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Three-dimensional free vibration analysis of functionally graded fiber reinforced cylindrical panels using differential quadrature method

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Abstract. Three dimensional solutions for free vibrations analysis of functionally graded fiber reinforced cylindrical panel are presented, using differential quadrature method (DQM). The orthotropic panel is simply supported at the edges and is assumed to have an arbitrary variation of reinforcement volume fraction in the radial direction. Suitable displacement functions that identically satisfy the simply supported boundary condition are used to reduce the equilibrium equations to a set of coupled ordinary differential equations with variable coefficients, which can be solved by differential quadrature method to obtain natural frequencies. The main contribution of this work is presenting useful results for continuous grading of fiber reinforcement in the thickness direction of a cylindrical panel and comparison with similar discrete laminate composite ones. Results indicate that significant improvement is found in natural frequency of a functionally graded fiber reinforced composite panel due to the reduction in spatial mismatch of material properties.

Keywords: DQM; functionally graded fiber reinforced; free vibrations; orthotropic; panel.

1. Introduction

Thin and thick shells as structural elements occupy a leadership position in civil, architectural, aeronautical, and marine engineering, since they give rise to optimum conditions for dynamic behavior, strength and stability. In other words, these structures support applied external forces efficiently by virtue of their geometrical shape. The study of the vibration of shells and panels of revolution is an important aspect in the successful applications of these structures. Functionally graded materials (FGM) are a class of composites that have a smooth and continuous variation of material properties from one surface to another and thus can alleviate the stress concentrations found in laminated composites. Typically these materials are made from a mixture of ceramic and metal, or a combination of different materials. Extensive research work has been carried out on this new class of composites since its concept was first introduced and proposed in the late 1980s in Japan.

Despite the evident importance in practical applications, investigations on the dynamic characteristics of FGM structures are still limited in number. Among those available, Bending and

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free vibration analysis of a smart functionally graded plate was carried out (Bian *et al.* 2006). Free vibrations of simply supported FGM cylindrical shell was investigated (Loy *et al.* 1999), which was later extended to cylindrical shells under various end supporting conditions (Pradhan *et al.* 2000). Vibration analysis of functionally graded shell was carried out, using a higher-order theory (Patel *et al.* 2005). The free vibrations analysis of functionally graded curved panels was studied by using a higher-order finite element formulation (Pradyumna and Bandyopadhyay 2008). Free vibration and dynamic instability of FGM cylindrical panels under combined static and periodic axial forces were studied by using a proposed semi-analytical approach (Yang and Shen 2003). Elastic response analysis of simply supported FGM cylindrical shell under low-velocity impact was presented (Gang *et al.* 1999). Vibrations and wave propagation velocity in a functionally graded hollow cylinder was studied (Shakeri *et al.* 2006). It was assumed the shell to be in plane strain condition and subjected to an axisymmetric dynamic loading. The free vibration of simply supported, fluid-filled orthotropic functionally graded cylindrical shells with arbitrary thickness was investigated (Chen *et al.* 2004). Optimal stacking sequence of laminated anisotropic cylindrical panel using genetic algorithm was studied, (Alibeigloo *et al.* 2007). In all the above studies, it was assumed that material properties follow a through-thickness variation according to a power-law distribution in terms of the volume fractions of constituents.

However, this paper is motivated by the lack of studies in the technical literature concerning to the 3- dimensional vibration analysis of functionally graded orthotropic panels. The aim is to analyze the influence of continuous grading of reinforcement and comparison of FGM orthotropic panel with discretely laminated 2 and 3 layer composite ones.

Many numerical methods are available in the literature to solve governing equations. An efficient Galerkin meshfree analysis of shear deformable cylindrical panels was carried out (Wang *et al.* 2008). In this paper, the solution is obtained by using numerical technique termed the generalized differential quadrature method (GDQ), which leads to a generalized eigenvalue problem. The mathematical fundamental and recent developments of GDQ method as well as its major applications in engineering are discussed in detail in book (Shu 2000). In recent years, the DQM has been become increasingly popular in the numerical solution for initial and boundary value problems (Bert and Malik 1996). The DQM can yield accurate solutions with relatively fewer grid points. The first application of DQM for composite plates was carried out (Bert *et al.* 1989). It was made use of DQM to determine vibration characteristics of generally laminated beams and cross-ply laminated plates subjected to cylindrical bending respectively (Chen and Bian 2003, Chen 2005). Combination of the state-space method and the technique of DQ based on the three dimensional elasticity theory were used for static and free vibrations analysis of laminated cylindrical panels (Alibegloo and Shakeri 2009, 2007). Three dimensional temperature dependent free vibration analysis of functionally graded material curved panels resting on two parameter elastic foundation was studied through using a hybrid semi-analytic differential quadrature method (Farid *et al.* 2009). 2-D higher- order deformation theory was used for free vibration and stability of functionally graded shallow shells (Matsumaga 2008). Thus another aim of the present paper is to demonstrate the efficient application of the differential quadrature approach, by solving the equations of motion governing the functionally graded orthotropic cylindrical panels. In this paper the frequency characteristics of cylindrical panels with continuous grading of reinforcement volume fraction is obtained. Numerical results are presented and compared to the available results in the literatures for isotropic FGM cylindrical panel to validate the proficiency of DQ approach.

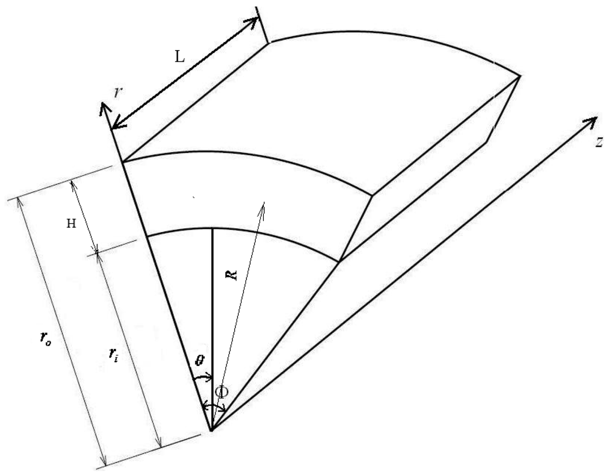

Fig. 1 Geometry of a cylindrical panel

2. Problem description

An FGM cylindrical panel with its co- ordinate system (r, θ, z) is shown in Fig. 1 where r, θ, z are in the radial, circumferential and axial directions of the panel. The solution presented here is applicable for arbitrary variation of material composition through the thickness of the shell. For orthotropic cylindrical panel, we assume the following specific power-law variation of the reinforcement volume fraction (Pelletier and Vel 2006).

$$V = V_i + (V_o - V_i)\left(\frac{r - r_i}{r_o - r_i}\right)^p \tag{1}$$

Where V_i and V_o are volume fraction of constituent material at the inner and outer surface respectively. The exponent "P" controls the volume fraction profile through the shell's thickness. The effective mechanical properties of the fiber reinforced cylindrical panel are obtained based on a micromechanical model as follows (Shen 2009, Vasiliev and Morozov 2001)

$$E_{11} = V_f E_{11}^f + V_m E_{11}^m \tag{2}$$

$$\frac{1}{E_{ii}} = \frac{V_f}{E_{ii}^f} + \frac{V_m}{E_{ii}^m} - V_f V_m \frac{v_f^2 E_{ii}^m / E_{ii}^f + v_m^2 E_{ii}^f / E_{ii}^m - 2 v^f v^m}{V_f E_{11}^f + V_m E_{11}^m} \qquad i = 2, 3 \tag{3}$$

$$\frac{1}{G_{ij}} = \frac{V_f}{G_{ij}^f} + \frac{V_m}{G_{ij}^m} \qquad i,j = 1, 2, 3 \quad i \neq j \tag{4}$$

$$v_{ij} = V_f v^f + V_m v^m \qquad i,j = 1, 2, 3 \quad i \neq j \tag{5}$$

Where E_{ij}, G_{ij}, v_{ij}, V_f and V_m are elasticity modulus, shear modulus, Poisson's ratio, fiber volume and matrix volume fractions respectively.

The mechanical constitutive relations, which relates the stresses to the strains are as follows

$$\begin{bmatrix} \sigma_z \\ \sigma_\theta \\ \sigma_r \\ \tau_{r\theta} \\ \tau_{zr} \\ \tau_{z\theta} \end{bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & 0 \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & 0 \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{C}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_z \\ \varepsilon_\theta \\ \varepsilon_r \\ \gamma_{r\theta} \\ \gamma_{zr} \\ \gamma_{z\theta} \end{bmatrix} \tag{6}$$

In the absence of body forces, the governing equations are as follows

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{z\theta}}{r\partial\theta} + \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} = \rho\frac{\partial^2 u_z}{\partial t^2}$$

$$\frac{\partial \tau_{\theta z}}{\partial z} + \frac{\partial \sigma_\theta}{r\partial\theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = \rho\frac{\partial^2 u_\theta}{\partial t^2}$$

$$\frac{\partial \tau_{zr}}{\partial z} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial\theta} + \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = \rho\frac{\partial^2 u_r}{\partial t^2} \tag{7}$$

Strain-displacement relations are expressed as

$$\varepsilon_\theta = \frac{u_r}{r} + \frac{\partial u_\theta}{r\partial\theta}, \qquad \varepsilon_r = \frac{\partial u_r}{\partial r}, \qquad \varepsilon_z = \frac{\partial u_z}{\partial z},$$

$$\gamma_{r\theta} = \frac{-u_\theta}{r} + \frac{\partial u_\theta}{\partial r} + \frac{\partial u_r}{r\partial\theta}, \qquad \gamma_{zr} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \qquad \gamma_{z\theta} = \frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{r\partial\theta} \tag{8}$$

Where u_r, u_θ and u_z are radial, circumferential and axial displacement components respectively.

Upon substitution Eq. (8) into (6) and then into (7), the following equations of motion as matrix form are obtained in term of displacement components

$$\begin{bmatrix} K_{1r} & K_{1\theta} & K_{1z} \\ K_{2r} & K_{2\theta} & K_{2z} \\ K_{3r} & K_{3\theta} & K_{3z} \end{bmatrix} \begin{Bmatrix} u_r \\ u_\theta \\ u_z \end{Bmatrix} = \begin{Bmatrix} \rho\ddot{u}_r \\ \rho\ddot{u}_\theta \\ \rho\ddot{u}_z \end{Bmatrix} \tag{9}$$

The following simply supported conditions are imposed at the edges of the cylindrical panel

$$u_r = 0, \quad \sigma_\theta = 0 \quad \text{at} \quad \theta = 0, \Phi$$

$$u_r = 0, \quad \sigma_z = 0 \quad \text{at} \quad z = 0, L \tag{10}$$

Moreover the inner and outer surfaces of the panel are traction free

$$\sigma_r = \tau_{rz} = \tau_{r\theta} = 0 \quad \text{at} \quad r = r_i, r_o \tag{11}$$

3. Semi-analytical solution

The following assumed solutions satisfy the simply supported opposite edges at $\theta = 0, \Phi$, $z = 0, L$

$$u_r = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_r(r)\sin(\beta_m\theta)\sin(p_n z)e^{i\omega t}$$

$$u_\theta = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_\theta(r)\cos(\beta_m\theta)\sin(p_n z)e^{i\omega t}$$

$$u_z = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_z(r)\sin(\beta_m\theta)\cos(p_n z)e^{i\omega t}$$

$$\beta_m = \frac{m\pi}{\Phi} \qquad (m = 1,2,...)$$

$$p_n = \frac{n\pi}{L} \qquad (n = 1,2,...) \tag{12}$$

Where "m" and "n" are circumferential and axial wave numbers respectively, and ω is the natural angular frequency of the vibration.

Upon substituting Eq. (12) into the governing Eq. (9), the coupled partial differential equations reduce to a set of ordinary differential equations as follows

$$\begin{aligned}
&-\bar{C}_{55}p_n\frac{\partial U_z}{\partial r} - \bar{C}_{55}p_n^2U_r + \bar{C}_{44}\frac{1}{r^2}\beta_mU_\theta - \bar{C}_{44}\frac{1}{r}\beta_m\frac{\partial U_\theta}{\partial r} - \bar{C}_{44}\frac{1}{r^2}\beta_m^2U_r - \frac{\partial\bar{C}_{13}}{\partial r}p_nU_z\\
&-\bar{C}_{13}p_n\frac{\partial U_z}{\partial r} + \frac{\partial\bar{C}_{23}}{\partial r}\frac{1}{r}U_r - \frac{\partial\bar{C}_{23}}{\partial r}\frac{1}{r}\beta_mU_\theta - \bar{C}_{23}\frac{1}{r}\beta_m\frac{\partial U_\theta}{\partial r} + \frac{\partial\bar{C}_{33}}{\partial r}\frac{\partial U_r}{\partial r} + \bar{C}_{33}\frac{\partial^2U_r}{\partial r^2}\\
&-\bar{C}_{13}\frac{1}{r}p_nU_z + \bar{C}_{33}\frac{1}{r}\frac{\partial U_r}{\partial r} + \bar{C}_{12}\frac{1}{r}p_nU_z - \bar{C}_{22}\frac{1}{r^2}U_r + \bar{C}_{22}\frac{1}{r^2}\beta_mU_\theta = -\omega^2\rho U_r\\
&-\bar{C}_{66}p_n^2U_\theta - \bar{C}_{66}\frac{1}{r}p_n\beta_mU_z - \bar{C}_{12}\frac{1}{r}\beta_mp_nU_z + \bar{C}_{22}\frac{1}{r^2}\beta_mU_r - \bar{C}_{22}\frac{1}{r^2}\beta_m^2U_\theta + \bar{C}_{23}\frac{1}{r}\beta_m\frac{\partial U_r}{\partial r}\\
&-\frac{\partial\bar{C}_{44}}{\partial r}\frac{1}{r}U_\theta + \frac{\partial\bar{C}_{44}}{\partial r}\frac{\partial U_\theta}{\partial r} + \frac{\partial\bar{C}_{44}}{\partial r}\frac{1}{r}\beta_mU_r + \bar{C}_{44}\frac{\partial^2U_\theta}{\partial r^2} + \bar{C}_{44}\frac{1}{r}\beta_m\frac{\partial U_r}{\partial r} - \bar{C}_{44}\frac{1}{r^2}U_\theta\\
&+\bar{C}_{44}\frac{1}{r}\frac{\partial U_\theta}{\partial r} + \bar{C}_{44}\frac{1}{r^2}\beta_mU_r = -\omega^2\rho U_\theta\\
&-\bar{C}_{11}p_n^2U_z + \bar{C}_{12}\frac{1}{r}p_nU_r - \bar{C}_{12}\frac{1}{r}\beta_mp_nU_\theta + \bar{C}_{13}p_n\frac{\partial U_r}{\partial r} - \bar{C}_{66}\frac{1}{r}\beta_mp_nU_\theta - \bar{C}_{66}\frac{1}{r^2}\beta_m^2U_z\\
&+\frac{\partial\bar{C}_{55}}{\partial r}\frac{\partial U_z}{\partial r} + \frac{\partial\bar{C}_{55}}{\partial r}p_nU_r + \bar{C}_{55}\frac{\partial^2U_z}{\partial r^2} + \bar{C}_{55}p_n\frac{\partial U_r}{\partial r} + \bar{C}_{55}\frac{1}{r}\frac{\partial U_z}{\partial r} + \bar{C}_{55}\frac{1}{r}p_nU_r = -\omega^2\rho U_z
\end{aligned} \tag{13}$$

A semi-analytical procedure with the aid of DQM was recently developed (Chen and Bian 2003). In this method the n_{th}. Order partial derivative of a continuous function $f(x, z)$ with respect to x at a given point x_i can be approximated as a linear sum of weighted function values at all of the discrete points in the domain of x, i.e.

$$\frac{\partial f^n(x_i,z)}{\partial x^n}=\sum_{k=1}^{N}c_{ik}^{n}f(x_{ik},z)\qquad (i=1,2,\ldots,N,n=1,2,\ldots,N-1)\tag{14}$$

Where N is the number of sampling points, and c_{ij}^{n} is the x_i dependent weight coefficients (Chen and Bian 2003). Eq. (14) being applied to Eq. (13), the following equations at an arbitrary sampling point x_i are then obtained

$$\begin{gathered}
-\bar{C}_{55}^{n}p_n\sum_{k=1}^{N}c_{ik}^{(1)n}U_{Zk}^{n}-\bar{C}_{55}^{n}p_n{}^2U_{ri}^{n}+\bar{C}_{44}^{n}\frac{1}{r^2{}_n}\beta_mU_{\theta i}^{n}-\bar{C}_{44}^{n}\frac{1}{r_n}\beta_m\sum_{k=1}^{N}c_{ik}^{(1)n}U_{\theta k}^{n}-\bar{C}_{44}^{n}\frac{1}{r^2{}_n}\beta_m{}^2U_{ri}^{n}\\
-\bar{C}_{13}^{n}p_n\sum_{k=1}^{N}c_{ik}^{(1)n}U_{Zk}^{n}-\bar{C}_{23}^{n}\frac{1}{r_n}\beta_m\sum_{k=1}^{N}c_{ik}^{(1)n}U_{\theta k}^{n}+\bar{C}_{33}^{n}\sum_{k=1}^{N}c_{ik}^{(2)n}U_{rk}^{n}-\bar{C}_{13}^{n}\frac{1}{r_n}p_nU_{zi}^{n}\\
+\bar{C}_{33}^{n}\frac{1}{r_n}\sum_{k=1}^{N}c_{ik}^{(1)n}U_{rk}^{n}+\bar{C}_{12}^{n}\frac{1}{r_n}p_nU_{zi}^{n}-\bar{C}_{22}^{n}\frac{1}{r_n^2}U_{ri}^{n}+\bar{C}_{22}^{n}\frac{1}{r_n^2}\beta_mU_{\theta i}^{n}=-\omega^2\rho_nU_{ri}^{n}\\
+\bar{C}_{23}^{n}\frac{1}{r_n}\beta_m\sum_{k=1}^{N}c_{ik}^{(1)n}U_{rk}^{n}+\bar{C}_{44}^{n}\sum_{k=1}^{N}c_{ik}^{(2)n}U_{\theta k}^{n}+\bar{C}_{44}^{n}\frac{1}{r_n}\beta_m\sum_{k=1}^{N}c_{ik}^{(1)n}U_{rk}^{n}\\
-\bar{C}_{66}^{n}p_n{}^2U_{\theta i}^{n}-\bar{C}_{66}^{n}\frac{1}{r_n}p_n\beta_mU_{zi}^{n}-\bar{C}_{12}^{n}\frac{1}{r_n}\beta_mp_nU_{zi}^{n}+\bar{C}_{22}^{n}\frac{1}{r_n^2}\beta_mU_{ri}^{n}-\bar{C}_{22}^{n}\frac{1}{r_n^2}\beta_m{}^2U_{\theta i}^{n}\\
-\bar{C}_{44}^{n}\frac{1}{r^2{}_n}U_{\theta i}^{n}+\bar{C}_{44}^{n}\frac{1}{r_n}\sum_{k=1}^{N}c_{ik}^{(1)n}U_{\theta k}^{n}+\bar{C}_{44}^{n}\frac{1}{r_n^2}\beta_mU_{ri}^{n}=-\omega^2\rho_nU_{\theta i}^{n}\\
-\bar{C}_{11}^{n}p_n{}^2U_{zi}^{n}+\bar{C}_{12}^{n}\frac{1}{r_n}p_nU_{ri}^{n}-\bar{C}_{12}^{n}\frac{1}{r_n}\beta_mp_nU_{\theta i}^{n}+\bar{C}_{13}^{n}p_n\sum_{k=1}^{N}c_{ik}^{(1)n}U_{rk}^{n}-\bar{C}_{66}^{n}\frac{1}{r_n}\beta_mp_nU_{\theta i}^{n}\\
-\bar{C}_{66}^{n}\frac{1}{r_n^2}\beta_m{}^2U_{zi}^{n}+\bar{C}_{55}^{n}\sum_{k=1}^{N}c_{ik}^{(2)n}U_{Zk}^{n}+\bar{C}_{55}^{n}p_n\sum_{k=1}^{N}c_{ik}^{(1)n}U_{rk}^{n}+\bar{C}_{55}^{n}\frac{1}{r_n}\sum_{k=1}^{N}c_{ik}^{(1)n}U_{Zk}^{n}+\bar{C}_{55}^{n}\frac{1}{r_n}p_nU_{ri}^{n}=-\omega^2\rho_nU_{zi}^{n}
\end{gathered}\tag{15}$$

In the above equation $c_{ik}^{(1)}$ and $c_{ik}^{(2)}$ are the weighting coefficients of the first and second order derivatives. In a similar manner the boundary conditions can be discretized. For this purpose, using Eq. (11) and DQ discretization rule for special derivatives, the boundary conditions at $r = r_i$ and $r = r_o$ become

$$\begin{gathered}
\bar{C}_{23}^{1}\frac{1}{a}U_{r1}^{1}+\bar{C}_{33}^{1}c_{11}^{(1)1}U_{r1}^{1}+\bar{C}_{33}^{1}c_{1N}^{(1)1}U_{rN}^{1}-\bar{C}_{23}^{1}\frac{1}{a}\beta_mU_{\theta 1}^{1}-\bar{C}_{13}^{1}p_nU_{z1}^{1}+\bar{C}_{33}^{1}\sum_{k=2}^{N-1}c_{1k}^{(1)1}U_{rk}^{1}=0\\
\bar{C}_{44}^{1}\frac{1}{a}\beta_mU_{r1}^{1}-\bar{C}_{44}^{1}\frac{1}{a}U_{\theta 1}^{1}+\bar{C}_{44}^{1}c_{11}^{(1)1}U_{\theta 1}^{1}+\bar{C}_{44}^{1}c_{1N}^{(1)1}U_{\theta N}^{1}+\bar{C}_{44}^{1}\sum_{k=2}^{N-1}c_{1k}^{(1)1}U_{\theta K}^{1}=0\\
\bar{C}_{55}^{1}c_{11}^{(1)1}U_{z1}^{1}+\bar{C}_{55}^{1}\sum_{k=2}^{N-1}c_{1k}^{(1)1}U_{zk}^{1}+\bar{C}_{55}^{1}c_{1N}^{(1)1}U_{zN}^{1}+\bar{C}_{55}^{1}p_nU_{r1}^{1}=0
\end{gathered}$$

$$\bar{C}^n_{33} c^{(1)n}_{N1} U^n_{r1} + \bar{C}^n_{23}\frac{1}{b}U^n_{rN} + \bar{C}^n_{33} c^{(1)n}_{NN} U^n_{rN} - \bar{C}^n_{23}\frac{1}{b}\beta_m U^n_{\theta N} - \bar{C}^n_{13} p_n U^n_{zN} + \bar{C}^n_{33}\sum_{k=2}^{N-1} c^{(1)n}_{Nk} U^n_{rk} = 0$$

$$\bar{C}^n_{44}\frac{1}{b}\beta_m U^n_{rN} + \bar{C}^n_{44} c^{(1)n}_{N1} U^n_{\theta 1} - \bar{C}^n_{44}\frac{1}{b}U^n_{\theta N} + \bar{C}^n_{44} c^{(1)n}_{NN} U^n_{\theta N} + \bar{C}^n_{44}\sum_{k=2}^{N-1} c^{(1)n}_{Nk} U^n_{\theta k} = 0$$

$$\bar{C}^n_{55} c^{(1)n}_{N1} U^n_{z1} + \bar{C}^n_{55}\sum_{k=2}^{N-1} c^{(1)n}_{Nk} U^n_{zk} + \bar{C}^n_{55} c^{(1)n}_{NN} U^n_{zN} + \bar{C}^n_{55} p_n U^n_{rN} = 0 \tag{16}$$

In order to carry out the eigenvalue analysis, domain and boundary degrees of freedom are separated and in vector forms they are denoted as (d) and (b), respectively. Based on this definition, the matrix form of the equilibrium equations and the related boundary conditions take the following form

$$\begin{bmatrix} [A_{bb}] & [A_{bd}] \\ [A_{db}] & [A_{dd}] \end{bmatrix} \begin{Bmatrix} \{U_b\} \\ \{U_d\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ -\omega^2[M]\{U_d\} \end{Bmatrix} \tag{17}$$

Where $\{U_d\}$ and $\{U_b\}$ are as follows

$$\{U_d\} = \{\{U_{rd}\},\{U_{\theta d}\},\{U_{zd}\}\}^T \tag{18}$$

$$\{U_b\} = \{\{U_{rb}\},\{U_{\theta b}\},\{U_{zb}\}\}^T \tag{19}$$

Eliminating the boundary degrees of freedom, this equation becomes

$$([A] + \omega^2[M])\{U_d\} = \{0\} \tag{20}$$

Where

$$[A] = [A_{dd}] - [A_{db}][A_{bb}]^{-1}[A_{bd}] \tag{21}$$

The above eigenvalue system of equations can be solved to find the natural frequencies of the FGM orthotropic cylindrical panels.

4. Results and discussion

For numerical computation, sampling points with the following coordinates are used (Shu and Richards 1992).

$$x_i = \frac{1}{2}\left(1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right) \qquad i = 1, 2, \ldots, N \tag{22}$$

To validate the analysis, numerical results for FGM isotropic cylindrical shell are compared with similar ones in the literature, see Table 1. The comparison shows that the present results agree well with those in the literatures. Besides fast rate of convergence of the method is quite evident and it is found that only seven DQ grid ($N = 7$) in the thickness direction can yield accurate results.

Further validation of the present results for FGM cylindrical panel are shown in Table 2.

Table 1 Comparison of the normalized natural frequency ($\Omega = \omega R\Phi\sqrt{\rho_m H/D}$, $D = E_m H^3/(12(1-\nu_m^2))$ for various "P" of an isotropic FGM cylindrical panel (m, $n = 1$, $L/R = 1$, $H/R = 0.1$, for outer surface $E_m = 70\times10^9$ pa, $\rho_m = 2702$ kg/m^3, $\nu_m = 0.3$, for inner surface $E_c = 380\times10^9$ pa, $\rho_c = 3000$ kg/m^3, $\nu_c = 0.3$)

	$P = 0$	$P = 0.2$	$P = 0.5$	$P = 1$	$P = 2$
(Pradyumna and Bandyopadhyay 2008)	51.5216	47.5968	43.3019	38.7715	34.3338
(Farid *et al.* 2009)	52.0521	48.0822	43.7281	39.144	34.6756
Present Results, for $N = 5$	52.03335	48.19178	43.78813	38.7294	33.07764
Present Results, for $N = 7$	52.05211	48.14951	43.79246	39.10217	34.7555
Present Results, for $N = 13$	52.05214	48.07752	43.7327	39.14399	34.6754
Present Results, for $N = 20$	52.05214	48.07579	43.7232	39.14413	34.6755

Table 2 Comparison of the normalized natural frequency for various L/R and L/H ratios

		"P"				
		0	0.5	1	4	10
$L/H = 2$	$L/R = 0.5$					
(Matsumaga 2008)		0.9334	0.8213	0.7483	0.6011	0.5461
(Farid *et al.* 2009)		0.9187	0.8013	0.7263	0.5267	0.5245
Present Results		0.924953	0.801888	0.7253725	0.579019	0.530157
$L/H = 2$	$L/R = 1$					
(Matsumaga 2008)		0.9163	0.8105	0.7411	0.5967	0.5392
(Farid *et al.* 2009)		0.8675	0.7578	0.6875	0.5475	0.4941
Present Results		0.885794	0.766774	0.693543	0.553117	0.506557
$L/H = 5$	$L/R = 0.5$					
(Matsumaga 2008)		0.2153	0.1855	0.1678	0.1413	0.1328
(Farid *et al.* 2009)		0.2113	0.1814	0.1639	0.1367	0.1271
Present Results		0.212934	0.181731	0.163836	0.137443	0.129626
$L/H = 5$	$L/R = 1$					
(Matsumaga 2008)		0.2239	0.1945	0.1769	0.1483	0.1385
(Farid *et al.* 2009)		0.2164	0.1879	0.1676	0.1394	0.1286
Present Results		0.215457	0.184812	0.167146	0.139172	0.130017

Table 3 Mechanical properties of the orthotropic material

	Cu	W
E (Gpa)	115.0	400.0
v	0.31	0.28
r (kg/m^3)	8960	19300

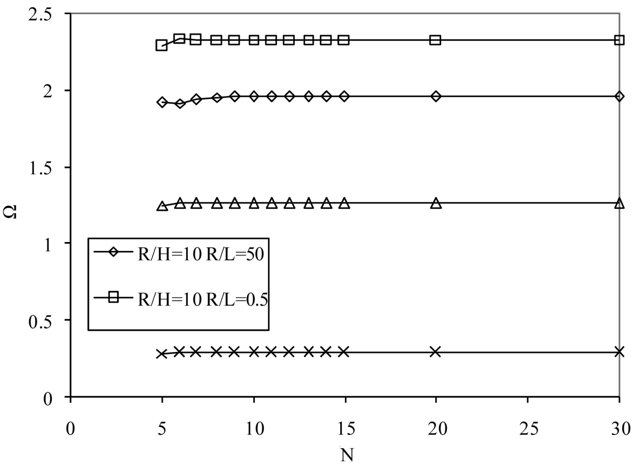

Fig. 2 Convergency of the normalized natural frequency of an orthotropic panel for various *R/H* and *R/L* ratios ($\Phi = \pi/6$, "P" = 1, m = 1, n = 1)

In this table, comparison is made for different *L/R* and *L/H* ratios and as it is observed there is good agreement between the results. In this section, we characterize the response of orthotropic shell panels with graded fiber volume fractions. The orthotropic panel consists of continuous tungsten reinforcement fibers in a copper matrix (W/Cu). These material combinations have found widespread use in high performance application (Miracle 2001). The relevant material properties for the constituent materials are listed in Table 3.

For this orthotropic material the fibers are oriented at $\phi = 0$, with respect to the axial direction of the panel shell. Here it is assumed that the functionally graded shell has a linear variation beginning at $V_w = 0$ (0% Tungsten, 100% Copper) on the inner surface of the shell to $V_w = 0.75$ (75% Tungsten, 25% Copper) on the outer surface. The normalized natural frequency is, $\Omega = \omega L\sqrt{\rho^i/E^i}$ (ρ^i, E^i are mechanical properties of Copper). A convergence study of the normalized natural frequency is shown in Fig. 2 for various *R/H* (*R* = mean radius of the panel, *H* = thickness of the panel) and *R/L* ratios. As it is noticed, fast rate of convergence of the method is observed and it is found that only seven DQ grid in the thickness direction can yield accurate results. It can also be seen for the considered system the formulation is stable while increasing the number of points and that the use of 30 points guarantees convergence of the procedure.

We now turn our attention to the comparison of the FGM orthotropic cylindrical panel with discretely laminated 2-layer, 3-layer panel containing [0/0.75], [0/0.375/0.75] volume fractions respectively, as shown in Table 4.

The effect of radius to thickness ratio, *S*, on the normalized natural frequency is shown in Fig. 3.

Table 4 Material volume fractions of 2-layer, 3-layer and FGM

Type of cylindrical panel		Material volume fractions
2 Layers	1st lamina	0% Tungsten, 100% Copper
	2nd lamina	75% Tungsten, 25% Copper
3 Layers	1st lamina	0% Tungsten, 100% Copper
	2nd lamina	37.5% Tungsten, 62.5% Copper
	3rd lamina	75% Tungsten, 25% Copper
FGM	inner surface	0% Tungsten, 100% Copper
	outer surface	75% Tungsten, 25% Copper

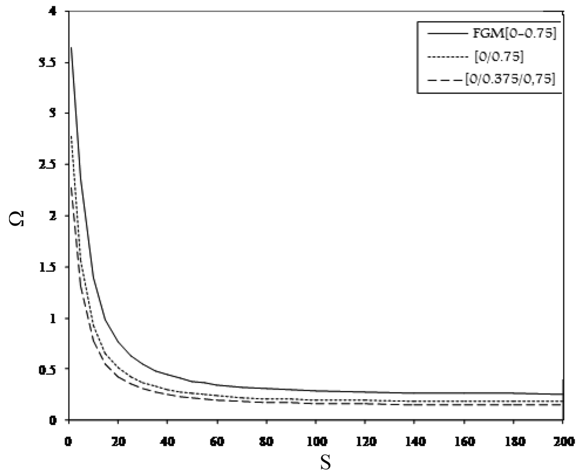

Fig. 3 Effect of S on the normalized natural frequency for orthotropic FGM, 2-layer and 3-layer ($m=1$, $n=1$, $R/L=1$, $\Phi=\pi/6$, "P" = 1)

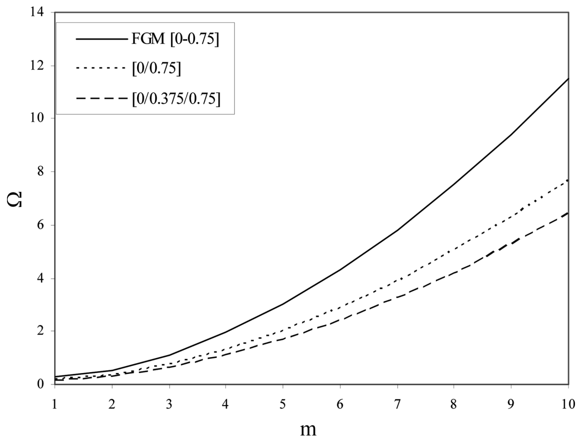

Fig. 4 Variation of the normalized natural frequency against circumferential wave m ($S=100$, $R/L=1$, $n=1$, "P" = 1, $\Phi=\pi/6$)

As it is observed the normalized natural frequency decreases sharply with increasing the S ratio for thick panels and remain unaltered for thin ones. It is also noticed the natural frequency of the FGM orthotropic panel is about 30% higher than the similar discrete laminated 2 and 3 layers. The natural frequency decreases with increasing the layers. Fig. 4 shows the variations of the normalized natural frequency with the circumferential wave numbers "m" for FGM, 2-layer, 3-layer cylindrical panel. The natural frequency increases with increasing the circumferential wave number. It is also seen that the natural frequencies of a FGM orthotropic panel is about 30% higher than similar 2-layer laminated composite one.

The influence of constituent volume fractions is studied by varying the volume fractions of W/Cu. This is carried out by varying the volume of the power-law exponent "P". Fig. 5 shows the influence of the constituent volume fractions "P" on the fundamental natural frequency with various circumferential wave number "m". As "P" increases, the natural frequency decreases. The natural

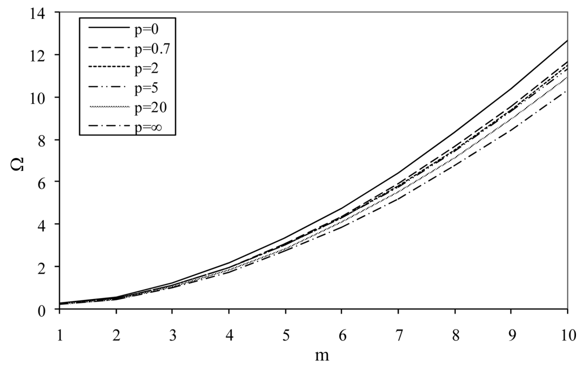

Fig. 5 Variation of the normalized natural frequency against circumferential wave number for various "*P*" (S = 100, n = 1, R/L = 1, $\Phi = \pi/6$)

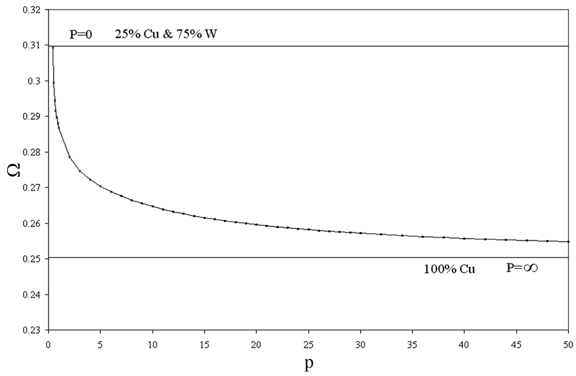

Fig. 6 Variation of the normalized natural frequency against "*P*" (m = 1, n = 1, S = 100, R/L = 1, $\Phi = \pi/6$)

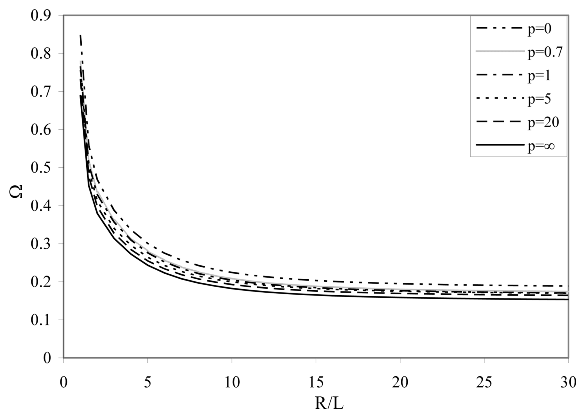

Fig. 7 Variations of the normalized natural frequency against R/L ratio for Various "*P*" (n = 1, m = 1, H/L = 0.05, $\Phi = \pi/6$)

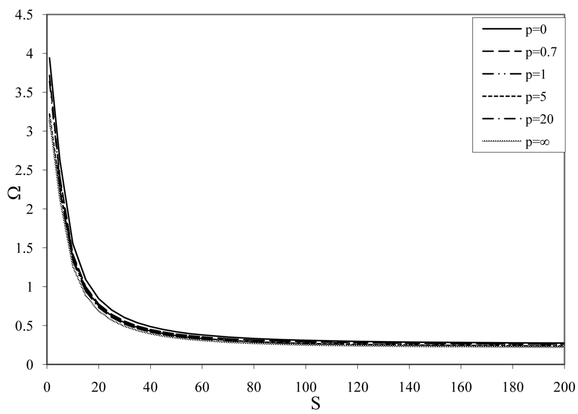

Fig. 8 Effect of S on the natural frequency for different volume fractions (n = 1, m = 1, L = 1 m, $\Phi = \pi/6$)

frequency increases from 100% Cu ("*P*" = ∞) to 25% Cu ("*P*" = 0).

Effect of power law exponent on the normalized first natural frequency is presented in Fig. 6. According to the Fig. 6 Ω decreases quickly with increasing the power-law exponent "*P*" and then remains constant. As it is noticed, Ω for different power-law exponent "*P*" varies between "*P*" = ∞ (100% Cu) and "*P*" = 0 (25% Cu & 75% W).

Fig. 7 shows the variations of the normalized natural frequency versus R/L ratio at different volume fractions ("*P*"). It is noticed the normalized natural frequency converges to a constant value at larger values of R/L ratio.

Effect of S ratio on the normalized natural frequency for the panel with different constituent volume fractions is shown in Fig. 8. It is seen that the normalized natural frequency decreases as the S increases. Ω decreases quickly with increasing the S, for thick panel and remains unaltered for the thin one It is also seen that volume fraction almost doesn't affect on the natural frequency for thin panels. According to the Fig. 9, it results the normalized natural frequency of the FGM panel

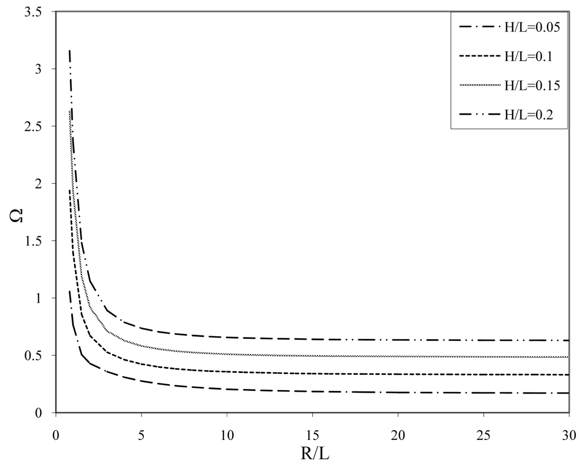

Fig. 9 Influence of *R/L* on the natural frequency for various values of *H/L* ratios ($n = 1$, $m = 1$, "P" = 1, $\Phi = \pi/6$)

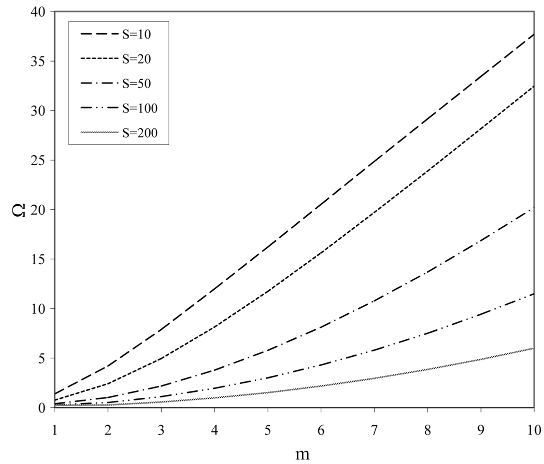

Fig. 10 Influence of the circumferential wave number on the normalized natural frequencies at different S ratios ($n = 1$, $R/L = 1$, "P" = 1, $\Phi = \pi/6$)

with linear fiber reinforced remains unaltered for larger values of *R/L* at different *H/L* ratios.

Finally the influence of the circumferential wave number on the normalized natural frequency is shown in Fig. 10 at different S ratios for linear grading fiber reinforced ("P" = 1) composite panel. It is apparent that the frequencies are higher at lower S ratios.

5. Conclusions

The Differential Quadrature Method has been used to study 3-dimensional free vibration analysis of continuous grading reinforced panels. The dynamic equilibrium equations are discretized with the present method giving a standard linear eigenvalue problem. The effectiveness of this method in predicting free vibration behavior of FGM grading reinforced panels was checked by comparing its results for isotropic FGM condition with corresponding numerical results available in the literatures. From this study, some conclusions can be made:

· It is shown that only seven DQ grid in the thickness direction can yield accurate results.

· The formulation is stable while increasing the number of points and that the use of 30 points guarantees convergence of the procedure.

· We investigated the advantages of using functionally graded fiber reinforced composite panel with graded fiber volume fractions over traditional discretely laminated composite panel. Results indicate that significant improvement is found in natural frequency of a functionally graded fiber reinforced composite panel due to the reduction in spatial mismatch of material properties and natural frequency.

· It is shown, upon increasing the middle surface radius to thickness ratio, S, in a thick grading reinforced panels the normalized natural frequency decreases rapidly and finally reaches the constant value in the thin panel and this behavior is similar to discretely laminated composite panels.

· It is shown, upon increasing the middle surface radius to length ratio (*R/L*), in a grading reinforced panel, the normalized natural frequency decreases rapidly and finally reaches a constant value.

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