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An improved parametric formulation for the variationally correct distortion immune three-noded bar element

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Abstract. A new method of formulation of a class of elements that are immune to mesh distortion effects is proposed here. The simple three-noded bar element with an offset of the internal node from the element center is employed here to demonstrate the method and the principles on which it is founded upon. Using the function space approach, the modified formulation is shown here to be superior to the conventional isoparametric version of the element since it satisfies the completeness requirement as the metric formulation, and yet it is in agreement with the best-fit paradigm in both the metric and the parametric domains. Furthermore, the element error is limited to only those that are permissible by the classical projection theorem of strains and stresses. Unlike its conventional counterpart, the modified element is thus not prone to any errors from mesh distortion. The element formulation is symmetric and thus satisfies the requirement of the conservative nature of problems associated with all self-adjoint differential operators. The present paper indicates that a proper mapping set for distortion immune elements constitutes geometric and displacement interpolations through parametric and metric shape functions respectively, with the metric components in the displacement/strain replaced by the equivalent geometric interpolation in parametric co-ordinates.

Keywords: metric and parametric; unsymmetric and symmetric formulations; element distortion; best-fit paradigm; variational correctness; orthogonal projection; function spaces; modified shape functions.

1. Introduction

Distortion sensitivity of isoparametric elements is an important issue in finite element analysis, especially in stress analysis problems near cut-outs or corners where significant element distortion and high stress gradients are inevitable, leading to large errors in stress values. While the finite element method is credited with the ability to negotiate arbitrary external boundaries of elements as often necessary, the isoparametric element formulations, which depend on the mathematical mapping of the physical domains of metric co-ordinates onto those of parametric co-ordinates, do indeed suffer the consequences of extreme distortions. Mesh distortions can often induce unacceptably large errors in finite element results.

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Studies on the deterioration of performance of isoparametric elements under mesh distortions have been made (Stricklin *et al.* 1977, Backlund 1978, Gifford 1979). It has been observed that isoparametric elements, based on identical sets of Lagrangian shape functions in parametric coordinates for both geometry and displacement interpolations, performed extremely well for regular meshes but degraded rapidly under mesh distortions. To reduce distortion effects in isoparametric elements, Rajendran and co-workers (Rajendran and Liew 2003, Rajendran and Subramanian 2004) have proposed the use of dual shape functions for the displacement field, *viz.,* the *compatibility/continuity* enforcing *parametric* functions and *completeness* enforcing *metric* functions. Functions of the former type are used as test functions representing virtual displacements, while those of the latter type are used as trial functions for the differential equations. This kind of unsymmetric formulations effectively eliminates the source of the problem; a Jacobian which becomes a general function of the parametric co-ordinates in the denominator of the strain-displacement vector. However, despite its efficacy in reducing the undesirable effects of distortions, the unsymmetric formulations do not reflect a precise physical description of the system and they inevitably violate the conservation of energy requirement of any self-adjoint system.

After a critical study of the effects of distortion, Felippa (2006) poses an intriguing question "Stay Cartesian or go natural?", to which Prathap (2007) responds in a meaningful way; "Stress Cartesian (metric) but strain natural (parametric)" through the Petrov Galerkin approach. According to Prathap, the improvement in the performance of the element is possible because completeness in the metric functions as trial functions helps to reduce errors in the stresses, while continuity is enforced through the test functions.

Prathap *et al.* (2007) have examined the conventional symmetric isoparametric (PP), symmetric metric (MM) and the unsymmetric (PM) formulations of the three noded bar element in the light of variational correctness (Prathap and Mukherjee 2003, Mukherjee and Jafarali 2010, Kumar and Prathap 2008) that is based on the best-fit paradigm which essentially springs from the projection theorems founded in the first principles of the finite element method (Strang and Fix 1973, Norrie and De Vries 1978). They have observed that the conventional three-noded isoparametric (PP) elements satisfy the best-fit rule for regular geometry, but violate it when the internal node is at an offset from the element center. Furthermore, the symmetric metric-metric (MM) elements do not violate the best-fit rule in the metric domain.

In the classical isoparametric formulation, the Jacobian appears as a function of the parametric coordinates in the denominator of the algebraic expression for the strain interpolations inside the *distorted* element. The performance of the element then becomes dependent on the characteristics of the Jacobian itself. Furthermore, under such circumstances, the best-fit paradigm in the parametric domain gets violated, indicating that under distortion, isoparametric elements are not really variationally correct. This can be appreciated if one considers the fact that the best-fit paradigm of finite element analysis is valid in the metric domain, and will remain so even in the parametric domain only if a proper mapping is performed under element distortion.

Recognizing the fact that the basic reason for mesh distortion sensitivity in isoparametric formulations actually lies in the *improper* interpolations of element displacement using identical sets of classical parametric Lagrangian shape functions for geometry and displacement, the present paper proposes a new kind of formulation that is symmetric and variationally correct in both the parametric and metric domains and one that retains the best of both the worlds of metric and the parametric formulations even under severe distortions. Since any proper MM formulation (in natural/rectilinear co-ordinates) can preserve completeness even under element distortion, it is actually cast in terms of an updated formulation in the parametric domain. The guiding wisdom behind this observation is that "completeness is a stronger requirement than conformity and in many practical cases, it is a sufficient condition for convergence (Norrie and De Vries 1978)". The simple three-noded bar element with an offset of the internal node is chosen for the purpose of demonstrating the validity of the best-fit rule even under mesh distortion provided proper geometry interpolation is taken into account for the displacement interpolation.

2. The three-noded distorted bar element in the parametric and metric domains

In this section, the consequences of the distorted quadratic three noded bar element of only axial degrees of freedom are discussed. A three noded bar element of length $2L$ and three axial degrees of freedom is shown in Fig. 1. The internal node (of nodal index 2) is not necessarily in the middle position of the bar, and a non-dimensional distortion parameter τ is used to indicate the relative offset of this node from the central position.

The geometry and displacement interpolations (x and u^h respectively) for the 3-noded distorted bar element are given as below.

Nodal Positions

$$x_1, x_2, x_3 = -L, \tau L, L \qquad \xi_1, \xi_2, \xi_3 = -1, 0, 1 \tag{1}$$

Parametric

$$x = \sum_{i=1}^{3} N_i x_i = \tau.L(1-\xi^2) + L\xi$$

$$u^h = \sum_{i=1}^{3} N_i u_i = [N_1 \;\; N_2 \;\; N_3]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [N]\{\delta^e\} \tag{2a,b}$$

Metric

$$\bar{u}^h = \sum_{i=1}^{3} M_i u_i = [M_1 \;\; M_2 \;\; M_3]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [M]\{\delta^e\} \tag{3}$$

where $\{\delta^e\} = [u_1 \;\; u_2 \;\; u_3]^T$ is the nodal displacement vector. The *parametric* shape functions are

$$N_1 = -\xi(1-\xi)/2, \quad N_2 = 1-\xi^2, \quad N_3 = \xi(1+\xi)/2 \tag{4}$$

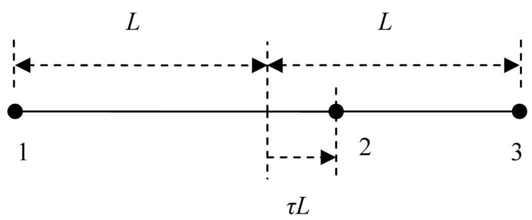

Fig. 1 The quadratic bar element with offset in internal node position

and the *metric* shape functions are

$$M_1 = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}, \quad M_3 = \frac{(x-x_3)(x-x_1)}{(x_2-x_3)(x_2-x_1)}, \quad M_3 = \frac{(x-x_2)(x-x_1)}{(x_3-x_2)(x_3-x_1)} \tag{5}$$

The corresponding axial strain interpolations in the element are given as

Parametric

$$\varepsilon^h = \frac{du^h}{dx} = \frac{du^h/d\xi}{dx/d\xi} = \frac{1}{2L(1-2\tau\xi)}[(-1+2\xi) \;\; -4\xi \;\; (1+2\xi)]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= [B_1 \;\; B_2 \;\; B_3] = [B_p]\{\delta^e\} \tag{6}$$

Metric

$$\bar{\varepsilon}^h = \frac{d\bar{u}^h}{dx} = \frac{1}{2L}\left[\left(-1+2\frac{x/L}{1+\tau}\right) \;\; -4\frac{x/L}{1-\tau^2} \;\; \left(1+2\frac{x/L}{1-\tau}\right)\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [B_M]\{\delta^e\} \tag{7}$$

Note that the corresponding strain-displacement matrices $[B_p]$ and $[B_M]$ are identical for zero distortion ($\tau = 0$), i.e., when the internal node 2 sits exactly at the element center.

Using the relation from the parametric representation of geometry as given by Eq. (2a)

$$\frac{x}{L} = \tau(1-\xi^2)+\xi \tag{8}$$

and substituting this in the metric strain expression of Eq. (7) we get the following expression for the metric strain, expressed in terms of parametric co-ordinates

$$\bar{\varepsilon}^h = \frac{d\bar{u}^h}{dx} = \frac{1}{2L}\left[\left(-1+2\frac{\tau(1-\xi^2)+\xi}{1+\tau}\right) \;\; -4\frac{\tau(1-\xi^2)+\xi}{1-\tau^2} \;\; \left(1+2\frac{\tau(1-\xi^2)+\xi}{1-\tau}\right)\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= [\bar{B}_1 \;\; \bar{B}_2 \;\; \bar{B}_3]\{\delta^e\} = [B_{PC}]\{\delta^e\} \tag{9}$$

Eqs. (9) and (7) are identical since each of these expresses the strain $\bar{\varepsilon}^h$ but in two different coordinates, x and ξ. Eq. (7) expresses the metric strain in terms of a *linear* function of the metric co-ordinate x and Eq. (9) expresses the same as a *quadratic* function of the natural co-ordinate ξ. This is an obvious consequence of the warping in geometry, as given by the quadratic interpolation of the metric coordinate x in terms of ξ (Eq. (2a)). Thus we have the following identity for this element for all values of the distortion parameter τ

$$[B_{PC}] = [B_M] \tag{10}$$

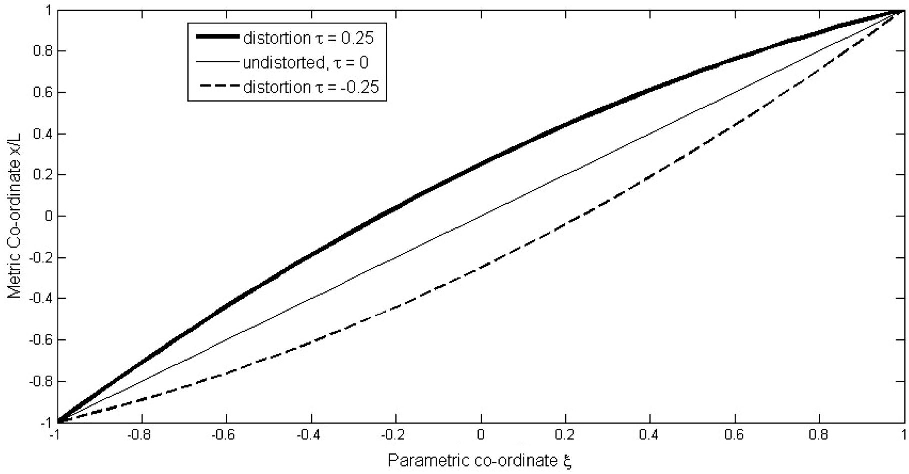

Fig. 2 Variation of metric co-ordinate with the parametric co-ordinate for three distorted positions of internal node of the 3 noded bar element

where

$$[B_{PC}] = \frac{1}{2L}\left[\left(-1+2\frac{\tau(1-\xi^2)+\xi}{1+\tau}\right) \quad -4\frac{\tau(1-\xi^2)+\xi}{1-\tau^2} \quad \left(1+2\frac{\tau(1-\xi^2)+\xi}{1-\tau}\right)\right] \tag{11}$$

and

$$[B_M] = \frac{1}{2L}\left[\left(-1+2\frac{x/L}{1+\tau}\right) \quad -4\frac{x/L}{1-\tau^2} \quad \left(1+2\frac{x/L}{1-\tau}\right)\right] \tag{12}$$

Eqs. (6) and (9) can now be compared. The conventional isoparametric strain ε^h (without consideration of the nodal distortion in element geometry) is a linear function while the metric strain $\bar{\varepsilon}^h$ (with consideration of the nodal distortion in element geometry) is a quadratic function of the parametric co-ordinate ξ. Furthermore, the parametric strain of Eq. (6) is relatively more difficult to handle by Gaussian integration because of the Jacobian $J = L(1-2\tau\xi)$ that appears in the denominator of $[B_P]$ as a function of the natural co-ordinate. It can be observed from Eq. (8) that under a finite non-zero distortion τ, the metric co-ordinate x is actually a quadratic (instead of a linear) function of the parametric co-ordinate ξ, indicating that the ξ coordinate is "warped" with respect to the metric coordinate x. In the particular case of the undistorted element, ($\tau = 0$) this relationship is reduced to a linear one, and the warping vanishes. This is shown in Fig. 2.

3. Finite element strains within the three-noded bar element as orthogonal projections onto subspaces

Thus for the *distorted* bar element, the best-fit paradigm of finite element analysis in both the parametric and the metric formulations can be realized if one obtains finite element strain solutions that agree with the orthogonal projections of the analytical strains onto the proper strain-displacement *subspace* B_{PC} (*quadratic* in the parametric ξ domain) and also onto the *subspace* B_M (*linear* in the metric x domain). For undistorted elements ($\tau = 0$), this subspace B_{PC} reduces to the subspace B_P (*linear* in ξ) of the conventional isoparametric element.

The subspace B_{PC} is the 'corrected' parametric strain-displacement subspace because it incorporates the necessary correction (due to warping in ξ) for making the formulation of a distorted element variationally correct, as will be demonstrated in the later sections, in contrast to subspace B_P that is employed in the formulation of the conventional isoparametric element that violates the best-fit rule under distortion.

3.1 Orthogonal basis vectors spanning the corrected parametric B_{PC} subspace

Eq. (11) provides the expression for the strain-displacement matrix $[B_{PC}]$ in the parametric domain as

$$[B_{PC}] = \frac{1}{2L}\left[\left(-1+2\frac{\tau(1-\xi^2)+\xi}{1+\tau}\right) \ -4\frac{\tau(1-\xi^2)+\xi}{1-\tau^2} \ \left(1+2\frac{\tau(1-\xi^2)+\xi}{1-\tau}\right)\right]$$

The inner product of the element in the B_{PC} subspace in the ξ domain is defined as

$$<a,b> = \int_{\xi=-1}^{1} \{a\}^T[D]\{b\}.J.d\xi \tag{13}$$

where $[D]$ is the element rigidity matrix and J is the determinant of the Jacobian. For the bar element, the rigidity is just $[D] = EA$, where E is the Young's Modulus of the bar material and A is the sectional area of cross section of the bar, and the Jacobian for the present distorted element is $J = L(1-2\tau\xi)$.

For convenience of algebraic work for the integration, the following substitution with $\eta = x/L$ is made so that from Eq. (8) one has

$$\eta = \tau(1-\xi^2)+\xi \tag{14a}$$

This leads to

$$d\eta = (1-2\tau\xi).d\xi \Rightarrow L.d\eta = J.d\xi \tag{14b}$$

Using the Gram-Schmidt process, it can be shown that the two-dimensional B_{PC} subspace can be spanned by two orthogonal basis vectors given as

$$v_1 = \tau(1-\xi^2)+\xi \qquad v_2 = 1 \qquad \text{where} \quad <v_1,\, v_2> = 0 \tag{15a, b, c}$$

3.2 Orthogonal basis vectors spanning the metric B_M subspace

Eq. (12) provides the expression for the strain-displacement matrix $[B_M]$ in the metric domain as

$$[B_M] = \frac{1}{2L}\left[\left(-1+2\frac{x/L}{1+\tau}\right) \ -4\frac{x/L}{1-\tau^2} \ \left(1+2\frac{x/L}{1-\tau}\right)\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{16}$$

The inner product in the metric space is defined as

$$<a,\, b>_M = \int_{x=x1}^{x_3} \{a\}^T[D]\{b\}dx = \int_{-L}^{L}\{a\}^T[D]\{b\}dx \tag{17}$$

Using the Gram-Schmidt process, it can be shown that the two-dimensional B_M subspace can be spanned by two orthogonal basis vectors given as

$$v_1^M = \frac{x}{L} \quad v_2^M = 1 \quad \text{where} \quad \langle v_1^M, v_2^M \rangle_M = 0 \tag{18a,b,c}$$

3.3 Orthogonal projections of analytical strain onto the corrected parametric B_{PC} and metric B_M subspaces

According to the best-fit paradigm, the FEM strain $\bar{\varepsilon}^h$ can be obtained as the orthogonal projection of the analytical strain ε onto the two-dimensional function subspace B_{PC} as

$$\bar{\varepsilon}^h = \sum_{i=1}^{2} \frac{\langle \varepsilon, v_i \rangle}{\langle v_i, v_i \rangle} v_i \tag{19a}$$

or equivalently, on the two-dimensional B_M subspace as

$$\bar{\varepsilon}^h = \sum_{i=1}^{2} \frac{\langle \varepsilon, v_i^M \rangle_M}{\langle v_i^M, v_i^M \rangle_M} v_i^M \tag{19b}$$

The geometrical projection is presented in simplified form in Fig. 3. The analytical strain ε is derived from the analytical displacement u which is the exact solution of the same differential equation upon which the finite element formulation is being based through the weak form. For the axially loaded bar element, the differential equation is

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = q(x) \tag{20a}$$

where $q(x)$ is the distributed axial load intensity acting on the bar. The analytical axial strain in the bar is given by

$$\varepsilon = \frac{du}{dx} \tag{20b}$$

Any variationally correct finite element solution for element strain appears as the orthogonal projection of the corresponding analytical strain onto the strain-displacement function space (Prathap and Mukherjee 2003, Mukherjee and Jafarali 2010).

The following inner product expressions are useful for evaluating the best-fit strains as given in Eqs. (19a) and (19b),

$$\langle v_1, v_1 \rangle = \int_{-1}^{1} EA\{\tau(1-\xi^2)+\xi\}^2 .J.d\xi = \int_{-1}^{1} EA\eta^2 .L.d\eta = \frac{2}{3}EAL$$

$$\langle v_2, v_2 \rangle = \int_{-1}^{1} EA.J.d\xi = \int_{-1}^{1} EA.L.d\eta = 2EAL$$

$$\langle v_1^M, v_1^M \rangle_M = \int_{-L}^{L} EA.\left(\frac{x}{L}\right)^2 dx = \frac{2}{3}EAL \quad \langle v_2^M, v_2^M \rangle_M = \int_{-L}^{L} EAdx = 2EAL$$

The orthogonal projection $\bar{\varepsilon}^h$ of the analytical strain ε given by Eqs. (19a, b) satisfies the

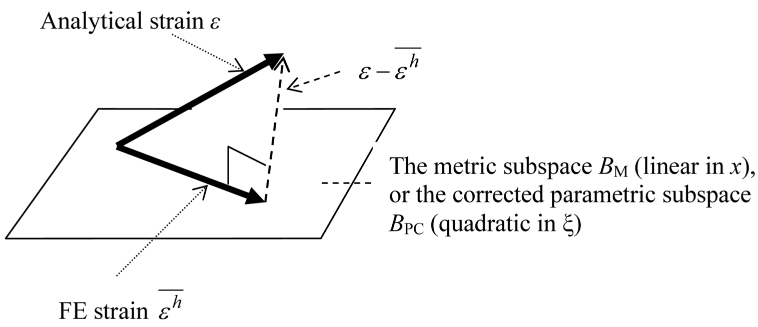

Fig. 3 The finite element (FE) strain as an orthogonal projection of the analytical strain onto the strain-displacement subspace for variationally correct formulations

following best-fit rule that follows from the Pythagorean Theorem

$$\|\varepsilon - \overline{\varepsilon}^h\|^2 = \|\varepsilon\|^2 - \|\overline{\varepsilon}^h\|^2 \tag{21}$$

i.e.,

Energy of the Error = Error of the Energy

In Eq. (21), the norms of the strain vectors are given by

$$\|\varepsilon\| = \sqrt{<\varepsilon, \varepsilon>}, \quad \|\overline{\varepsilon}^h\| = \sqrt{<\overline{\varepsilon}^h, \overline{\varepsilon}^h>}, \quad \|\varepsilon - \overline{\varepsilon}^h\| = \sqrt{<(\varepsilon - \overline{\varepsilon}^h), (\varepsilon - \overline{\varepsilon}^h)>}$$

Finite element formulations that yield results according to Eq. (21) are variationally correct. From Fig. 3, it is obvious that this equation ensures that, of all the strains in the strain-displacement subspace, the best-fit strain (which agrees with the variationally correct finite element strain) actually guarantees a solution of *minimum* dispersion, or error, of the strain in the element.

The objective of the present work is to develop a procedure to model parametric formulations of an element, which under distortion, performs better than the conventional isoparametric element. In the following sections, it will be demonstrated through some chosen problems that the metric formulation (MM) based modified bar element MBAR3, which can also be cast as the *improved* parametric element (IPP) in the parametric domain, satisfies the best-fit rule even under distortion caused by the offset in the position of the internal node from the central position, and therefore is variationally correct. However, the conventional isoparametric (PP) element, under distortion, violates the best-fit paradigm, given by Eq. (21), and is therefore variationally incorrect. It will also be shown that the metric (MM) element yields better results (of lower errors for major portion of the element) than the conventional isoparametric (PP) element.

The modified parametric form of the MBAR3 element essentially employs $[B_{PC}]$ as the strain-displacement matrix in the parametric domain, and a set of modified appropriate interpolation functions for the displacement, but maintaining the conventional Lagrangian shape functions in the parametric domain for interpolation of element geometry. In contrast, the conventional isoparametric BAR3 element employs identical sets of interpolation functions for both geometry and displacement and employs $[B_P]$ as the strain-displacement matrix.

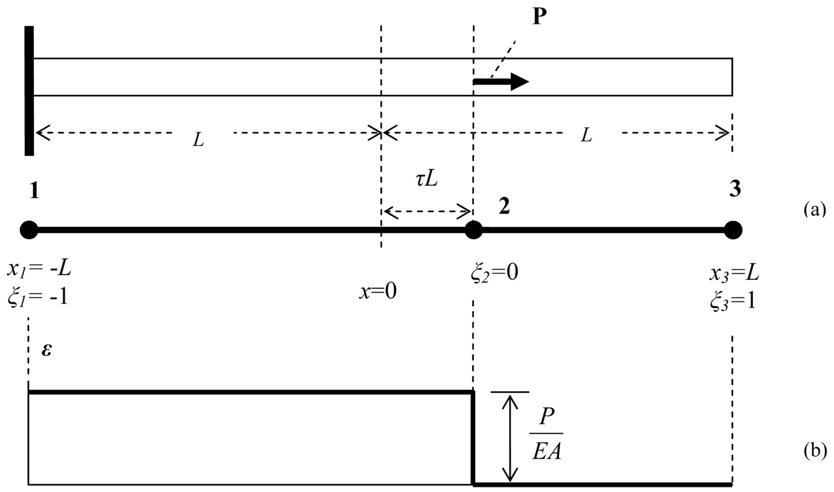

Fig. 4 A cantilever bar with a concentrated axial load P (a) a single 3 noded bar element representing the cantilever bar with offset of the internal node (where the load P acts) from the central position, (b) analytical axial strain variation in the bar due to concentrated axial load acting at the internal node

4. Demonstrative problems

4.1 Bar subjected to a concentrated load

A uniform cantilever bar loaded with a concentrated load P acting on it, is to be analyzed using the conventional isoparametric BAR3 and the modified MBAR3 elements. The bar is modeled as a single three-noded element of length $2L$ with distortion of the internal node. The loaded bar, its finite element model and the analytical strain is presented in Fig. 4.

The analytical strain of the bar is given by

$$\begin{aligned} \varepsilon &= P/EA \quad \text{for} \quad x_1 \le x \le x_2 \quad \text{i.e.,} \quad -1 \le \xi \le 0 \\ &= 0 \quad \text{for} \quad x_2 < x \le x_3 \quad \text{i.e.,} \quad 0 < \xi \le 1 \end{aligned} \tag{22}$$

A discontinuity of the analytical strain can be noticed at $x = x_2$ (or $\xi = 0$) where the concentrated load P is lumped. Finite element analysis of the problem using the IPP element yields an element strain that is equal to the best fit strain, given by

$$\bar{\varepsilon}^h = \sum_{i=1}^{2} \frac{<\varepsilon, v_i>}{<v_i, v_i>} v_i = \frac{\left(\frac{P}{EA}\frac{EAL(\tau^2-1)}{2}\right)}{2EAL/3}\{\tau(1-\xi^2)+\xi\} + \frac{\left(\frac{P}{EA}\cdot EAL(\tau+1)\right)}{2EAL}\cdot 1$$

$$\bar{\varepsilon}^h = \frac{P}{2EA}\left\{1 + \frac{3}{2}[\tau(1-\xi^2)+\xi](\tau^2-1)+\tau\right\}$$

$$\bar{\varepsilon}^h = \frac{P}{2EA}\left\{1 + \frac{3x}{2L}(\tau^2-1)+\tau\right\} \tag{23a, b}$$

Eq. (23) indicates that the projection of the analytical strain for the distorted element is *quadratic* in the natural co-ordinate ξ, but *linear* in the metric co-ordinate x. The expressions above involve the following integrals which are achieved in piecewise manner due to the sudden strain (and stress) discontinuity in the element.

$$<\varepsilon, v_i> = \int_{\xi=-1}^{1} \{\varepsilon\}^T[D]\{v_i\}.J.d\xi = \int_{\xi=-1}^{0} \frac{P}{EA}.EA.v_i.J.(d\xi) + \int_{\xi=0}^{1} 0.EA.v_i.J.d\xi = \int_{\eta=-1}^{\tau} \frac{P}{EA}.EA.v_i.L.d\eta$$

The nodal displacements are consequences of the integrals of strains which are given by the areas covered under the strain curves. Area covered by the best-fit strain within the element is given by

$$A_{BF} = \int_{x=-L}^{L} \bar{\varepsilon}^h.dx = \int_{\xi=-1}^{1} \bar{\varepsilon}^h.Jd\xi = \int_{\eta=-1}^{1} \bar{\varepsilon}^h.Ld\eta = \int_{\eta=-1}^{1} \frac{P}{2EA}\left\{1 + \frac{3}{2}\eta(\tau^2 - 1) + \tau\right\}Ld\eta = \frac{PL}{EA}(1 + \tau)$$

Area covered by the exact strain distribution will be

$$A = \int_{x=-L}^{L} \varepsilon.dx = \int_{\xi=-1}^{1} \varepsilon.Jd\xi = \int_{\xi=-1}^{0} \frac{P}{EA}.Jd\xi + \int_{\xi=0}^{1} 0.Jd\xi = \int_{\eta=-1}^{\eta=\tau} \frac{P}{EA}.Ld\eta + 0 = \frac{PL}{EA}(\tau+1) = A_{BF}$$

Since the spatial integrals (with respect to metric coordinate x) of the best-fit strain and the analytical strains are identical for arbitrary distortions, it can be concluded that for this problem the exact nodal displacements are recovered by the finite element computations. However, it must be pointed out here that in general, finite element computations do not always recover the exact nodal displacements (Mukherjee and Jafarali 2010).

A comparison of the solutions of the finite element strains by the conventional isoparametric (PP) formulation using the $[B_P]$ matrix and the improved parametric formulation IPP, (which is effectively the same as the metric MM formulation of the bar element, but cast in the parametric domain) using the $[B_{PC}]$ matrix is shown in Fig. 5 for a distortion parameter $\tau = 0.25$. Under distortion, the isoparametric PP solution violates the best-fit rule in the parametric domain ξ, as shown earlier (Prathap *et al.* 2007) but the metric MM and its equivalent IPP solutions conserve the best-fit rule in both the metric (x) and the parametric (ξ) domains in accordance with Eq. (21), and are therefore variationally correct. The IPP formulation, even with a distorted element, actually yields best-fit strains (orthogonal projections onto the B_{PC} subspace) in the parametric domain ξ. Fig. 6 presents the finite element strains obtained by the proposed IPP formulation as best-fits of the analytical strains for three selected values of the distortion parameter τ. As indicated by Eq. (23), the finite element strain by the IPP formulation (best-fit) is in general a quadratic function of the same parameter ξ for arbitrary distorted positions of the internal node. For an undistorted element ($\tau = 0$), it reduces to a linear function of the parametric co-ordinate ξ. Fig. 7 shows that the metric MM element yields best-fit strains that are linear in the metric co-ordinate x for arbitrary distortion τ.

It can be observed from Fig. 5 that the overall performance of the variationally correct MBAR3 element, based on the metric (MM) formulation, or the improved parametric (IPP) element is superior to that of the conventional isoparametric element (PP). The maximum error in the strains (and consequently stresses) as given by the conventional isoparametric (PP) element at the free (right) end exceeds the strain error at any other position by the MBAR3 element based on the MM (or the IPP) formulation. Though the strain errors of the PP element are less than that of the IPP

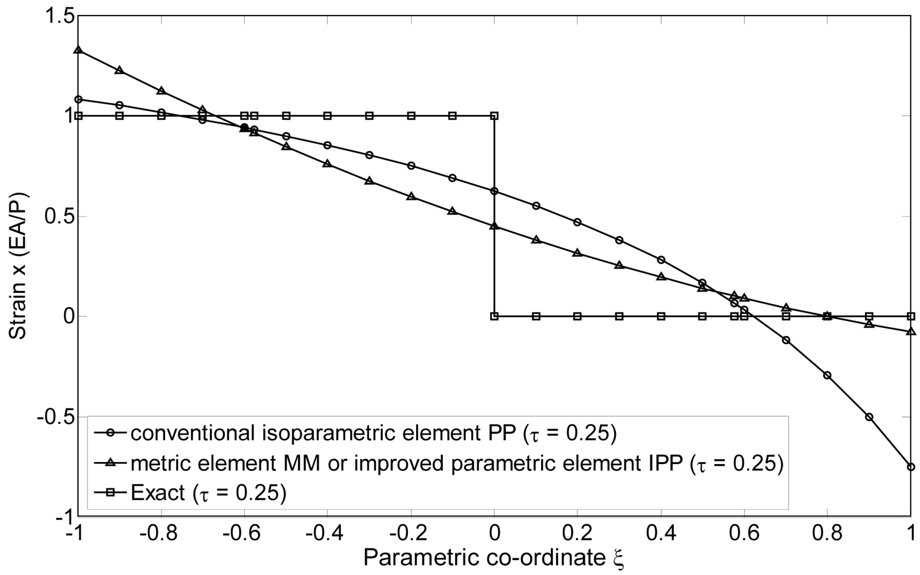

Fig. 5 Comparison of the axial strains obtained by finite element analysis of the bar loaded with a concentrated load P at the internal node of the single distorted three-noded element representing the bar. The conventional isoparametric (PP) element and the improved parametric (IPP) element (based on the metric interpolation functions) have been used, both with a distortion parameter $\tau = 0.25$. The strains are scaled up with a non-dimensional factor EA/P. The strain variation of the MM element is the best-fit to the exact strain, but that of the PP element violates the best-fit paradigm

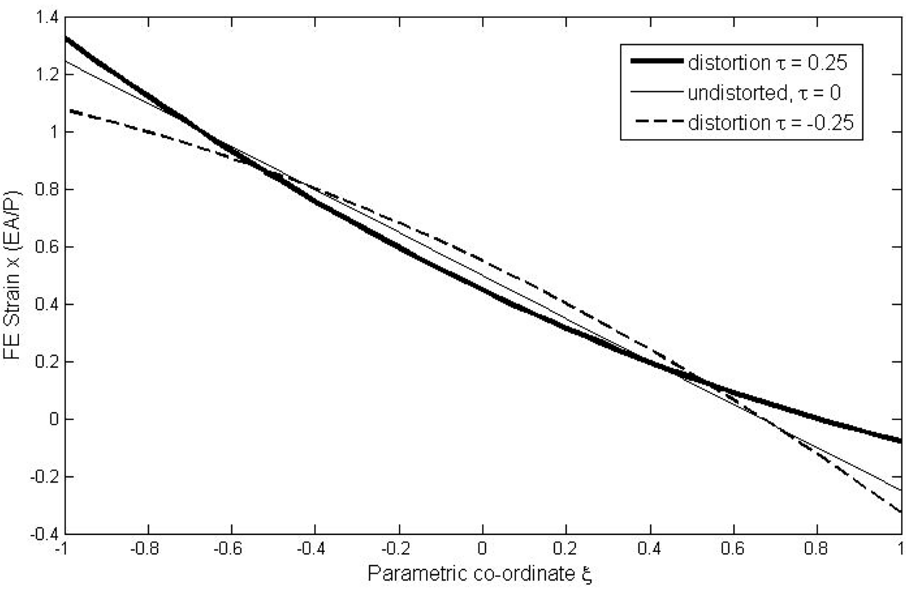

Fig. 6 Variation of the strains in the parametric co-ordinate ξ, as obtained from finite element analysis of the bar with load P at the internal node of the single distorted IPP (or MM) element, for various values of the distortion parameter τ. The strains are scaled up with a non-dimensional factor EA/P

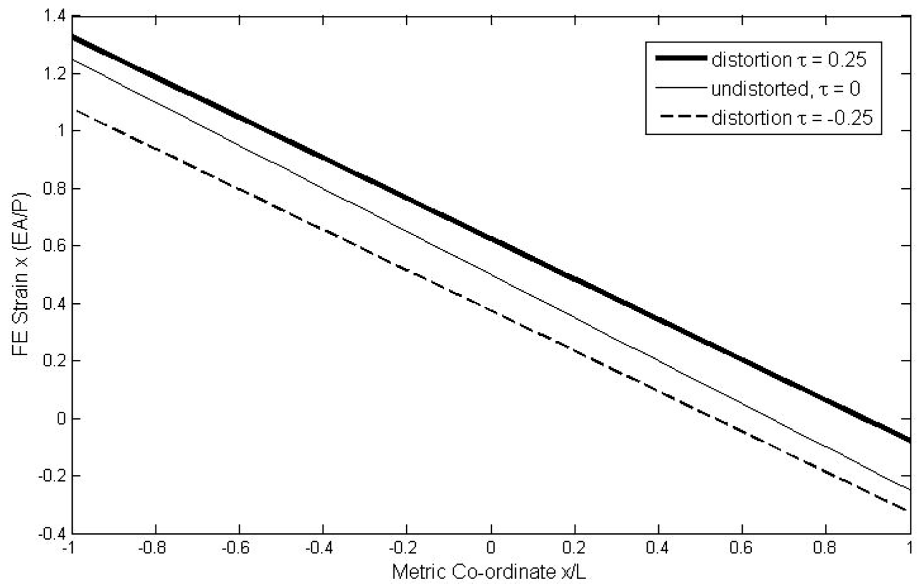

Fig. 7 Linear variation of the strain in the metric co-ordinate x obtained by finite element analysis of the bar with load P at the internal node of the single MM element representing the bar, for various values of the distortion parameter τ. The strains are scaled up with a non-dimensional factor EA/P

element at some locations (at the fixed end, for instance), the best-fit nature of the IPP actually smoothens the strain error over the entire element so that large strain/stress errors at certain locations can be avoided.

4.2 Bar subjected to distributed loading

A uniform cantilever bar subjected to distributed axial loading $q(x)$ is to be analyzed using the conventional isoparametric BAR3 and the modified MBAR3 elements. The bar is again modeled as a single 3 noded element with distortion of the internal node, as shown in Fig. 8. Two loading cases are considered here for analysis; a uniformly distributed load and a linearly varying load $q(x)$.

For the uniformly distributed load, the load intensity is constant

$$q = q_0 \tag{24a}$$

The analytical strain for this uniform loading is

$$\varepsilon = \frac{q_0 L}{EA}\left(1-\frac{x}{L}\right) = \frac{q_0 L}{EA}[1-\{\tau(1-\xi^2)+\xi\}] \tag{24b}$$

Note that while the analytical strain ε is a linear function of the metric co-ordinate x, it is in general a quadratic function of the parametric coordinate ξ for a distorted element, because of the relationship $(x/L) = \{\tau(1-\xi^2)+\xi\}$. For the undistorted element ($\tau = 0$), the linear relationships of ξ with both x and ε are restored.

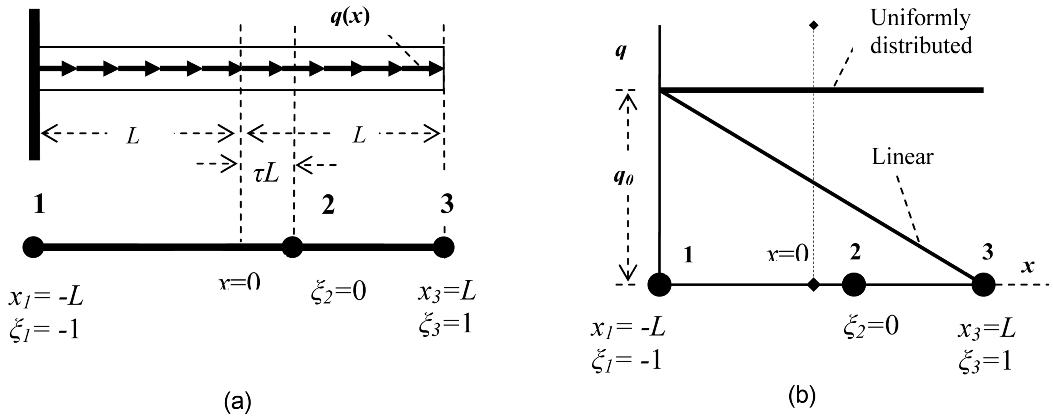

Fig. 8 A cantilever bar with a distributed axial load $q(x)$ per unit length, (a) a single 3 noded MBAR3 element representing the cantilever bar with offset of the internal node from the central position, (b) the two kinds of load distributions $q(x)$ used for the analysis; a uniformly distributed loading and a distribution linearly varying in position x

For this particular case, the metric element (MM), (using linear interpolation in the metric co-ordinate x to approximate the strain) yields the exact strain in the element because it corresponds to a linear best-fit (in the metric coordinate x) of the exact strain, which is also linear in x. Thus the improved parametric (IPP) element captures the exact strain for this case. The projection formula is used here to determine the best-fit strain

$$\bar{\varepsilon}^h = \sum_{i=1}^{2} \frac{<\varepsilon,\, v_i>}{<v_i,\, v_i>} v_i = \frac{q_0 L}{EA}[1 - \{\tau(1-\xi^2) + \xi\}] = \frac{q_0 L}{EA}\left(1 - \frac{x}{L}\right) \tag{24c}$$

For a load distribution varying linearly with metric co-ordinate x (Fig. 8(b)), the expression for the loading is given by

$$q = \frac{q_0}{2}\left(1 - \frac{x}{L}\right) = \frac{q_0}{2}[1 - \{\tau(1-\xi^2) + \xi\}] \tag{25a}$$

The analytical strain for this loading is given by

$$\varepsilon = \frac{q_0 L}{4EA}\left\{1 - \left(\frac{x}{L}\right)\right\}^2 = \frac{q_0 L}{4EA}[1 - \{\tau(1-\xi^2) + \xi\}]^2 \tag{25b}$$

The projection formula is used here to determine the best-fit strain, which exactly agrees with the finite element strain distribution as obtained by the IPP and the MM elements

$$\bar{\varepsilon}^h = \sum_{i=1}^{2} \frac{<\varepsilon,\, v_i>}{<v_i,\, v_i>} v_i = \frac{q_0 L}{3EA}\left[1 - \frac{3}{2}\{\tau(1-\xi^2) + \xi\}\right] = \frac{q_0 L}{3EA}\left(1 - \frac{3}{2}\left\{\frac{x}{L}\right\}\right) \tag{25c}$$

The results of the finite element analyses of the cantilever bar subjected to the two types of distributed loading using a single three-noded distorted element are presented in Fig. 9. A distortion parameter of $\tau = 0.25$ is used for the offset of the central node from the element center.

For the uniformly distributed loading, the improved parametric IPP element captures the exact strain distribution (in accordance with Eq. (24c). For the distributed load varying linearly in the

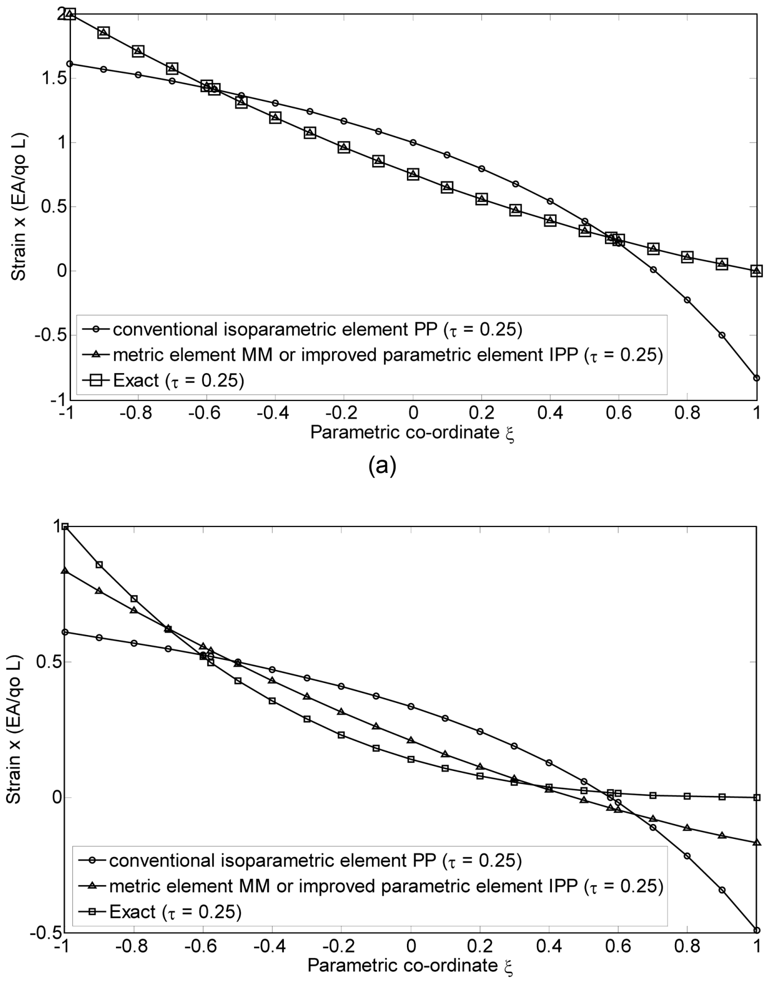

Fig. 9 Comparison of the variations of the axial strains (scaled up with a non-dimensional factor EA/q_0L) obtained by the conventional three-noded isoparametric element (PP formulation) and the distortion immune MBAR3 element (improved parametric, or IPP, formulation) for the bar under two different kinds of distributed loading, (a) for uniformly distributed load and (b) for a linearly varying distributed load. Distortion parameter of the element is $\tau = 0.25$

metric co-ordinate x, this element gives the best-fit strain to the exact strain distribution (in accordance with Eq. (25c). It can be observed that the IPP (or MBAR3) element outperforms the conventional isoparametric PP (or BAR3) element. It yields strain variations of errors that are less than that obtained by the conventional BAR3 element for the major portion of the element.

Furthermore, since the cantilever beam is of uniform sectional rigidity, (EA constant), the spatial integrals (with respect to the metric co-ordinate x) of the exact strain and the best-fit strain curves in the element are equal. Thus the MBAR3 should give the exact nodal displacements at the nodal points (Mukherjee and Jafarali 2010), since nodal displacements are effectively associated with the spatial integrals of the strains in the element.

Improved performance in the prediction of strain implies improved performance in the prediction of stresses. From the practical point of view, the proposed MBAR3 element is superior to the conventional isoparameteric BAR3 element in predicting stresses with reasonable accuracy with relatively coarse meshes and element distortions.

5. Improved formulation of the distorted MBAR3 element with modified displacement function

It has been demonstrated that variationally correct *distorted* bar element is immune to the effects of distortion, and gives the best-fit strain in the both the metric and parametric domains. Under distortion, performance of the conventional isoparametric BAR3 element deteriorates and the element violates the best-fit paradigm. These undesirable effects however vanish in the undistorted ($\tau = 0$) isoparametric element of regular geometry.

In this section, a formal procedure to formulate the improved parametric element (IPP) for the axially loaded bar is proposed. For the purpose, the metric MM element is being cast in the parametric domain with suitable modifications over the conventional isoparametric element BAR3. This modified element, called as the MBAR3, is immune to the effects of element distortion, and reduces to the conventional isoparametric formulation for the undistorted case, i.e., when the internal node of the element exactly sits in its central position.

The improved parametric (IPP) formulation proposes a modified set of interpolation functions in the metric domain for approximating the displacement in this improved MBAR3 element, but the element geometry is still described by the standard Lagrangian shape functions in the ξ domain as used in the conventional BAR3. The proposed set of modified displacement functions has the capability to counter the effects of nodal distortions by incorporating the resulting warping effect of the parametric coordinate. Since these modified displacement functions in the *parametric* domain are generated from the Lagrangian shape functions in the *metric* domain, this element should yield results that are identical to the metric results (with appropriate transformation rule between the parametric and metric co-ordinates), despite arbitrary distortions of the element. Thus the MBAR3 element retains all the advantages offered by the conventional isoparamteric BAR3 element, but is superior to it because unlike the conventional isoparametric element, it is immune to element distortions.

5.1 The modified displacement interpolation

The metric form of the displacement function, satisfying completeness and continuity is given by

$$\bar{u}^h = \sum_{i=1}^{3} M_i u_i \tag{26}$$

where the *metric* shape functions are given by Eq. (5)

$$M_1 = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}, \quad M_2 = \frac{(x-x_3)(x-x_1)}{(x_2-x_3)(x_2-x_1)}, \quad M_3 = \frac{(x-x_2)(x-x_1)}{(x_3-x_2)(x_3-x_1)}$$

Using the nodal co-ordinates $x = -L, x_2 = \tau L, x_3 = L$ the metric shape functions can be expressed as

$$M_1 = -\frac{\left(\frac{x}{L}-\tau\right)\left(1-\frac{x}{L}\right)}{2(1+\tau)},\ M_2 = \frac{\left\{1-\left(\frac{x}{L}\right)^2\right\}}{1-\tau^2},\ M_3 = \frac{\left(\frac{x}{L}-\tau\right)\left(1+\frac{x}{L}\right)}{2(1-\tau)} \tag{27}$$

The element geometry is still described by the conventional geometric interpolation with the *parametric* shape functions

$$x = \sum_{i=1}^{3} N_i(\xi)x_i = \tau.L(1-\xi^2)+L\xi \tag{28}$$

where the conventional shape functions $N_i(\xi)$ are given by Eq. (4).

Using the geometry interpolation of Eq. (28), the metric shape functions of the metric co-ordinate can be expressed as modified shape functions of the parametric co-ordinate

$$M_1 = -\frac{\left(\frac{\sum_{i=1}^{3} N_i(\xi)x_i}{L}-\tau\right)\left(1-\frac{\sum_{i=1}^{3} N_i(\xi)x_i}{L}\right)}{2(1+\tau)} = -\frac{\{\tau(1-\xi^2)+\xi-\tau\}[1-\{\tau(1-\xi^2)+\xi\}]}{2(1+\tau)}$$

$$M_1 = -\frac{\{-\tau\xi^2+\xi\}\{1-\xi-\tau(1-\xi^2)\}}{2(1+\tau)} = \overline{N}_1(\xi) \tag{29a}$$

$$M_2 = \frac{\left\{1-\left(\frac{\sum_{i=1}^{3} N_i(\xi)x_i}{L}\right)^2\right\}}{1-\tau^2} = \frac{[1-\{\tau(1-\xi^2)+\xi\}^2]}{1-\tau^2} = \overline{N}_2(\xi) \tag{29b}$$

$$M_3 = \frac{\left(\frac{\sum_{i=1}^{3} N_i(\xi)x_i}{L}-\tau\right)\left(1+\frac{\sum_{i=1}^{3} N_i(\xi)x_i}{L}\right)}{2(1-\tau)} = +\frac{\{-\tau\xi^2+\xi\}\{1+\xi+\tau(1-\xi^2)\}}{2(1-\tau)} = \overline{N}_3(\xi) \tag{29c}$$

Thus the modified form of the displacement function in the parametric space can be expressed as

$$u_C^h = \sum_{i=1}^{3} \overline{N}_i(\xi)u_i = [\overline{N}_1\ \ \overline{N}_2\ \ \overline{N}_3]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [\overline{N}]\{\delta^e\} \tag{30}$$

where the modified shape functions $\overline{N}_i$ are given as in Eqs. (29a, b, c). The subscript C denotes that the displacement interpolation is a corrected form over the conventional one of Eq. (2b). Note that the modified shape functions $\overline{N}_i$ share the same properties as the original shape functions N_i, viz.,

$$\overline{N}_i(\xi_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad \xi_1, \xi_2, \xi_3 = -1, 0, 1$$

$$\sum_{i=1}^{3} \overline{N}_i(\xi) = 1 \tag{31a, b}$$

The variations of the modified shape functions $\overline{N}_i(\xi)$, ($i = 1, 2, 3$) with ξ for various values of the distortion parameter τ are shown in Fig. 10.

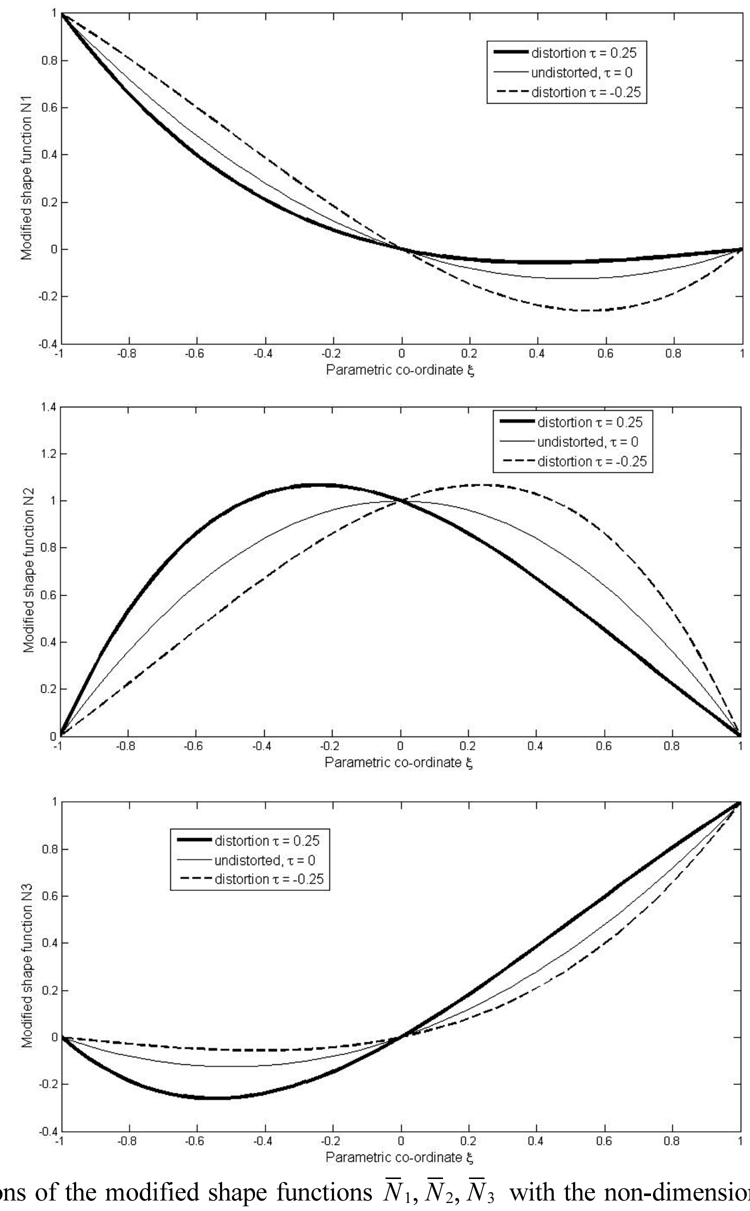

Fig. 10 The variations of the modified shape functions $\overline{N}_1, \overline{N}_2, \overline{N}_3$ with the non-dimensional co-ordinate ξ for distortion parameters $\tau = 0.25$, 0 and -0.25

5.2 Element strain with modified displacement

The axial strain in the corrected parametric domain of the bar element from the displacement interpolation (Eq. (30)) is given by

$$\varepsilon_C^h = \frac{du_C^h}{dx} = \frac{du_C^h}{d\xi}\cdot\frac{d\xi}{dx} = \frac{1}{L(1-2\tau\xi)}\sum_{i=1}^{3}\left\{\frac{d\bar{N}_i}{d\xi}u_i\right\} \tag{32}$$

Using the following expressions for the derivatives of the modified shape functions

$$\begin{aligned}\frac{d\bar{N}_1}{d\xi} &= -\frac{(1-2\tau\xi)}{2(1+\tau)}\{1-\tau+2(\tau\xi^2-\xi)\}\\ \frac{d\bar{N}_2}{d\xi} &= -\frac{2.(1-2\tau\xi)}{1-\tau^2}\{\tau(1-\xi^2)+\xi\}\\ \frac{d\bar{N}_3}{d\xi} &= +\frac{(1-2\tau\xi)}{2(1-\tau)}\{1+\tau-2(\tau\xi^2-\xi)\}\end{aligned} \tag{33a, b, c}$$

the element strain can be expressed as

$$\varepsilon_C^h = \frac{1}{L(1-2\tau\xi)}\sum_{i=1}^{3}\left\{\frac{d\bar{N}_i}{d\xi}u_i\right\}$$

$$= \frac{1}{2L}\left[\left(-1+2\frac{\tau(1-\xi^2)+\xi}{1+\tau}\right)\ -4\frac{\tau(1-\xi^2)+\xi}{1-\tau^2}\ \left(1+2\frac{\tau(1-\xi^2)+\xi}{1-\tau}\right)\right]\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix}$$

$$= [\bar{B}_1\ \bar{B}_2\ \bar{B}_3]\{\delta^e\} = [B_{PC}]\{\delta^e\} \tag{34}$$

The variations of the components $\bar{B}_i(\xi)$, $(i = 1, 2, 3)$ of the corrected strain-displacement matrix $[B_{PC}]$ for a distortion parameter $\tau = 0.25$ are shown in Fig. 11. Again, the subscript C denotes that the strain interpolation is a corrected form in parametric co-ordinates over the conventional one of Eq. (6). Thus by using the modified shape functions for displacement interpolation (Eq. (30)) in the parametric domain, one can retrieve both the appropriate displacement and strain expressions as the metric counterparts as given earlier by Eqs. (3) and (9) respectively. With these modified shape functions, strain results in the MBAR3 element (unlike those of the conventional BAR3 element) obtained through a finite element analysis, should follow the best-fit rule even in the parametric domain ξ for all permissible values of the distortion parameter τ.

The modified shape functions, despite their relatively complicated forms, provide advantages over the conventional shape functions for distorted elements. Even under distortion, they provide the same degree of completeness and continuity to the elements just as the regular ones, and the denominator of the final expressions of the element strain does not involve any Jacobian as function of the parameter ξ. This is especially advantageous for two reasons. Firstly, numerical integration by Gaussian quadrature becomes simple and accurate since only simple polynomials (without any polynomial denominator) become the integrands. Secondly, element performance is unaffected by the values or signs of the Jacobians from distortion.

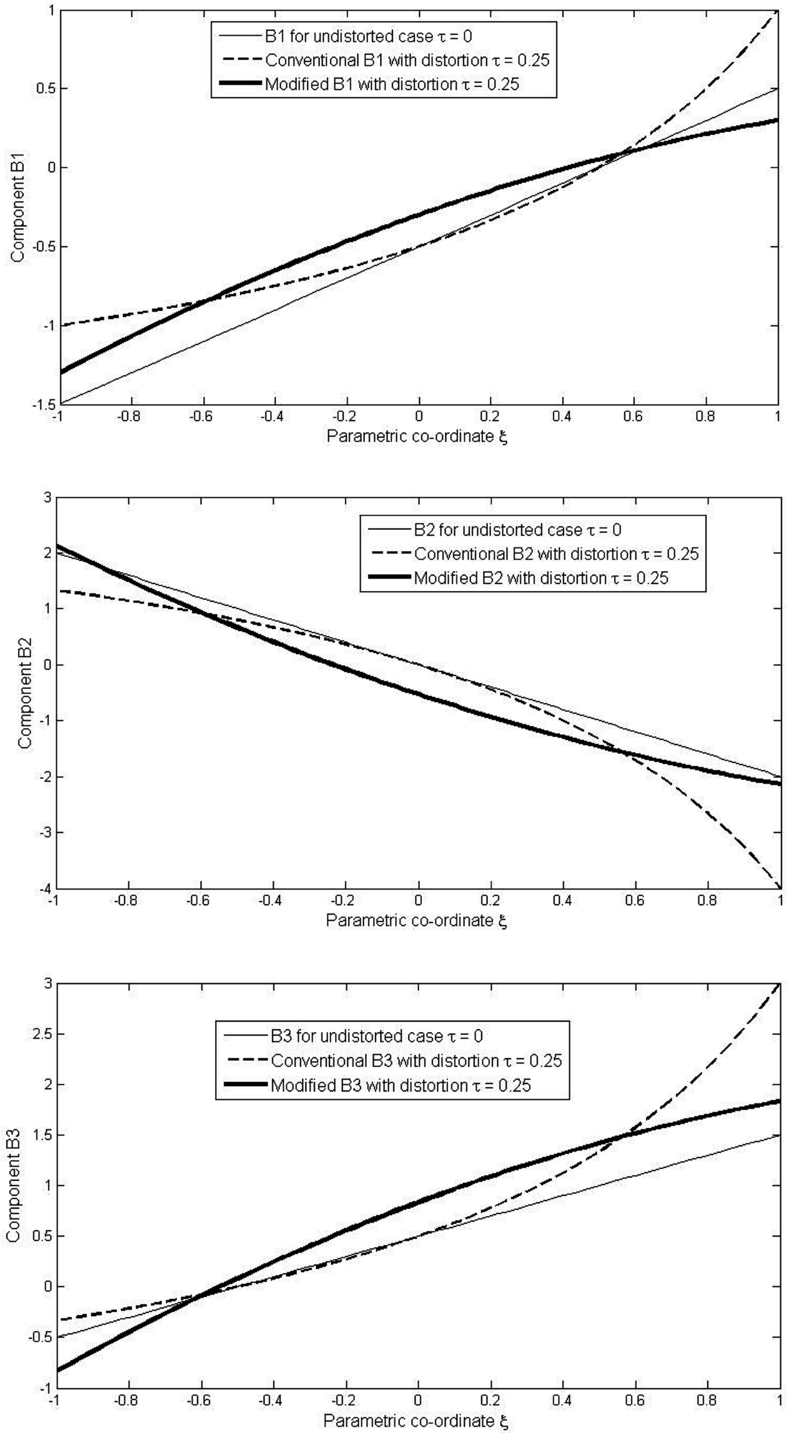

Fig. 11 The variations of the components of the conventional $[B_P]$ of the isoparametric PP element and the modified $[B_{PC}]$ matrices of the IPP element with distortion $\tau = 0.25$. The half-length L of the element used here is 1 unit of length

5.3 Corrected forms for the element stiffness and generalized forces for the distorted element

The improved parametric formulation IPP employs the modified set of interpolation functions $\overline{N}_i$ for displacement (Eq. (30)) but retains the conventional Lagrangian shape functions N_i for geometric interpolation (Eq. (28)). Thus the correct stiffness matrix and the generalized force vector for the distortion immune MBAR3 element are given by the following expressions respectively

$$[K^e] = \int_{-1}^{1}[B_{PC}]^T[D][B_{PC}]Jd\xi \qquad [D] = EA$$

$$\{F^e\} = \int_{-1}^{1}[\overline{N}]^T q(\xi).Jd\xi \tag{35a, b}$$

Here $[\overline{N}] = [\overline{N}_1\ \ \overline{N}_2\ \ \overline{N}_3]$ is the modified shape function matrix employed for displacement interpolation instead of the conventional shape function $[N]$ of the isoparametric element. Furthermore, unlike the conventional isoparametric formulation that employs $[B_P]$ as the strain-displacement matrix, the modified formulation of the MBAR3 element employs $[B_{PC}]$ as the strain-displacement matrix that does not have the Jacobian J (as a function of the distortion τ and the parametric co-ordinate ξ) in the denominator. Note that for a constant sectional rigidity EA, the integrand for the stiffness matrix involves only polynomials of degree four, so that a 3 point Gaussian quadrature scheme can be used to integrate exactly.

6. Topological justification for the MBAR3 formulation

The present formulation of the modified bar element MBAR3 can be justified on the basis of topological arguments. The mapping of the metric strain $\overline{\varepsilon}^h$ of the normed metric subspace B_M into the parametric strain ε_C^h of the normed parametric subspace B_{PC} is mathematically expressed by

$$T(\overline{\varepsilon}^h) = \varepsilon_C^h \tag{36}$$

where T is the mapping operator. Furthermore, since strains $\overline{\varepsilon}^h$ and ε_C^h are actually identical, Eq. (36) leads to

$$\|T(\overline{\varepsilon}^h)\| = \|\varepsilon_C^h\| = \|\overline{\varepsilon}^h\| \tag{37}$$

where $\|\cdot\|$ denotes the norm of a vector in the respective space, as defined by the inner product. In topological parlance, Eq. (37) expresses the condition for the two normed linear subspaces B_M and B_{PC} to be *isometrically isomorphic* (Simmons 1963). This terminology enables us to give a precise meaning to the statement that one normed linear space is essentially the same as another. Eq. (37) effectively stipulates the general mathematical criterion for legitimate mapping from the metric to the parametric spaces for elements under distortion, so that the best-fit paradigm remains valid both at the metric and parametric domains. Conventional isoparametric elements deviating from regular geometry violate Eq. (37), and therefore they inadvertently violate the best-fit paradigm under distortion.

7. Conclusions

The present paper examines, in the light of variational correctness through function space projections, the performance of the various formulations of the three noded bar element under distortion in the position of the internal node from its regular central position. Completeness with variational correctness in the metric formulation even under element distortion is an attractive feature that has been utilized here to develop an improved MBAR3 element with appropriate modified shape functions maintaining the C^0 continuity. The projection formula has been invoked to estimate the variationally correct finite element results (using MBAR3 elements) for a simple specimen bar problem subjected to various kinds of loading, through orthogonal projections of the analytical strain onto the correct strain displacement subspace in the parametric domain. It has been demonstrated that while the conventional isoparametric BAR3 element violates the best-fit rule under element distortion, the modified MBAR3 element with its modified shape functions is immune to such distortions and is variationally correct. A topological criterion for legitimate mapping in elements immune to distortion is also presented.

Isoparametric elements, in general, suffer from the consequences of unavoidable mesh distortions from regular geometry, and hence often yield undesirable errors in the estimation of stresses and strains in the element. For the axially loaded bar problem, the superior performance of the proposed MBAR3 element against the conventional isoparametric element is demonstrated here. The present approach seems to be a promising one that can be employed to improve upon such conventional elements so that they can be immune to distortional effects.

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