{\kappa C_{12}D_{22}Q_{22}}{A_{22}(C_{11}D_{22}-C_{12}D_{12})}M_x \\ &\quad - z\frac{C_{12}Q_{22}}{C_{11}D_{22}-C_{12}D_{12}}M_x - \frac{z}{1+\kappa z}\frac{\kappa D_{22}Q_{12}}{C_{11}D_{22}-C_{12}D_{12}}M_x \end{aligned} \tag{41}$$

The stress formulas derived by the curved laminate analysis that neglected the nonlinear terms (Whitney 1987) are given by

$$\sigma_1 = \frac{A_{22}Q_{11} - A_{12}Q_{12}}{A_{11}A_{22} - A_{12}^2}N_x + z\frac{D_{22}Q_{11} - D_{12}Q_{12}}{D_{11}D_{22} - D_{12}D_{12}}M_x \tag{42a}$$

$$\sigma_2 = \frac{A_{22}Q_{12} - A_{12}Q_{22}}{A_{11}A_{22} - A_{12}^2}N_x + z\frac{D_{22}Q_{12} - D_{12}Q_{22}}{D_{11}D_{22} - D_{12}D_{12}}M_x \tag{42b}$$

Eqs. (42a), (42b) are the same as those derived by the classic lamination theory.

The bending stress, σ_1, distributions through the thickness (i.e., z axis) at the location of $\theta = \pi/2$ for the radius $R = 10$ mm and 5 mm are, respectively, illustrated in Figs. 3 and 4, where the predictions by Whitney (1987) are also plotted for comparisons. Unlike the linear stress distributions by Whitney analysis, the stress distributions by the present analysis are in the form of nonlinear distributions. By comparison of Figs. 3 and 4, the difference of predictions by the present and Whitney analyses becomes more significant as the thickness-radius (h/R) ratio increases. As seen in Figs. 3, 4, the maximum bending stress occurs at the inner radius due to the presence of geometrical curvature κ of the curved laminate. The bending stresses predicted by two analyses are most different at the location of inner radius. The difference of the bending stresses at the inner radius predicted by two analyses is plotted as a function of the h/R ratio, as depicted in Fig. 5, which shows that the difference becomes larger as the h/R ratio increases. It implies that the analysis by Whitney no longer predicts the bending stresses accurately for curved laminates as the h/R ratio is getting larger and the present curved lamination theory is required for proper stress analyses.

The curved structure has been analyzed by using finite element analysis (FEA) method by Mangala *et al.* (2002), Kundu *et al.* (2007) and Hu *et al.* (2007). Here, the FEA analysis has also

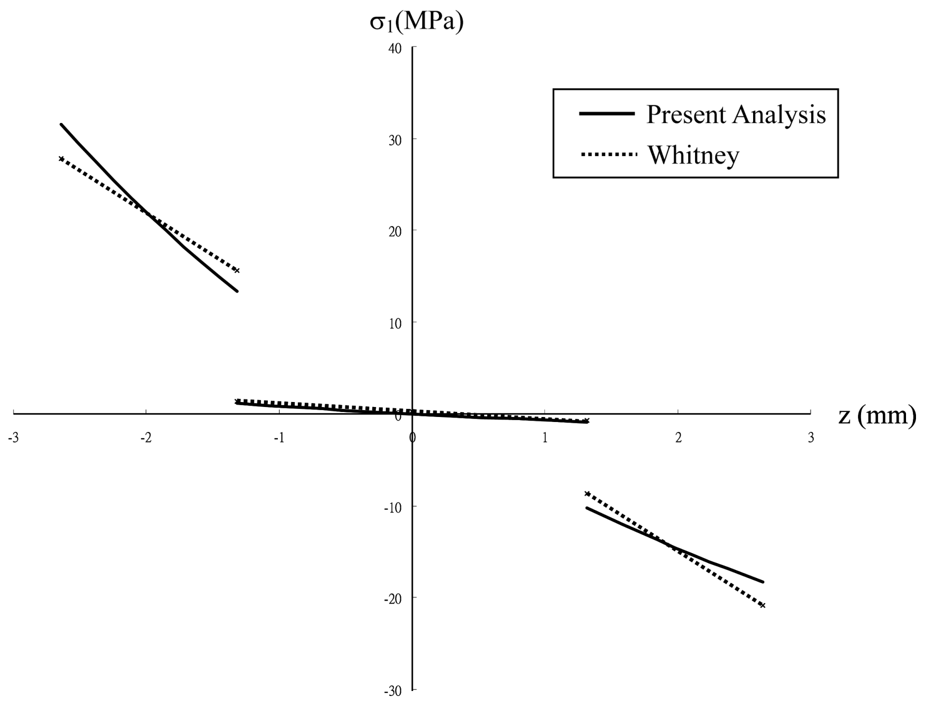


Fig. 3 The bending stress distributions, σ_1, through the laminate thickness for $[0_{10}/90_{10}]_s$ curved laminate with $R = 10$ mm

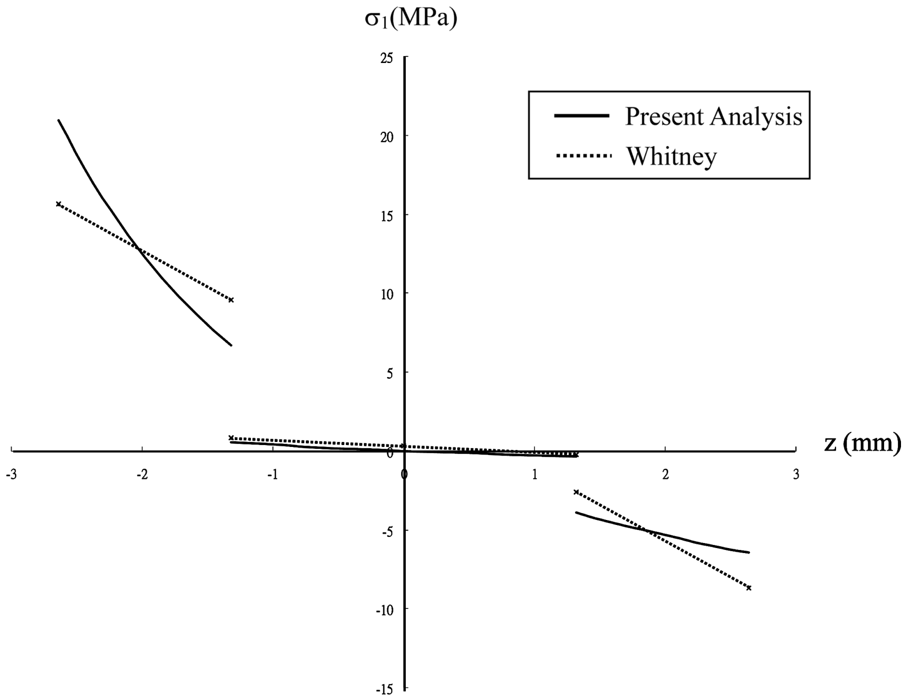


Fig. 4 The bending stress distributions, σ_1, through the laminate thickness for $[0_{10}/90_{10}]_s$ curved laminate with $R = 5$ mm

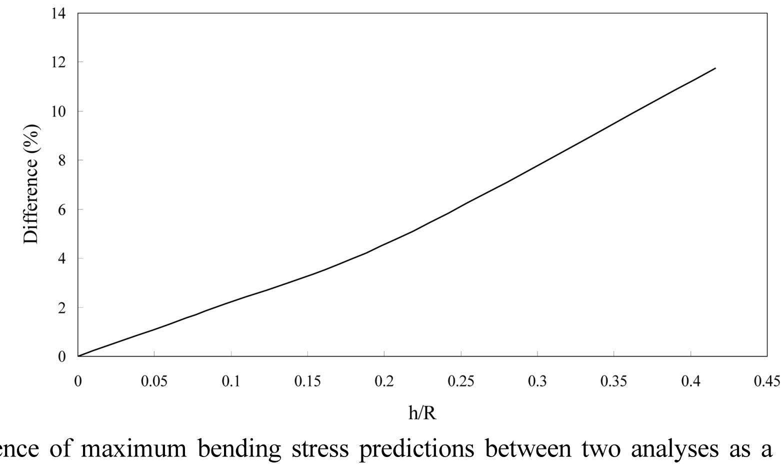


Fig. 5 The difference of maximum bending stress predictions between two analyses as a function of the *h/R* ratio

been conducted to investigate the curvature effect on the stress predictions for curved laminates. For a unidirectional 90° curved laminate with the radius $R = 10$ mm and $h/R = 1$ under the loading of $q = 12$ kN/m at one end with the other end fixed, FEA by NASTRAN predicts the through-thickness bending stress distribution σ_x at the location of $\theta = \pi/2$ and plotted in Fig. 6, where the results of the present and Whitney are illustrated for comparison. As analogous to the theoretical

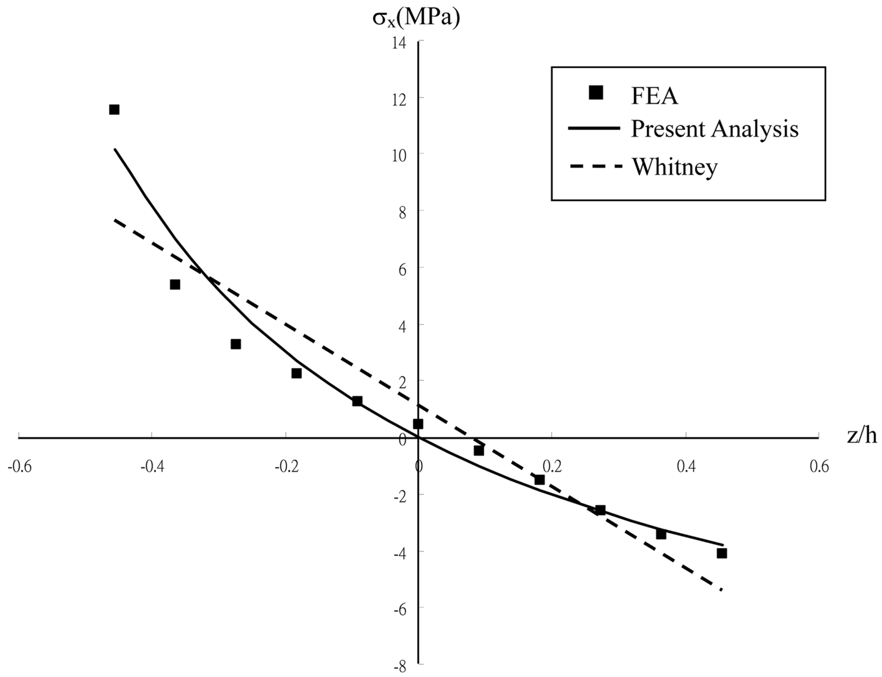


Fig. 6 The bending stress distributions, σ_x, through the laminate thickness for a unidirectional 90° curved laminate with $R = 10$ mm and $h/R = 1$

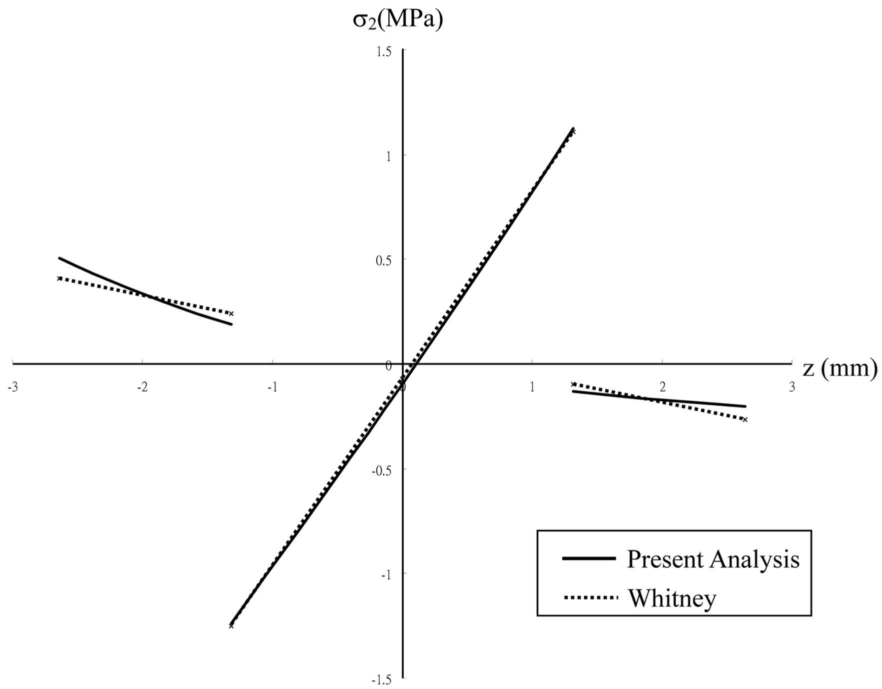


Fig. 7 The stress distributions, σ_2, through the laminate thickness for $[0_{10}/90_{10}]_s$ curved laminate with $R = 10$ mm

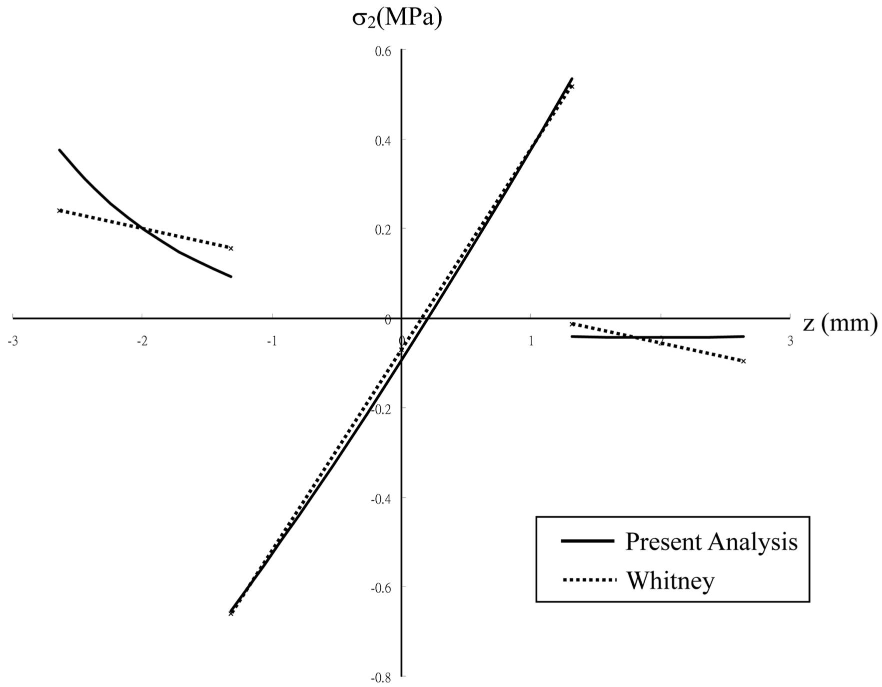


Fig. 8 The stress distributions, σ_2, through the laminate thickness for $[0_{10}/90_{10}]_s$ curved laminate with $R = 5$ mm

predictions as shown in Figs. 3 and 4, the FEA results also indicates that the bending stress shifts to higher tensile stress at the inner radius and shifts to smaller compressive stress at the outer radius as predicted by the present analysis rather than the linear bending stress distribution predicted by Whitney. The FEA results confirm the curvature effect on the stress calculations for curved laminates predicted by the present analysis.

The σ_2 stress distributions along the z axis at the location of $\theta = \pi/2$ for the radius $R = 10$ mm and 5 mm by two analyses are, respectively, illustrated in Figs. 7, 8, where the predictions by Whitney are also plotted for comparisons. Similar to the bending stresses of σ_1, the difference of predictions by the present and Whitney (1987) analyses becomes more significant as the h/R ratio increases.

6. Conclusions

1. A curved lamination theory that describes the linear elastic response of a curved laminate subjected to stretching and bending has been developed in the present paper. Similar to the classic 6×6 ABD matrix constitutive relation of a flat laminate, the new 6×6 ABCD matrix constitutive relation between force resultants, moment resultants, mid-plane strains and deformed curvatures for a curved laminate has been formulated. This new curved laminate theory will provide the fundamental basis for the analyses of curved laminated structures. The thermal behavior of a curved laminate has also been formulated in the present analysis.
2. The present analysis indicates that the nonlinear terms are critical in the analyses of the curved laminated structures. The effects of the non-flat geometries on the stresses and displacements of a

curved laminate are reflected by those nonlinear terms. In the present analysis, a curved lamination theory has been developed by taking into account the nonlinear teams in the analysis.

3. Unlike the classic ABD constitutive relation of a flat symmetric laminate, in which the vanishing of B matrix gives an uncoupling between stretching and bending, the stretching and bending coupling still exists for the curved symmetric laminate. The coupling effect is caused by the curvature effect of the curved laminate.
4. The application of the curved laminate theory is demonstrated by the stress calculations of a curved laminate under the extension loading. In addition, the stress predictions are compared to those by the analysis that neglected the nonlinear terms. The results show that the curved laminate analysis that neglected the nonlinear terms cannot properly reflect the curvature effect and can no longer predict the stresses accurately as the curvature becomes noticeable.
5. The finite element analysis (FEA) has been conducted to investigate the curvature effect on the stress predictions for curved laminates. As analogous to the theoretical predictions, the FEA results also indicates that the bending stress shifts to higher tensile stress at the inner radius and shifts to smaller compressive stress at the outer radius as predicted by the present analysis rather than the linear bending stress distribution predicted by the analysis that neglected the nonlinear terms. The FEA results confirm the curvature effect on the stress calculations for curved laminates predicted by the present analysis.
6. In additional to the curvature effect, the transverse shear stress should also be considered for the curved plate and laminate as the ratio *h/R* is getting larger (Sun and Kelly 1988, Kedward *et al*. 1989). It is the future work to combine the effects of curvature and transverse shear stress for the analyses of thick curved plates and laminates.

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