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# The numerical solution of dynamic response of SDOF systems using cubic B-spline polynomial functions

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**Abstract.** In this paper, we present a new explicit procedure using periodic cubic B-spline interpolation polynomials to solve linear and nonlinear dynamic equation of motion governing single degree of freedom (SDOF) systems. In the proposed approach, a straightforward formulation was derived from the approximation of displacement with B-spline basis in a fluent manner. In this way, there is no need to use a special pre-starting procedure to commence solving the problem. Actually, this method lies in the case of conditionally stable methods. A simple step-by-step algorithm is implemented and presented to calculate dynamic response of SDOF systems. The validity and effectiveness of the proposed method is demonstrated with four examples. The results were compared with those from the numerical methods such as Duhamel integration, Linear Acceleration and also Exact method. The comparison shows that the proposed method is a fast and simple procedure with trivial computational effort and acceptable accuracy exactly like the Linear Acceleration method. But its power point is that its time consumption is notably less than the Linear Acceleration method especially in the nonlinear analysis.

**Keywords:** B-spline; numerical solution; direct integration; explicit; dynamic equation; nonlinear; stability.

## 1. Introduction

Solving the differential equation of motion governing the SDOF systems is done through various methods. Because most of the time, the loading function is not a specific one, numerical methods are our only option to solve this differential equation. Time integration methods are the most suitable methods for nonlinear problems in structural dynamics and for dynamic analysis of very large structures in which the equilibrium equations are solved at discrete times. In general, they involve a solution of the complete set of equilibrium equations at each time increment. In the case of nonlinear analysis, it may be necessary to reform the stiffness values for the complete structural system for each time step. Also, iteration may be required within each time increment. It can take a significant amount of time to solve structural systems with just a few hundred degrees-of-freedom. Thus, today, we are mostly interested in those numerical methods which not only provide acceptable accuracy and stability, but also solve a problem in the least possible time. The simulation of a complex dynamic system requires a high efficient algorithm of time integration, with high accuracy

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and limited amount of computation. This requirement has attracted many researchers (Belytschko *et al.* 1983, Ebeling *et al.* 1997).

Generally, there are two basic categories of step-by-step integration methods. One is explicit (Chang 1997, 2010, Chung *et al.* 1994, Mullen *et al.* 1983, Rio *et al.* 2005) and the other is implicit (Hilber *et al.* 1977, Houbolt 1950, Newmark 1959, Park 1975, Wilson *et al.* 1973). A method is explicit if the equation of motion of the current time step is not used in determining the current step displacement while it is implicit if that is involved (Bathe 1996). The most significant advantage of explicit methods is that it is unnecessary to solve a system of equations or to involve any iterative procedure in each time step, thus it usually requires considerably less computational effort per time step and less storage is required than for implicit methods (Dokainish *et al*. 1989). This also leads to an easy implementation of explicit methods. Almost all of the explicit time integration schemes are conditionally stable and for a few of them with unconditional stability, the consistency is conditional. It is the major disadvantage of explicit methods. Consequently, a very small time step and thus a very large number of time steps may be needed in a time history analysis. This may not be a disadvantage since the use of a very small time step can easily overcome the difficulty caused by the linearization errors for nonlinear systems. In addition, explicit algorithms are very efficient for shock response and wave propagation problems in which the contribution of intermediate and high frequency structural modes to the response is important. The Newmark explicit method, Central difference method and explicit Runge-Kutta method are the very commonly used explicit methods. Meanwhile, since the implementation of an explicit method is much simpler than an implicit method for performing pseudo-dynamic tests (Chang 1997, 2002), some explicit methods have been developed for pseudo-dynamic tests.

In this study, a new explicit method is proposed using a family of piecewise polynomial approximations called B-spline. Piecewise polynomial approximations are fundamental to geometric modeling, computer graphics, data fitting, and finite element methods. For most applications, B-splines have become a widely accepted standard because of their flexibility and computational efficiency. In the recent years, Caglar *et al.* (2006a, b, 2009) used B-splines with various degrees to solve several mathematical boundary values problems (BVPs). In the field of dynamic problems, Liu (2002) employed piecewise Birkhoff interpolation polynomials for the solution of dynamic response of MDOF systems. Liu (2001) extended a procedure to smoothen varying loading cases by using the piecewise second or third degree Lagrange polynomial for linear SDOF systems. Inoue and Sueoka (2002) presented a step-by-step integration scheme by utilizing the cardinal B-spline. This very method just organized conventional implicit methods such as Newmark-$\beta$ method and Wilson-$\theta$ method, etc., to provide a simple computation procedure so that the step-by-step integration can be carried out efficiently.

Regarding the potential ability of B-spline basis functions in interpolation and approximation and also their order (degree) elevation property, using this function in the field of numerical calculation is inevitable. We can introduce a B-spline curve as a linear combination of Basis functions by specifying its order, control points and knot vector (Rogers 2001). As the theory of B-spline is a very active field of approximation and solving differential equations, we tried to use this function as a basis. Here, we describe the construction of cubic B-spline bases to approximate the response of SDOF equation of motion as an initial value problem. Because the SDOF equation of motion is a second order differential equation, we decided to use cubic B-spline interpolation function. Here, cubic refers to the order of B-spline. In this way, using periodic cubic B-spline function results in a new explicit procedure to find linear and nonlinear dynamic response of SDOF system. The use of single frequency facilitated the evaluation of the accuracy of the computed responses solved at regular time increments during ground shaking. The consequence of implementing this approach is a single-step explicit and straight forward formulation in a fluent manner. And as a result, the displacement, velocity and acceleration values are submitted independently. Hence, decreasing the solution formulation, reserving time during the analysis.

In section 4, we have investigated the numerical stability of this proposed method. This method is, actually, one of the conditionally stable methods where its stability condition is the same as Linear Acceleration method. The validity of the proposed method is illustrated with four examples in section 5 where structural dynamic response for each one was characterized by the computed time-history response of displacement, velocity, and acceleration. The results of these examples show that the accuracy of this proposed step-by-step method depends on the time step value and frequency characteristics of the ground motion or the load applied to the system. Actually, the goal of this study is to introduce a new methodology to be a base for further studies on B-splines in order to make use of them in dynamic analysis.

## 2. Overview of the B-splines

### 2.1 Piecewise polynomial

Generally, consider the piecewise polynomial, let $[a, b] \subset \mathbb{R}$ be a finite interval and we introduce a set of partition $\Omega = \{t_0, t_1, t_2, \ldots, t_n\}$ of $[a, b]$ as knot vector, where $t_i$ is called knot of the partition. The set of piecewise polynomials of degree $d$ (or order $k$) defined on a partition $\Omega$ is denoted by $P_d(\Omega_n)$ in each subinterval $I_i = [t_{i-1}, t_i]$ is a $d^{\text{th}}$ degree polynomial. Specifically, consider the type of basis, B-spline, for our spline interpolation function, for which we only use the equidistant partition. Moreover, we extend the set of knots by taking

$$h = \frac{b-a}{n}, \quad t_0 = a, \quad t_i = t_0 + ih, \quad \text{where} \quad i = \pm 1, \pm 2, \pm 3, \ldots$$

A detailed description of B-spline functions generated by subdivisions can be found in De Boor (1978).

### 2.2 Implicit definition

Let $\{\Omega_n\}$ be a partition of $[a, b] \subset \mathbb{R}$. A B-spline of degree $d$ is a spline from $S_d(\Omega_n)$ with minimal support and the partition of unity holding. Let $B_{i,d}(t)$ denote the B-spline of degree $d$, where $i \in \mathbb{Z}$ and then we have the following properties:

- Supp $(B_{i,d}) = [t_i, t_{i+d+1}]$ *or* $[t_i, t_{i+k}]$,
- $B_{i,d}(t) \geq 0 \ \forall \ t \in R$ (non-negativity),
- $\sum_{i=-\infty}^{\infty} B_{i,d}(t) = 1 \ \forall \ t \in R$ (partition of unity).

### 2.3 Explicit definition

Let $\{\Omega_n\}$ be a partition of $[a, b] \subset R$, the zero degree B-spline are defined as follows

$$B_{i,0}(t) = \begin{cases} 1, & t_i \le t \le t_{i+1} \\ 0, & otherwise \end{cases} \tag{1}$$

and for degree $d$, it is defined as recursive formula in the following form

$$B_{i,d}(t) = \left(\frac{t - t_i}{t_{d+i} - t_i}\right) B_{i,d-1}(t) + \left(\frac{t_{d+i+1} - t}{t_{d+i+1} - t_{i+1}}\right) B_{i+1,d-1}(t) \tag{2}$$

In this formula $B(t)$ is a polynomial of degree $d$ on each interval $t_i \le t \le t_{i+1}$ as $B(t)$ and its derivatives of order 1, 2,…, $d$–1 are all continuous over the entire domain. Eq. (2) clearly shows that the choice of knot vector has a significant influence on the B-spline curve. The only requirement for a knot vector is that, it satisfies the relation $t_i \le t_{i+1}$ i.e., it is a monotonically increasing series of real numbers (Rogers 2001).

Fundamentally, based on the arrangement of the knot vectors, B-spline functions are categorized in periodic and open types where each type can have a uniform or non-uniform flavor (Rogers 2001). In a uniform knot vector, knot values are evenly spaced. In this work, as it was mentioned in section 2.1, we preferred to make use of periodic and uniform types. Thus, for a specified order of B-spline, periodic uniform knot vectors yield periodic uniform basis functions for which

$$B_{i,d}(t) = B_{i-1,d}(t-1) = B_{i+1,d}(t+1) \tag{3}$$

Furthermore, in periodic type each basis function is simply a translation of the other one and the range of nonzero function values spreads with increasing order. Thus, the basis function provides support on the interval $t_i$ to $t_{i+d+1}$.

For a uniform knot vector beginning at 0 with integer spacing, usable parameter range is $t_d \le t \le t_{n-d}$. Thus, for the cubic B-spline $(d = 3)$ which we have used in this work, in order to start from $t_0 = 0$, as shown in Fig. 1, we have to consider the Basis functions from $B_{-3,3}$.

### 2.4 Cubic B-spline interpolation

The cubic B-spline interpolation is a linear combination of the cubic B-spline basis as follows

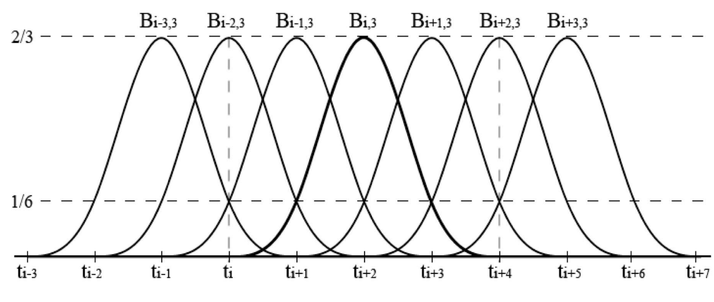



Fig. 1 Periodic cubic B-splines (usable range is from $t_i$ to $t_{i+4}$)

$$S_d(t) = \sum_{i=-3}^{n-1} C_i B_{i,3}(t) \tag{4}$$

where $C_i$ (control points) are unknown real coefficients and $B_{i,3}(t)$ are third degree (cubic) B-spline functions (De Boor 1978, Yingkang *et al.* 1995). A short description of cubic B-spline functions can be found in (Caglar *et al.* 2006a, b, 2009).

Here, we apply the recursive formula (2) to get to the third degree B-splines which are defined as

$$B_{i,3}(t) = \frac{1}{6h^3}\begin{cases}(t-t_i)^3, & t_i \le t \le t_{i+1}\\ h^3+3h^2(t-t_{i+1})+3h(t-t_{i+1})^2-3(t-t_{i+1})^3, & t_{i+1}\le t\le t_{i+2}\\ h^3+3h^2(t_{i+3}-t)+3h(t_{i+3}-t)^2-3(t_{i+3}-t)^3, & t_{i+2}\le t\le t_{i+3}\\ (t_{i+4}-t)^3, & t_{i+3}\le t\le t_{i+4}\end{cases} \tag{5}$$

where, $h = (b-a)/n$ and $t_i = t_0 + ih$.

In this paper, as $t_i$ refers to time values and generally starts from zero ($t_0 = 0$) and $h$ is time interval or $\Delta t$, so $B_{0,3}$ can be obtained as follows

$$B_{0,3}(t) = \frac{1}{6h^3}\begin{cases}t^3, & 0\le t\le h\\ -3t^3+12ht^2-12h^2t+4h^3, & h\le t\le 2h\\ 3t^3-24ht^2+60h^2t-44h^3, & 2h\le t\le 3h\\ -t^3+12ht^2-48h^2t+64h^3, & 3h\le t\le 4h\end{cases} \tag{6}$$

as $B_{i,3}(t) = B_{0,3}(t-ih)$, $i = -3,-2,-1,\ldots$.

Next we will have its first and second derivatives as $B'_{0,3}$ and $B''_{0,3}$, respectively. According to the previous discussions, we can simply obtain other $B_{i,3}$ s and their derivates just by transference.

## 3. Implementation of cubic B-spline to solve the differential equation of motion

A nonlinear differential equation of motion can be expressed as

$$m\ddot{u} + c\dot{u} + f_s(u,\dot{u}) = F(t) \tag{7}$$

subject to the initial conditions

$$u(0) = u_0,\ \dot{u}(0) = \dot{u}_0 \tag{8}$$

The applied force, $F(t)$, is given by a set of discrete values $F_i = F(t_i)$. The time interval $\Delta t_i = t_{i+1} - t_i$ is usually taken to be constant, although it is not necessary. The response is determined at the discrete time instant $t_i$, denoted as time *i*. The displacement, velocity and acceleration of SDOF system are $u_i, \dot{u}_i$ and $\ddot{u}_i$, respectively. These values, assumed to be known, satisfy Eq. (7) at time *i*

$$m\ddot{u}_i + c\dot{u}_i + (f_s)_i = F_i \tag{9}$$

where $(f_s)_i$ is the resisting force at time *i* and for a linear elastic system is $ku_i$ but would depend on

Table 1 Values of $B_i, B_i'$ and $B_i''$

| | $t_i$ | $t_{i+1}$ | $t_{i+2}$ | $t_{i+3}$ | $t_{i+4}$ |
|---|---|---|---|---|---|
| $B_i$ | 0 | 1/6 | 2/3 | 1/6 | 0 |
| $B_i'$ | 0 | $1/2\Delta t$ | 0 | $-1/2\Delta t$ | 0 |
| $B_i''$ | 0 | $1/\Delta t^2$ | $-2/\Delta t^2$ | $1/\Delta t^2$ | 0 |

the prior history of displacement and velocity in an inelastic system. For a linear system the equation of motion could be written as

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = F(t) \tag{10}$$

where $m$, $c$ and $k$ are the mass, damping characteristic and stiffness of the system, respectively (Chopra 1995).

The third-degree B-spline is used to construct a numerical solution for Eq. (10). Here we use Eq. (6), in which $h$ refers to $\Delta t$ (time interval). To solve the second-order boundary value problem, evaluation of $B_i, B_i'$ and at the nodal points (knots) is required. Their coefficients are summarized in the Table 1.

Let

$$u(t) = \sum_{i=-3}^{n-1} C_i B_{i,3}(t) \tag{11}$$

be the approximate solution of Eq. (10), therefore

$$\dot{u}(t) = \sum_{i=-3}^{n-1} C_i B_{i,3}'(t) \tag{12}$$

and

$$\ddot{u}(t) = \sum_{i=-3}^{n-1} C_i B_{i,3}''(t) \tag{13}$$

describe the approximate functions of the velocity and acceleration, respectively, where $C_i$s are unknown real coefficients (control points) and $B_i(t)$s are cubic B-spline basis functions. Let $t_0, t_1, t_2, \ldots, t_n$ be $n+1$ grid points in the time interval $[0, t_d]$, so that

$$t_i = i\Delta t, \quad i = 0, 1, 2, \ldots, n; \quad t_0 = 0, \quad t_n = t_d, \quad \Delta t = t_d/n$$

We can rewrite Eq. (10) as

$$\ddot{u}(t) + 2\xi\omega\dot{u}(t) + \omega^2 u(t) = \frac{F(t)}{m} \tag{14}$$

where $\xi$ is the damping ratio and $\omega$ is the natural frequency of the system.

It is required that the approximate solution (11) and its derivates (12, 13) satisfy the differential Eq. (14) at each point $t = t_j$.

Substituting the terms (11), (12) and (13) in Eq. (14), gives

$$\forall t_j \in [0, t_d], \quad \sum_{i=-3}^{n-1} C_i B''_{i,3}(t_j) + 2\xi\omega \sum_{i=-3}^{n-1} C_i B'_{i,3}(t_j) + \omega^2 \sum_{i=-3}^{n-1} C_i B_{i,3}(t_j) = \frac{F(t_j)}{m} \tag{15}$$

Factoring $C_i$ unknown coefficient, Eq. (15) can be written in an incremental form as

$$\sum_{i=-3}^{n-1} (B''_{i,3}(t_j) + 2\xi\omega B'_{i,3}(t_j) + \omega^2 B_{i,3}(t_j))C_i = \frac{F(t_j)}{m} \tag{16}$$

If we use Eq. (16) at any time instant $t_j$ (knots), $j = 0, 1, \ldots, n$, we would have $n+3$ unknown coefficients ($C_i$s) and $n+3$ equations while $n+1$ equations are located on force points and two of them are related to initial conditions.

Because in each $t_j$, only three Basis functions ($B_{j-3}, B_{j-2}$ and $B_{j-1}$) and their derivatives have nonzero values and the others are zero (see Fig. 1), we can develop (16) in each $t_j$ as follows

$$\begin{aligned} \frac{F(t_j)}{m} = {} & \{B''_{j-3,3}(t_j) + 2\xi\omega B'_{j-3,3}(t_j) + \omega^2 B_{j-3,3}(t_j)\}C_{j-3} + \\ & \{B''_{j-2,3}(t_j) + 2\xi\omega B'_{j-2,3}(t_j) + \omega^2 B_{j-2,3}(t_j)\}C_{j-2} + \\ & \{B''_{j-1,3}(t_j) + 2\xi\omega B'_{j-1,3}(t_j) + \omega^2 B_{j-1,3}(t_j)\}C_{j-1} \end{aligned} \tag{17}$$

According to the values of the spline function at the knots $(t_i)_{i=0}^{n}$ which have been determined in Table 1, (17) can be summarized as

$$\frac{F(t_j)}{m} = \alpha C_{j-3} + \beta C_{j-2} + \gamma C_{j-1}, \qquad j = 0, 1, 2, \ldots, n \tag{18}$$

where $\alpha, \beta$ and $\gamma$ are constant values as follows

$$\alpha = \left(\frac{1}{\Delta t^2} - \frac{\xi\omega}{\Delta t} + \frac{\omega^2}{6}\right), \quad \beta = \left(\frac{-2}{\Delta t^2} + \frac{2\omega^2}{3}\right), \quad \gamma = \left(\frac{1}{\Delta t^2} + \frac{\xi\omega}{\Delta t} + \frac{\omega^2}{6}\right)$$

Initial condition can be written as

$$u(t_0) = u_0 \Rightarrow \sum_{i=-3}^{n-1} C_i B_{i,3}(t_0) = u_0 \tag{19}$$

$$\dot{u}(t_0) = v_0 \Rightarrow \sum_{i=-3}^{n-1} C_i B'_{i,3}(t_0) = v_0 \tag{20}$$

For $t_0 = 0$, using the first condition results that

$$C_{-3}B_{-3,3}(t_0) + C_{-2}B_{-2,3}(t_0) + C_{-1}B_{-1,3}(t_0) + \underbrace{0 + 0 + \ldots + 0}_{n} = u_0 \Rightarrow$$

$$\frac{1}{6}C_{-3} + \frac{2}{3}C_{-2} + \frac{1}{6}C_{-1} = u_0 \tag{21}$$

and using second condition similarly gives

$$\frac{1}{2\Delta t}(C_{-1} - C_{-3}) = v_0 \tag{22}$$

The spline solution of Eq. (14) with the initial conditions is obtained by solving the following matrix equation. The matrix is constructed using Eqs. (21), (22) and (18). Then as a result, a system of $n+3$ linear equations in the $n+3$ unknowns $C_{-3}, C_{-2}, \ldots, C_{n-1}$ is obtained. This system can be written in the matrix-vector form as follows

$$\{F\} = [\psi]\{C\} \tag{23}$$

where

$$[F] = \frac{1}{m}[mu_0, mv_0, F(t_0), F(t_1), F(t_2), \ldots, F(t_n)]^T$$

$$\{C\} = [C_{-3}, C_{-2}, C_{-1}, C_o, \ldots, C_{n-1}]^T$$

and $\psi$ is a $(n+3)(n+3)$-dimensional matrix given by

$$\begin{bmatrix} 1/6 & 2/3 & 1/6 & 0 & \ldots & \ldots & \ldots & 0 \\ -1/2\Delta t & 0 & 1/2\Delta t & 0 & \ldots & \ldots & \ldots & 0 \\ \alpha & \beta & \gamma & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & \alpha & \beta & \gamma & 0 & \ldots & \ldots & 0 \\ \vdots & 0 & \alpha & \beta & \gamma & 0 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & 0 & \alpha & \beta & \gamma & 0 \\ 0 & 0 & \ldots & \ldots & 0 & \alpha & \beta & \gamma \end{bmatrix}$$

According to the sparse and bandwidth form of the matrix $\psi$, in order to find unknown coefficients ($C_i$s), there is no need to inverse $\psi$ completely.

At the beginning, to find the first three unknown coefficients (i.e., $C_{-3}, C_{-2}$ and $C_{-1}$) we can consider just the first three rows and columns of matrix $\psi$, as

$$\begin{Bmatrix} u_0 \\ v_0 \\ F(t_0)/m \end{Bmatrix} = \begin{bmatrix} 1/6 & 2/3 & 1/6 \\ -1/2\Delta t & 0 & 1/2\Delta t \\ \alpha & \beta & \gamma \end{bmatrix} \begin{Bmatrix} C_{-3} \\ C_{-2} \\ C_{-1} \end{Bmatrix} \tag{24}$$

so, solving the above equations, we get to

$$C_{-3} = \frac{1}{2\gamma - \beta + 2\alpha}\left\{-3\beta u_0 - \Delta t(4\gamma - \beta)v_0 + \frac{2F(t_0)}{m}\right\} \tag{25a}$$

$$C_{-2} = \frac{1}{2\gamma - \beta + 2\alpha}\left\{3(\gamma + \alpha)u_0 + \Delta t(\gamma - \alpha)v_0 - \frac{F(t_0)}{m}\right\} \tag{25b}$$

$$C_{-1} = \frac{1}{2\gamma - \beta + 2\alpha}\left\{-3\beta u_0 + \Delta t(\beta - 4\alpha)v_0 + \frac{2F(t_0)}{m}\right\} \tag{25c}$$

where $2\gamma - \beta + 2\alpha$ is equal to $6/\Delta t^2$. Then, to find the remainder of unknown coefficients (i.e., $C_0, C_1, \ldots, C_{n-1}$) we use Eq. (23) again and develop it from the fourth row to the end. Then as a result, we get to the below recursive relation for finding $C_i$ unknown values for $i = 0, 1, \ldots, n-1$

$$C_i = \frac{1}{\gamma}\left(\frac{F(t_{i+1})}{m} - \alpha C_{i-2} - \beta C_{i-1}\right), \quad i = 0, 1, 2, \ldots, n-1 \tag{26}$$

Here, $C_{i-1}$ and $C_{i-2}$ known values have been used to determine $C_i$ unknown value in an explicit form.

Now, having all unknown coefficients ($C_i$s) in hand, we can determine system displacement, velocity and acceleration values in each time point using the (11) to (13). As these terms show the piecewise polynomial functions of displacement, velocity and acceleration, we can easily find these values not only in each time instant $t_i$ (knots) but also in any other time point.

As only three Basis functions have nonzero values in each $t_i$, we can summarize the Eqs. (11) to (13) as follows

$$u(t_i) = \frac{1}{6}(C_{i-3} + 4C_{i-2} + C_{i-1}) \tag{27a}$$

$$\dot{u}(t_i) = \frac{1}{2\Delta t}(C_{i-1} - C_{i-3}) \tag{27b}$$

$$\ddot{u}(t_i) = \frac{1}{\Delta t^2}(C_{i-3} - 2C_{i-2} - C_{i-1}) \tag{27c}$$

As it is clearly denoted from the above relations, because all the $C_i$s depend on previous values already calculated, this procedure is an explicit one. In order to write a computer code, the complete algorithm used in this proposed method is given in Table 2.

## 4. Numerical stability

Step-by-step numerical integration methods transfer the state at the $t^{th}$ step to the $(t+\Delta t)^{th}$ step and this can be written as follows

$$\{\hat{X}_{t+\Delta t}\} = [A]\{X_t\} + [L]\{\hat{f}_{t+\upsilon}\} \tag{28}$$

where $\{\hat{X}_t\}$ is the displacement, velocity or acceleration is derived from last step. $[A]$ is the amplification matrix which transfers $\{\hat{X}_t\}$ to the next step. $\{\hat{f}_{t+\upsilon}\}$ is the external force at each step and $[L]$ is the load factor vector to relate external force to $\{\hat{X}_{t+\Delta t}\}$. Each quantity in (28) depends on specific integration scheme employed (Bathe 1996, Hughes 1987).

To investigate the stability of the proposed method, at first, it is required to find the amplification matrix. Thus we have to construct a relation in which the values of displacement, velocity and acceleration at the end of each time step are written in terms of those very values at the beginning. Therefore, if we solve the Eqs. (27) in order to get to the $C_{i-1}, C_{i-2}$ and $C_{i-3}$, we will have

Table 2 Step-by-step numerical solution of equation of motion using cubic B-spline

***A. Initial calculation:***

1- Determine stiffness $k$, mass $m$, and damping ratio $\xi$ of the system.
2- Specify the force value applied to the system in each time instant.
3- Determine initial value of displacement $u_0$ and velocity $v_0$.
4- Select appropriate time step ($\Delta t < \Delta t_{critical}$) and calculate constant parameters $\alpha$, $\beta$ and $\gamma$ as

$$\alpha = \left(\frac{1}{\Delta t^2} - \frac{\xi\omega}{\Delta t} + \frac{\omega^2}{6}\right)$$

$$\beta = \left(\frac{-2}{\Delta t^2} + \frac{2\omega^2}{3}\right)$$

$$\gamma = \left(\frac{1}{\Delta t^2} + \frac{\xi\omega}{\Delta t} + \frac{\omega^2}{6}\right) \qquad \text{where } \omega = \sqrt{\frac{k}{m}}$$

5- Using the below terms determine the values of three unknown coefficient ($C_{-3}$, $C_{-2}$ and $C_{-1}$);

$$C_{-3} = \frac{\Delta t^2}{6}\left\{-3\beta u_0 - \Delta t(4\gamma - \beta)v_0 + \frac{2F(t_0)}{m}\right\}$$

$$C_{-2} = \frac{\Delta t^2}{6}\left\{3(\gamma + \alpha)u_0 + \Delta t(\gamma - \alpha)v_0 - \frac{F(t_0)}{m}\right\}$$

$$C_{-1} = \frac{\Delta t^2}{6}\left\{-3\beta u_0 + \Delta t(\beta - 4\alpha)v_0 + \frac{2F(t_0)}{m}\right\}$$

***B. For each time step (i = 0, 1, ..., n):***

1- Calculate displacement, velocity and acceleration simultaneously by

$$u(t_i) = \frac{1}{6}\left(C_{i-3} + 4C_{i-2} + C_{i-1}\right)$$

$$\dot{u}(t_i) = \frac{1}{2\Delta t}\left(C_{i-1} - C_{i-3}\right)$$

$$\ddot{u}(t_i) = \frac{1}{\Delta t^2}\left(C_{i-3} - 2C_{i-2} + C_{i-1}\right)$$

2- **Just for nonlinear analysis**; update the values of stiffness, $k$ and mass, $m$ and as its result $\alpha$, $\beta$ and $\gamma$ will be updated according to the values of $u_i$ and $\dot{u}_i$.
3- Calculate unknown coefficient $C_i$ from $i = 0$ to $(n-1)$ by

$$C_i = \frac{1}{\gamma}\left(\frac{F(t_{i+1})}{m} - \alpha C_{i-2} - \beta C_{i-1}\right)$$

---

$$C_{i-1} = u_i + \Delta t\dot{u}_i + \frac{\Delta t^2}{3}\ddot{u}_i \tag{29a}$$

$$C_{i-2} = u_i - \frac{\Delta t^2}{6}\ddot{u}_i \tag{29b}$$

$$C_{i-3} = u_i - \Delta t\dot{u}_i + \frac{\Delta t^2}{3}\ddot{u}_i \tag{29c}$$

Setting the Eq. (29a) at the current time ($t$) equal to the Eq. (29b) at the next time $(t + \Delta t)$, we will get to

$$C_{i-1} = u_i + \Delta t\dot{u}_i + \frac{\Delta t^2}{3}\ddot{u}_i = u_{i+1} - \frac{\Delta t^2}{6}\ddot{u}_{i+1} \tag{30}$$

Then, if we arrange the above equation in terms of $u_{i+1}$, it can be expressed as

$$u_{i+1} = u_i + \Delta t\dot{u}_i + \frac{\Delta t^2}{6}(\ddot{u}_{i+1} + 2\ddot{u}_i) \tag{31}$$

Similarly, if we do this process for the Eq. (29b) at the current time ($t$) and the Eq. (29c) at the next time $(t+\Delta t)$, we will get to an equation. Then, if we use (31) instead of $u_{i+1}$ in this equation, and then arrange the outcome in terms of $\dot{u}_{i+1}$, we get to

$$\dot{u}_{i+1} = \dot{u}_i + \frac{\Delta t}{2}(\ddot{u}_{i+1} + \ddot{u}_i) \tag{32}$$

Now, having the Eqs. (31) and (32) in hand, it is possible to make the amplification matrix. The amplification matrix is obtained by solving the equation of SDOF system (14) in $t+\Delta t$ time instant as follows

$$\ddot{u}_{t+\Delta t} + 2\xi\omega\dot{u}_{t+\Delta t} + \omega^2 u_{t+\Delta t} = f_{t+\Delta t} \tag{33}$$

where $f_{t+\Delta t} = F_{t+\Delta t}/m$. Substituting Eqs. (31) and (32) into (33), an equation is obtained with $\ddot{u}_{t+\Delta t}$ as the only unknown. Solving it for $\ddot{u}_{t+\Delta t}$ and substituting into Eqs. (31) and (32), the following relationship of the form (28) is obtained as

$$\begin{Bmatrix} \ddot{u}_{t+\Delta t} \\ \dot{u}_{t+\Delta t} \\ u_{t+\Delta t} \end{Bmatrix} = [A]\begin{Bmatrix} \ddot{u}_t \\ \dot{u}_t \\ u_t \end{Bmatrix} + \{L\}f_{t+\Delta t} \tag{34}$$

where

$$A = \begin{bmatrix} -\kappa - \frac{\mu}{3} & \frac{-1}{\Delta t}(2\kappa - \mu) & \frac{-\mu}{\Delta t^2} \\ \Delta t\left(\frac{1}{2} - \frac{\kappa}{2} - \frac{\mu}{6}\right) & \left(1 - \kappa - \frac{\mu}{2}\right) & \frac{-\mu}{2\Delta t} \\ \frac{\Delta t^2}{3}\left(1 - \frac{\kappa}{2} - \frac{\mu}{6}\right) & \Delta t\left(1 - \frac{\kappa}{3} - \frac{\mu}{6}\right) & \left(1 - \frac{\mu}{6}\right) \end{bmatrix} \quad \text{and} \quad L = \begin{Bmatrix} \frac{\mu}{\omega^2\Delta t^2} \\ \frac{\mu}{2\omega^2\Delta t} \\ \frac{\mu}{6\omega^2} \end{Bmatrix}$$

$$\mu = \left(\frac{1}{\omega^2\Delta t^2} + \frac{1}{6} + \frac{\xi}{\omega\Delta t}\right)^{-1}; \quad \kappa = \frac{\xi\mu}{\omega\Delta t}$$

According to the fact that the stability of an integration method is determined by examining the behavior of the numerical solution for arbitrary initial conditions, we consider the integration of Eq. (33) when no load is satisfied; i.e., $f = 0$ (Bathe 1996).

Stability analysis can be performed by solving eigenproblem of amplification matrix. The eigenvalues and eigenvectors of $A$ are calculated using $([A] - \lambda[I])\{\Phi\} = \{0\}$ or $|A - \lambda I| = 0$. It is now possible to write $A$ in terms of its eigenvalues and eigenvectors $[A] = [\Phi][\lambda][\Phi]^{-1}$. Here, $[\Phi]$ consists of the eigenvectors of $A$, and $[\lambda]$ is a diagonal matrix consisting the eigenvalues of $A$. Now, to have a stable solution, the norm of the elements of $[\lambda]$ should not be more than unity

$$\rho(A) = \max(\|\lambda_1\|, \|\lambda_2\|, \|\lambda_3\|) \leq 1$$

In this equation, $\rho(A)$ is the spectral radius which is a function of time step length, $\Delta t$. As the spectral radius slightly changes with the variation of damping ratio, here, we assume $\xi = 0$ in amplification matrix and it results in $0.55T_0$ for the maximum value of $\Delta t$ in the case of $\rho(A) = 1$. Here, $T_0$ is the period of the system.

Thus, the condition of stability is $\Delta t_{critical} \leq 0.55T_0$ in this proposed method which is exactly the same as condition of stability in Linear Acceleration method (Wilson-$\theta$ method with $\theta = 1$) (Wilson *et al.* 1973). Of course, it can be resulted from the sameness of the amplification matrix for both methods. Also the other reason for this sameness is because the relations (31) and (32) are the same as those in the Linear Acceleration method.

## 5. Numerical examples

In this section, the validity and effectiveness of the proposed method is demonstrated with four examples, the first and the second examples have linear behavior and the third and fourth have nonlinear behavior.

Example 1: *A portal frame with linear behavior under a harmonic loading in the ceiling level;*

Fig. 2 shows a SDOF portal frame with linear behavior under a sinusoidal load. Frame elevation is 4 m and the columns are I-shaped (IPB300) with the moment of inertial equal to 25170 $cm^4$. Here, Young's modulus of material is assumed to be $2.1 \times 10^6$ $kg/cm^2$. The loading function applied to this frame is a harmonic single frequency load equal to ($5\sin(3t)$) in tons. The mass of the frame is 5000 kg and damping ratio is assumed to be *5* percent ($\xi = 0.05$). In this example, $\Delta T = 0.1$ sec has been selected as time increment.

This example is analyzed by four methods including the Exact method, Duhamel integration method, Linear Acceleration method and Cubic B-spline method (proposed method). The results including the values of displacement, velocity and acceleration for all these four methods have been plotted as three time-history graphs in a time interval between 0 to 10 seconds as shown in Figs. 3 to 5.

As the graphs show, regarding the selected time increments ($\Delta T = 0.1$) and frequency of the

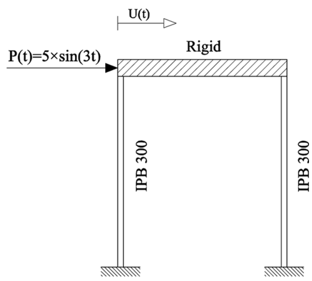



Fig. 2 A portal frame (Example 1)

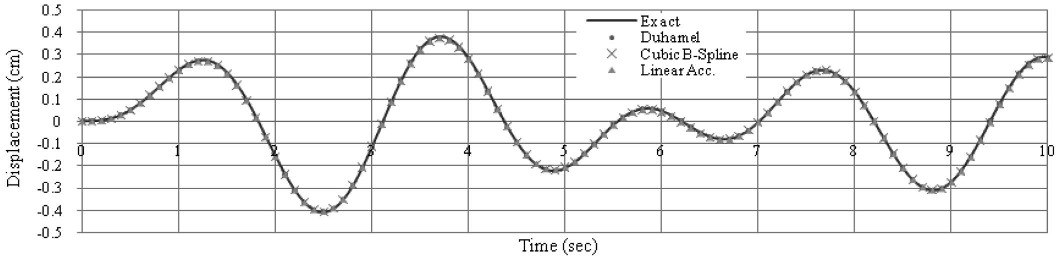



Fig. 3 Displacement time histories (cm)

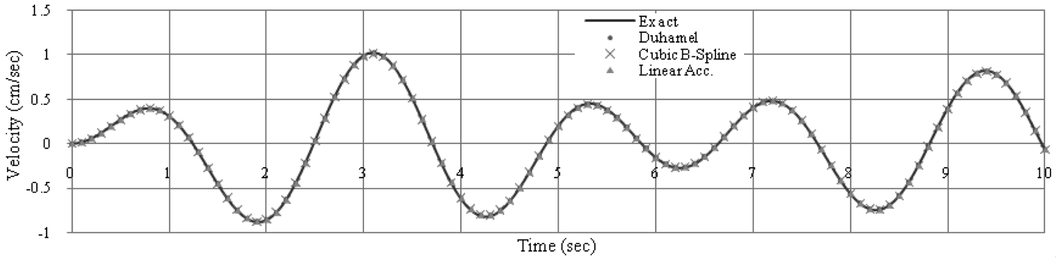



Fig. 4 Velocity time histories (cm/sec)

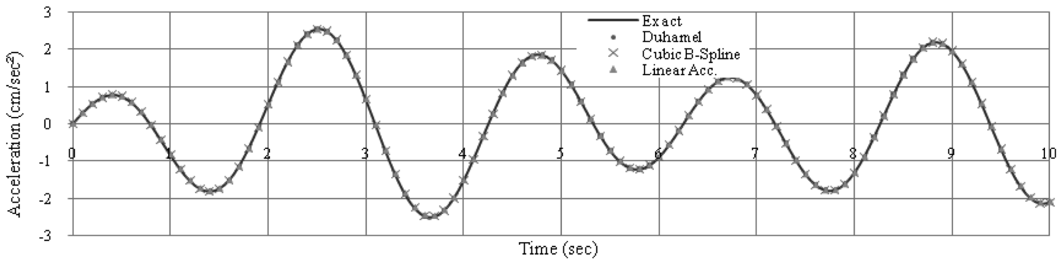



Fig. 5 Acceleration time histories (cm/sec$^2$)

Table 3 The peak time values of displacement, velocity and acceleration for all four methods

| Displacement | | | | | Velocity | | | | | Acceleration | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Time | Exact | Duhamel | Cubic B-spline | Linear Acc. | Time | Exact | Duhamel | Cubic B-spline | Linear Acc. | Time | Exact | Duhamel | Cubic B-spline | Linear Acc. |
| 1.2 | 0.27291 | 0.27086 | 0.27049 | 0.27049 | 0.8 | 0.39949 | 0.39649 | 0.39579 | 0.39579 | 0.4 | 0.78123 | 0.78237 | 0.78298 | 0.78298 |
| 2.5 | -0.41063 | -0.40756 | -0.40657 | -0.40657 | 1.9 | -0.88246 | -0.87585 | -0.87341 | -0.87341 | 1.4 | -1.82166 | -1.81460 | -1.81520 | -1.81520 |
| 3.7 | 0.38044 | 0.37759 | 0.37517 | 0.37517 | 3.1 | 1.02661 | 1.01890 | 1.01400 | 1.01400 | 2.5 | 2.55866 | 2.54650 | 2.54360 | 2.54360 |
| 4.9 | -0.22390 | -0.22223 | -0.21867 | -0.21867 | 4.3 | -0.81585 | -0.80973 | -0.80234 | -0.80234 | 3.7 | -2.50608 | -2.49480 | -2.48640 | -2.48640 |
| 5.9 | 0.05581 | 0.05540 | 0.05225 | 0.05225 | 5.3 | 0.45078 | 0.44740 | 0.43874 | 0.43874 | 4.8 | 1.86267 | 1.85600 | 1.84090 | 1.84090 |
| 6.6 | -0.08120 | -0.08060 | -0.08224 | -0.08224 | 7.2 | 0.48181 | 0.47820 | 0.48156 | 0.48156 | 6.7 | 1.25151 | 1.24930 | 1.25290 | 1.25290 |
| 7.7 | 0.22886 | 0.22715 | 0.22941 | 0.22941 | 8.3 | -0.74368 | -0.73811 | -0.74035 | -0.74035 | 7.7 | -1.78932 | -1.78260 | -1.79200 | -1.79200 |
| 8.8 | -0.31222 | -0.30988 | -0.30995 | -0.30995 | 9.4 | 0.82329 | 0.81712 | 0.81360 | 0.81360 | 8.7 | 2.05587 | 2.04660 | 2.04550 | 2.04550 |

applied load ($\omega = 3$) for this problem, the results from these four methods are nearly coincident. But for sake of a more accurate investigation, the resulted values of these methods in peak points are given in Table 3.

It is clear that, first of all, the proposed method provides us with the results having a very low rate of error compared to Duhamel and even the Exact method. Secondly, the results from the proposed method are exactly the same as the results from Linear Acceleration method. Selecting a cubic function for describing displacement brought a linear function for describing acceleration and therefore, the sameness of the results for these two methods is justifiable. Meanwhile, for solving this problem, the proposed method consumed a time equal to 66% of the needed time by Duhamel method and about 92% of the required time by Linear acceleration method.

Example 2: *A water reservoir with linear behavior under harmonic ground acceleration;*

Fig. 6 shows a SDOF water reservoir with linear behavior under a twenty cycles of sinusoidal acceleration with peak ground acceleration of 1 g and $T_g = 0.05$ sec (see Fig. 7). The period of this system is equal to 0.25 sec and damping ratio is assumed to be *5* percent ($\xi = 0.05$). In this example, $\Delta t$ = 0.01 sec has been selected as time increment.

This example has been also analyzed by the same methods which were used in the previous example. The results including the values of displacement, velocity and acceleration for all aforesaid methods have been plotted as three time-history graphs as shown in Figs. 8 to 10.

As the graphs show, regarding the selected time increment ($\Delta t = 0.01$) and frequency of the applied load ($\omega = 126$) for this problem, the result of Exact method for relative displacement and velocity has shown some differences with the others just at the peak and valley zones. They are nearly coincident in the other zones. In order to have an accurate investigation, resulted values of these methods in some peak and valley points have been given in Table 4.

In this example, as the applied load has high frequency content, the result from numerical methods (even Duhamel method) in peak points is, somehow, different from those related to the Exact method. But, like the results from the previous example, these differences can be neglected. Furthermore, outcomes from the proposed method are exactly the same as those from Linear Acceleration method. From the point of view of time consumption, the results of the previous

$X_{ground}(t) = A_{max} \times \sin(2\pi T_g t)$

And $A_{max} = 1g$

$g = 981$ cm/sec$^2$

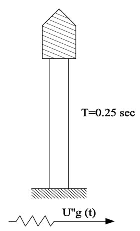



Fig. 6 A water reservoir

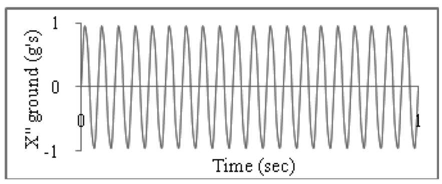



Fig. 7 Ground acceleration time-history

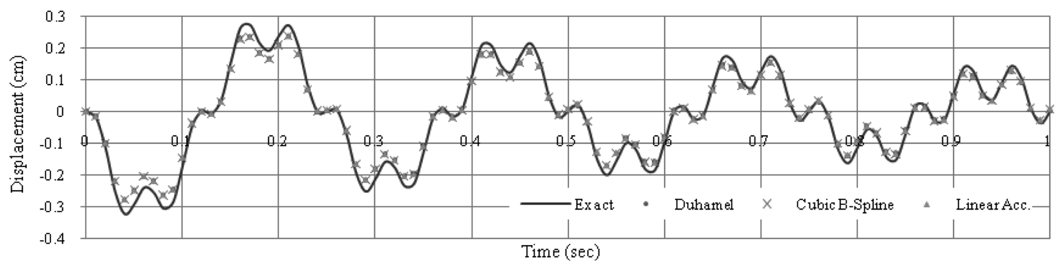



Fig. 8 Displacement time histories (cm)

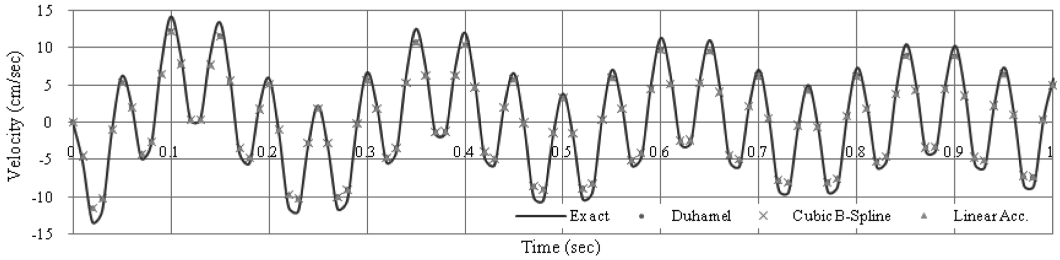



Fig. 9 Velocity time histories (cm/sec)

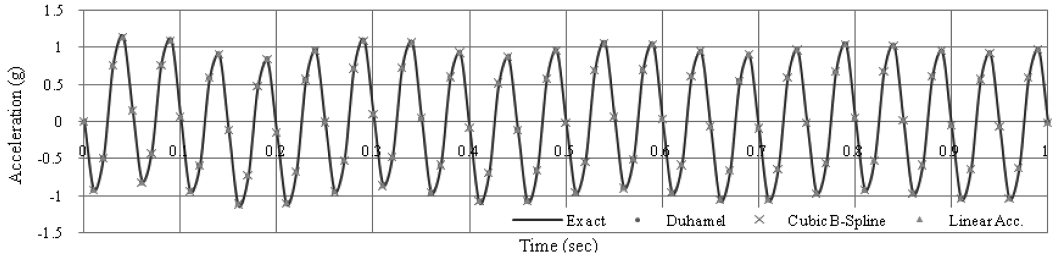



Fig. 10 Acceleration time histories (g)

Table 4 The peak and valley time values of displacement, velocity and acceleration for all four methods

| Displacement | | | | | Velocity | | | | | Acceleration | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Time | Exact | Duhamel | Cubic B-spline | Linear Acc. | Time | Exact | Duhamel | Cubic B-spline | Linear Acc. | Time | Exact | Duhamel | Cubic B-spline | Linear Acc. |
| 0.04 | -0.32164 | -0.27915 | -0.27799 | -0.27799 | 0.02 | -13.41709 | -11.61000 | -11.54400 | -11.54400 | 0.01 | -0.92533 | -0.92935 | -0.92959 | -0.92959 |
| 0.06 | -0.23788 | -0.20494 | -0.20466 | -0.20466 | 0.05 | 6.15160 | 5.32260 | 5.26660 | 5.26660 | 0.04 | 1.16104 | 1.13333 | 1.13262 | 1.13262 |
| 0.08 | -0.30394 | -0.26351 | -0.26344 | -0.26344 | 0.15 | 13.37989 | 11.57600 | 11.56100 | 11.56100 | 0.06 | -0.80371 | -0.82413 | -0.82426 | -0.82426 |
| 0.17 | 0.27349 | 0.23715 | 0.23589 | 0.23589 | 0.18 | -5.57061 | -4.81850 | -4.73470 | -4.73470 | 0.41 | -1.09979 | -1.08033 | -1.07870 | -1.07870 |
| 0.19 | 0.19352 | 0.16657 | 0.16686 | 0.16686 | 0.2 | 5.88580 | 5.09210 | 5.13100 | 5.13100 | 0.44 | 0.86498 | 0.87714 | 0.87637 | 0.87637 |
| 0.21 | 0.27230 | 0.23646 | 0.23775 | 0.23775 | 0.3 | 6.64217 | 5.74690 | 5.61720 | 5.61720 | 0.49 | 0.96532 | 0.96396 | 0.96073 | 0.96073 |
| 0.91 | 0.13983 | 0.12186 | 0.11855 | 0.11855 | 0.4 | 11.98822 | 10.37200 | 10.40300 | 10.40300 | 0.81 | -0.92161 | -0.92614 | -0.92549 | -0.92549 |
| 0.94 | 0.03823 | 0.03221 | 0.03327 | 0.03327 | 0.82 | -6.05032 | -5.23590 | -5.32250 | -5.32250 | 0.84 | 1.03571 | 1.02487 | 1.02752 | 1.02752 |
| 0.96 | 0.14510 | 0.12641 | 0.12938 | 0.12938 | 0.95 | 7.30910 | 6.32360 | 6.40870 | 6.40870 | 0.91 | -1.05122 | -1.03823 | -1.03649 | -1.03649 |

example are present here.

Example 3: *A SDOF system with nonlinear behavior under an earthquake ground shaking;*

A vertical cantilever tower with nonlinear behavior that supports a lumped weight at the top is considered; assuming that the tower mass is equal to 5000 kg, $\xi = 3\%$, and the force-deformation relation (stiffness) is $4000/\sqrt{|u|+1}$ kg/cm$^2$. The system is under the acceleration time-history of *El-Centro* ground shaking with the peak ground acceleration (*PGA*) equal to 0.32 g.

Figs. 11 to 13 show the analysis results including relative displacement, velocity and acceleration of the aforesaid system for both Linear Acceleration and B-spline (proposed) methods during the first 15 seconds of the earthquake.

In this example, because of the nonlinear behavior, the proposed method has been just compared to Linear Acceleration method. And, as the above graphs show, the outcomes from the proposed

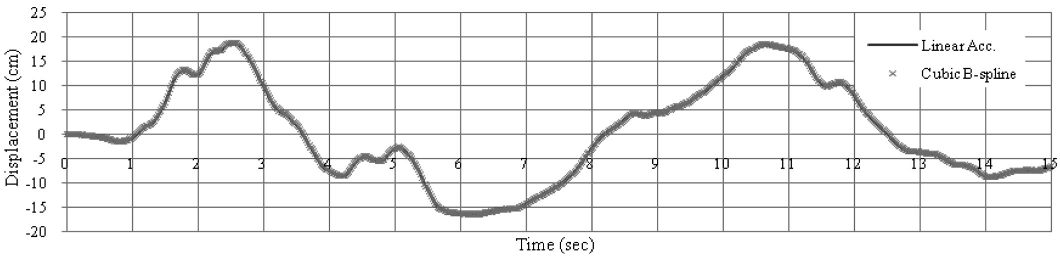



Fig. 11 Displacement time histories (cm)

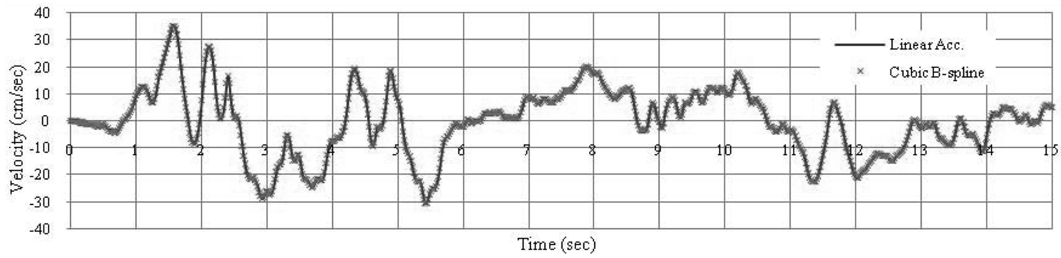



Fig. 12 Velocity time histories (cm/sec)

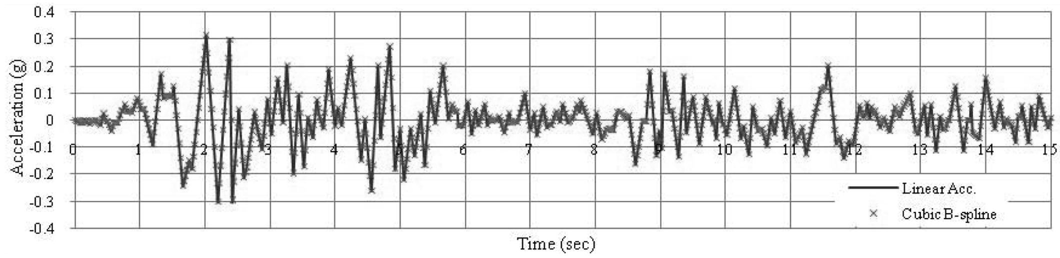


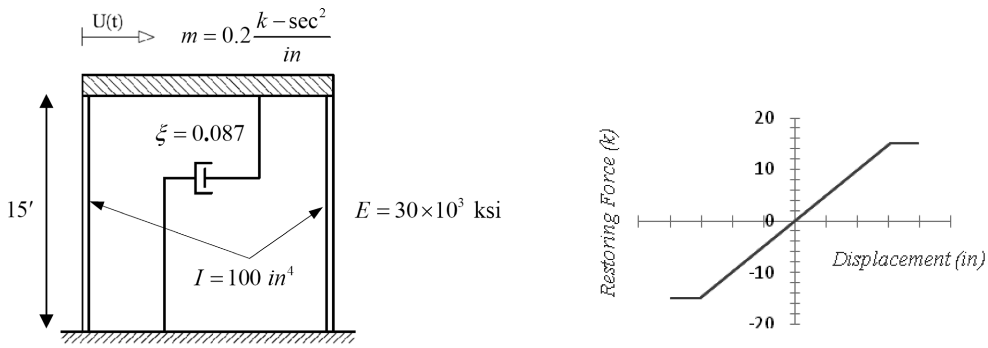

Fig. 13 Acceleration time histories (g)

method including relative displacement, velocity and acceleration, are exactly coincident with those from the compared method. The difference lies here that the needed time for solving this problem using Cubic B-spline method is about 87% of the consumed time by the Linear Acceleration method. This is the advantage of the proposed method.

Example 4: *A portal frame with elasto-plastic behavior under the acceleration time-history of El-Centro ground shaking;*

Fig. 14 shows a SDOF portal frame with nonlinear behavior under the acceleration time-history of *El-Centro* ground shaking with the peak ground acceleration (*PGA*) equal to 0.32 g. In this example, it is assumed that the damping coefficient ($\xi = 0.087$) is constant. So, nonlinearity is only limited to the stiffness changes.

Fig. 15 shows the analysis result (displacement time-history) of the above system for both Linear Acceleration and Cubic B-spline (proposed) methods during this 30-second earthquake. Fig. 16 also shows the hysteresis diagram of bilinear behavior of the system. As the graph shows, the result of proposed method is coincident with the result of Linear Acceleration method.




(A) Portal frame (B) Elasto-plastic behavior

Fig. 14 A portal frame with elasto-plastic behavior under earthquake ground shaking

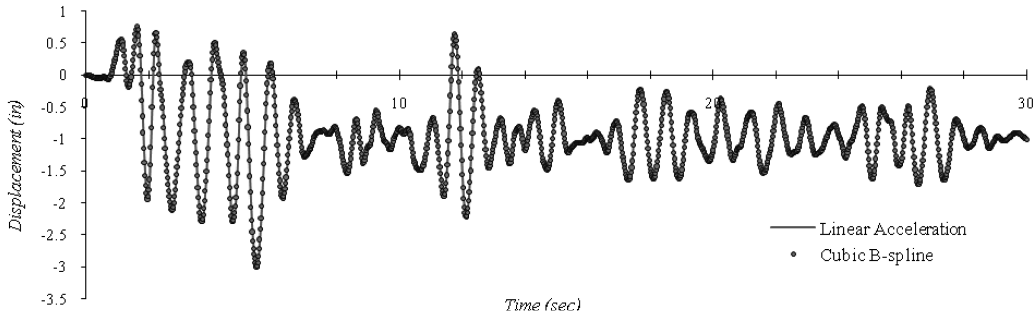



Fig. 15 Displacement time-histories

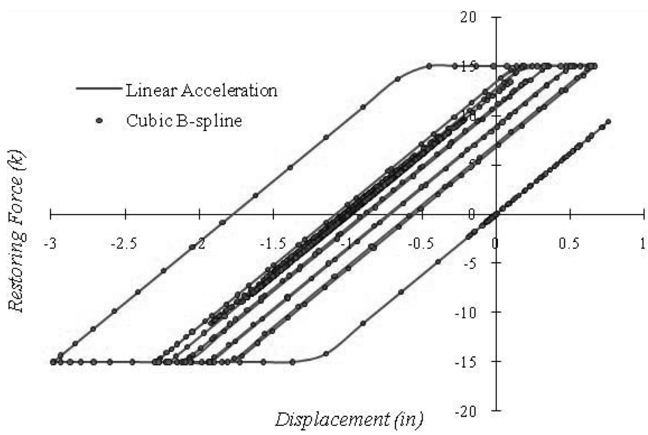



Fig. 16 Hysteresis diagram

## 6. Conclusions

In this paper, we introduced a new direct integration method to find dynamic response of SDOF systems using periodic cubic B-spline. The result of implementing B-spline basis function to solve differential equation of motion is an explicit, straightforward and fluent formulation with a simple algorithm for linear and nonlinear analyses. The proposed method is conditionally stable. The aforementioned method is accurate and fast to analyze, especially in the case of nonlinear analysis. The results from this method are completely coincident with those from the Linear Acceleration method, while solving a problem using the proposed method is less time comsuming than Linear Acceleration. The proposed method calculates the values of displacement, velocity and acceleration independently while the Linear Acceleration method does not act in this way. This property reduces the time consumption of the analysis. This method can be simply generalized to multi-degree-of-freedom (MDOF) systems, but as our goal has been just to introduce a new methodology, we have just discussed SDOF systems. According to the possibility of increasing the order in B-spline functions, the higher order B-spline functions can be used to improve the accuracy and convergence of the method. Finally, the authors foresee a very clear and inspiring future for this method, asserting that it can be used as an efficient method in dynamic analysis, particularly in nonlinear systems.

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