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The Homotopy Perturbation Method for free vibration analysis of beam on elastic foundation

Baki Ozturk*[1] and Safa Bozkurt Coskun[2]

[1]*Department of Civil Engineering, Faculty of Engineering and Architecture, Nigde University, 51100 Nigde, Turkey*
[2]*Department of Civil Engineering, Faculty of Engineering, Kocaeli University, 41380 Kocaeli, Turkey*

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Abstract. In this study, the homotopy perturbation method (HPM) is applied to free vibration analysis of beam on elastic foundation. This numerical method is applied on three different axially loaded cases, namely: 1) one end fixed, the other end simply supported; 2) both ends fixed and 3) both ends simply supported cases. Analytical solutions and frequency factors are evaluated for different ratios of axial load N acting on the beam to Euler buckling load, N_r. The application of HPM for the particular problem in this study gives results which are in excellent agreement with both analytical solutions and the variational iteration method (VIM) solutions for all the cases considered in this study and the differential transform method (DTM) results available in the literature for the fixed-pinned case.

Keywords: homotopy perturbation method; beam on elastic foundation; free vibration analysis.

1. Introduction

It is known that free vibration equation of the axially loaded beam on elastic foundation is a fourth-order partial differential equation which is difficult to solve analytically. For this particular engineering problem, numerical methods can provide approximate solutions rather than the analytical solutions of problems. Although, the governing equation seems to be a linear one, finding the eigen values for the free vibration analysis is still challenging. One may not simply obtain the eigen values sequentially and their corresponding eigen vectors even with a software. However, by using the homotopy perturbation method (HPM), eigen values for the problem would be easily obtained by implementing a simple procedure for the governing equation.

Free vibration equation of the beam on partially elastic foundation including only bending moment effect was analytically solved (Doyle and Pavlovic 1982) while the eigenvalues for free vibration of column-beam systems on elastic foundation were obtained using a numerical approach (West and Mafi 1984). The separation of variables technique was used to obtain the free vibration circular frequencies of piles partially embedded in soils (Çatal 2002). In addition, differential transform method (DTM) has been proposed to solve eigen value problems for free and transverse vibration problems of a rotating twisted Timoshenko beam under axial loading (Chen and Ho 1996,

*Corresponding author, Assistant Professor, E-mail: bakiozturk@nigde.edu.tr

1999). Furthermore, the DTM was also used to find the non-dimensional natural frequencies of tapered cantilever Bernoulli-Euler beam (Özdemir and Kaya 2006). Free vibration equations for one end fixed and other end simply supported beam on elastic foundation was solved by using the DTM for various axial loads acting on the beam (Çatal 2008). Meanwhile, the variational iteration method (VIM) was used to solve the free vibration equations of beam on elastic foundation for support conditions of one end fixed and other end simply supported, both ends fixed and both ends simply supported considering various case studies (Öztürk 2009). Recently, there have been also other studies which are helpful to better understand dynamic behavior of both infinite beams resting on elastic foundation (Raftoyiannis *et al.* 2010) and tapered column with pinned ends embedded in Winkler-Pasternak elastic foundation (Civalek and Öztürk 2010).

In this study, the homotopy perturbation method (HPM) is introduced to solve the free vibration equations of beam on elastic foundation for support conditions of one end fixed and other end simply supported, both ends fixed and both ends simply supported. The beam on elastic foundation is investigated for these three different support conditions considering various case studies. It has to be emphasized that in the VIM formulation the governing equation is multiplied by Lagrange multiplier for the corresponding iteration step and integrated afterwards which makes the integral evaluation very difficult at larger iteration numbers. On the contrary, only the governing equation is solved for each iteration step in the HPM formulation. In addition, the approximation for any iteration number is not obtained by adding up the solutions of all previous iterations which is the case for the VIM formulation. Hence, VIM requires much more computer time and memory which is not practical when compared to HPM. The issues explained above are the merits of HPM over VIM. It is also known that the HPM formulation is a simpler one in comparison with the DTM formulation.

The concept of homotopy perturbation method was first proposed by He (2000, 2004a) and has been applied to solve different engineering problems (He 2004b, 2005, 2006, Atay 2009, Coskun 2009). Detailed formulation of the method is provided below.

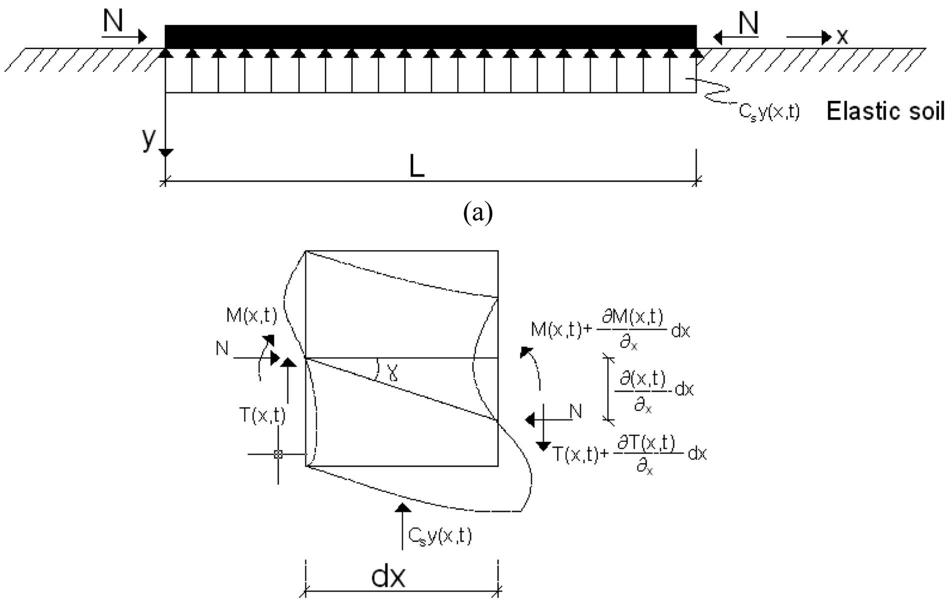
2. Problem formulation

Beam on elastic foundation and internal forces and deformations of differential segment of the beam having a length of dx are depicted in Figs. 1(a) and 1(b), respectively. The elastic foundation is idealized by Winkler model. Hence, the relationship between displacement function $y(x, t)$ of the beam on elastic foundation and the distributed force $q(x, t)$ applied on the elastic foundation beneath the beam can be written by $q(x, t) = C_S y(x, t)$. In this equation $C_s = C_o b$ where C_o is the modulus of sub grade reaction while b is the beam width.

The equilibrium equations of the internal forces acting on differential beam segment, dimensionless parameter z instead of position variable x with $0 \leq z \leq x/L$ and neglecting the second-order terms are used in order to define the motion equation of the beam on elastic foundation (Çatal 2002)

$$\left\{\phi_1^{iv}(z) + \left[\pi^2 N_r + \frac{(\overline{m}\omega^2 - C_S)\overline{k}L^2}{AG}\right]\phi_1^{ii}(z) + \frac{(C_S - \overline{m}\omega^2)L^4}{EI}\phi_1(z)\right\}\sin(\omega t + \theta) = 0 \tag{1}$$

In Eq. (1) $\phi_1(z)$ is dimensionless displacement function of the beam considering axial and shear



(a)

(b)

Fig. 1 (a) beam on elastic foundation and (b) internal forces and deformations of the beam on elastic foundation

forces; t is the time variable; θ is the phase angle; $N_r = NL^2/(\pi^2 EI)$ is the ratio of axial load N acting on the beam to Euler buckling load; $\overline{m}$ is the distributed mass of the beam; ω is the beam circular frequency; $\overline{k}$ is the shape factor for the shape of the beam section considered; L is the beam length; A, G, E, I are cross-section area, shear modulus, elastic modulus and moment of inertia of the beam; $\phi_1^{ii}(z) = d^2\phi_1(z)/dt^2$, $\phi_1^{iv}(z) = d^4\phi_1(z)/dt^4$, respectively. If the axial and shear force effects are neglected, the dimensionless equation of motion for the beam on elastic foundation becomes as follows (Tuma and Cheng 1983)

$$\left\{\phi_2^{iv}(z) + \frac{(C_S - \overline{m}\omega^2)L^4}{EI}\phi_2(z)\right\}\sin(\omega t + \theta) = 0 \tag{2}$$

In Eq. (2), $\phi_2(z)$ is the dimensionless displacement function of the beam neglecting axial and shear forces. Division of both sides of Eqs. (1) and (2) by $\sin(\omega t + \theta)$ gives the following equations

$$\left\{\phi_1^{iv}(z) + \left[\pi^2 N_r + \frac{(\overline{m}\omega^2 - C_S)\overline{k}L^2}{AG}\right]\phi_1^{ii}(z) + \frac{(C_S - \overline{m}\omega^2)L^4}{EI}\phi_1(z)\right\} = 0 \tag{3}$$

$$\left\{\phi_2^{iv}(z) + \frac{(C_S - \overline{m}\omega^2)L^4}{EI}\phi_2(z)\right\} = 0 \tag{4}$$

3. The homotopy perturbation method

In recent years, application of perturbation techniques in nonlinear problems has been increased by many researchers and engineers. In these studies, He's Homotopy Perturbation Method (He 2000, 2004a, b, 2005, 2006, Atay 2009, Coskun 2009) is one of the most promising for nonlinear problems. The HPM provides an analytical approximation for problems at hand. In order to apply this technique, the brief theoretical steps are given in following section. $L(u)+N(u)=f(r)$, $r \in \Omega$ with boundary conditions $B(u, \partial u/\partial n) = 0$, $r \in \Gamma$ where L is a linear operator, N is nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω and $f(r)$ is a known analytic function. Homotopy Perturbation technique, which was described in detail in (He 2000, 2006), defines homotopy as $v(r,p) \rightarrow \Omega \times [0,1] \rightarrow R$ which satisfies following inequalities

$$H(v,p) = (1-p)[L(v)-L(u_0)]+p[L(v)+N(v)-f(r)] = 0 \tag{5}$$

or

$$H(v,p) = L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)] = 0 \tag{6}$$

where $r \in \Omega$ and $p \in [0,1]$ is an imbedding parameter, u_0 is an initial approximation which satisfies the boundary conditions. Obviously, from Eq. (5) and Eq. (6), we have

$$H(v,0) = L(v)-L(u_0) = 0 \tag{7}$$

$$H(v,1) = L(v)+N(v)-f(r) = 0 \tag{8}$$

The changing process of p from zero to unity is that of $v(r,p)$ from u_0 to $u(r)$. In topology, this deformation $L(v)-L(u_0)$ and $L(v)+N(v)-f(r)$ are called homotopic. The basic assumption is that the solutions of Eq. (5) and Eq. (6) can be expressed as a power series in p such that

$$v = v_0+pv_1+p^2v_2+p^3v_3+\ldots \tag{9}$$

Hence, for the present problem the solution can be expressed as

$$\phi_1 = v_0+pv_1+p^2v_2+p^3v_3+\ldots \tag{10}$$

Approximate solution of $L(\phi_1)+N(\phi_1) = f(r)$, $r \in \Omega$ in view of Eq. (10) can be obtained from the following

$$\phi_1 = \lim_{p \to 1} v = v_0+v_1+v_2+\ldots \tag{11}$$

The convergence of the series in Eq. (11) has been proved in (He 2000, 2004a, b, 2005, 2006). Eq. (3) can be divided into two parts by introducing the linear and nonlinear operators as follows

$$L = \frac{d}{dz^4} \tag{12}$$

$$N = \left[\pi^2 N_r + \frac{(\overline{m}\omega^2-C_S)\overline{k}L^2}{AG}\right]\frac{d^2}{dz^2} + \frac{(C_S-\overline{m}\omega^2)L^4}{EI} \tag{13}$$

with

$$f(r) = 0 \tag{14}$$

Defined homotopy for the problem should satisfy Eq. (6) which becomes for the present case as

$$H(v,p) = L(v) - L(\phi_1^0) + pL(\phi_1^0) + p[N(v) - f(r)] = 0 \tag{15}$$

ϕ_1^0 is an initial solution satisfying boundary conditions for the problem. Inserting Eq. (10) into Eq. (15) with the help of the Eqs. (12)-(14), and equating the like powers of the parameter p and, then solving each equation yields the following iteration procedure.

$$p^0: \quad v_0^{iv} - (\phi_1^0)^{iv} = 0 \tag{16}$$

$$p^1: \quad v_1^{iv} + (\phi_1^0)^{iv} + \left[\pi^2 N_r + \frac{(\overline{m}\omega^2 - c_s)\overline{k}L^2}{AG}\right] v_0'' + \frac{(c_s - \overline{m}\omega^2)L^4}{EI} v_0 = 0 \tag{17}$$

$$p^2: \quad v_2^{iv} + \left[\pi^2 N_r + \frac{(\overline{m}\omega^2 - c_s)\overline{k}L^2}{AG}\right] v_1'' + \frac{(c_s - \overline{m}\omega^2)L^4}{EI} v_1 = 0 \tag{18}$$

$$p^3: \quad v_3^{iv} + \left[\pi^2 N_r + \frac{(\overline{m}\omega^2 - c_s)\overline{k}L^2}{AG}\right] v_2'' + \frac{(c_s - \overline{m}\omega^2)L^4}{EI} v_2 = 0 \tag{19}$$

$$\vdots$$

$$p^n: \quad v_n^{iv} + \left[\pi^2 N_r + \frac{(\overline{m}\omega^2 - c_s)\overline{k}L^2}{AG}\right] v_{n-1}'' + \frac{(c_s - \overline{m}\omega^2)L^4}{EI} v_{n-1} = 0 \tag{20}$$

After conducting n iterations, the result would be obtained by the use of Eq. (10). Based on the iteration formula given betwen Eqs. (16)-(20), fifteen iterations were conducted to perform the analysis of specified problem. The iteration formula is a simple approximation and it is expected to be an important contribution of HPM to the current problem.

Choice of an initial solution is very important in homotopy perturbation method. Hesameddini and Latifizadeh (2009a, b) discuss about an optimal choice of such an approximation. Coskun (2010) discusses about choosing a cubic polynomial as an optimal initial approximation for beam and column problems.

Hence, as an initial approximation, following solution of $L\phi = 0$ was used.

$$\phi_1^0(z) = Az^3 + Bz^2 + Cz + D \tag{21}$$

Once the iteration process is completed, four boundary conditions for each case are applied to the resulting approximation and hence four equations are obtained for each case analyzed which can be shown in matrix form as follows

$$[K(\omega)]\{A\} = 0 \tag{22}$$

where $\{A\} = \langle A\ B\ C\ D\rangle^T$. For a nontrivial solution, determinant of coefficient matrix must be zero. Determinant of coefficient matrix $[K(\omega)]$ yields a characteristic equation in terms of ω. Positive real roots of this equation are free vibration frequencies of the beams on elastic foundations shown in Fig. 2.

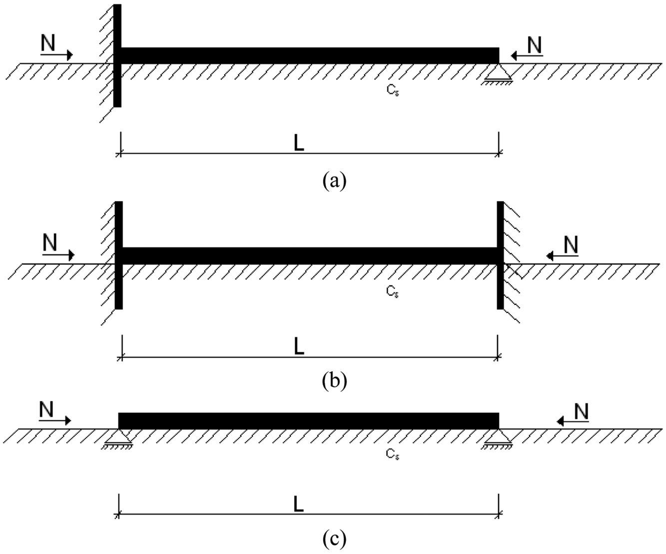


Fig. 2 (a) One end fixed and other simply supported beam on elastic foundation, (b) both ends fixed, (c) both ends pinned

4. Numerical study

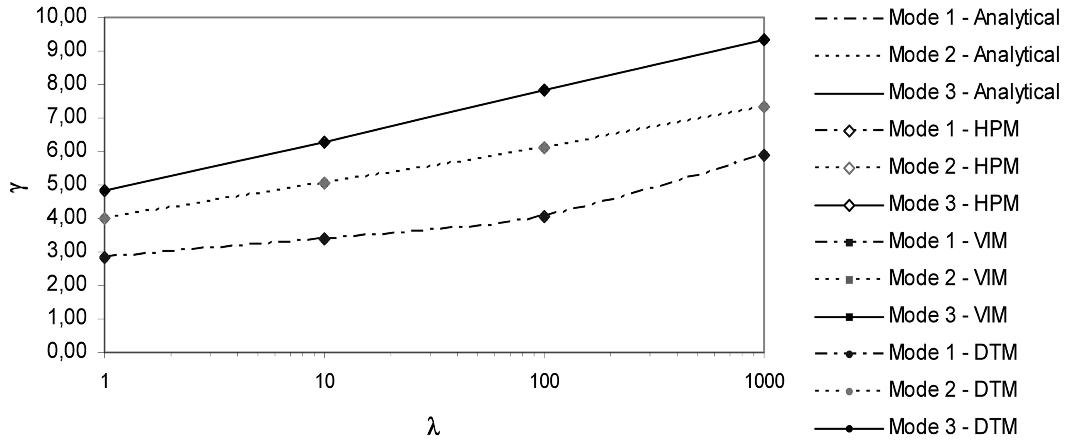
Free vibration of a one end fixed and one end pinned beam on elastic foundation by using DTM was previously analyzed (Çatal 2008). In addition, the variational iteration method (VIM) was used to solve the free vibration equations of beam on elastic foundation for support conditions of one end fixed and other end simply supported, both ends fixed and both ends simply supported considering various case studies (Öztürk 2009). In this study, three cases are considered which are shown in Fig. 2. Each case is analyzed by the use of HPM with the previously explained procedure. First case is a beam on elastic foundation with one end fixed and other end is pinned. Second case considers the same beam with both ends fixed and the third case is a beam with both ends pinned. Numerical values are chosen as the same used in (Çatal 2008). Hence, an IPB 500 steel profile resting on a Winkler foundation having a modulus of sub grade reaction of 50,000 kN/m^2 is considered. Other numerical values are as follows:

$I = 107.2 \times 10^{-5}\ \mathrm{m}^4$; $A = 2.39 \times 10^{-2}\ \mathrm{m}^2$; $\overline{m} = 0.19\ \mathrm{kN\ s^2/m}$; $\overline{k} = 3.705$;
$E = 2.1 \times 10^8\ \mathrm{kN/m^2}$; $G = 8.1 \times 10^7\ \mathrm{kN/m^2}$

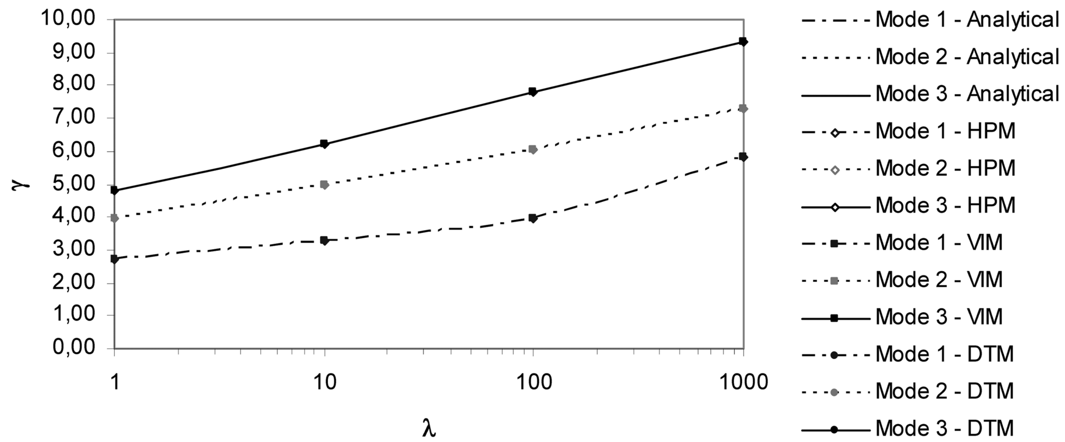
Frequency factors $\gamma = \sqrt[4]{\dfrac{\overline{m}\,\omega^2 L^4}{EI}}$ are calculated taking bending moment, shear and axial effects into consideration for $N_r = 0.25$, $N_r = 0.5$ and $N_r = 0.75$ due to circular frequencies of the beam

Variation of frequency factors are shown in between Figs. 3-5. Fig. 3 includes comparison with both analytical solution and the results obtained by using DTM (Çatal 2008). Figs. 4, 5 show the comparison of analytical solution and the HPM results obtained in this study.

The results shown in between Figs. 3 and 5 seem to be in excellent agreement with both DTM results for fixed-pinned case (Çatal 2008), and with analytical solutions and the VIM results (Öztürk 2009) for all the cases considered in this study.



(a)



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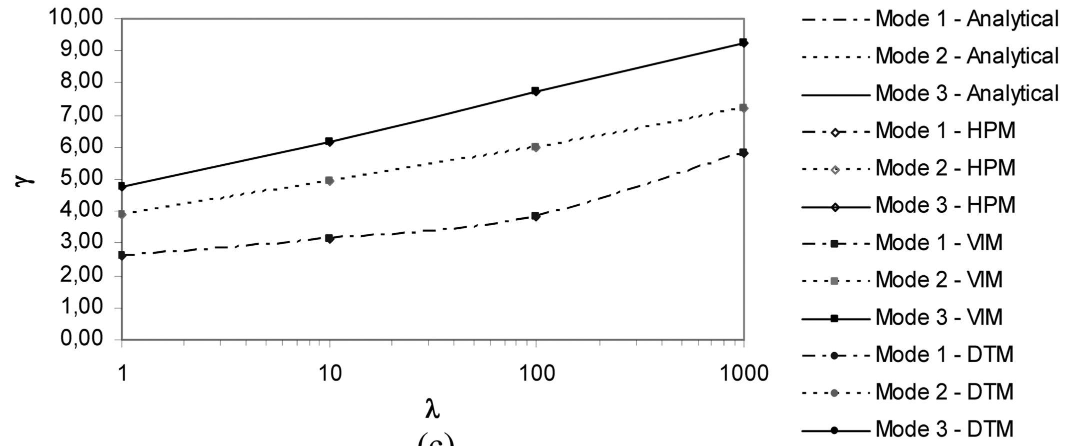
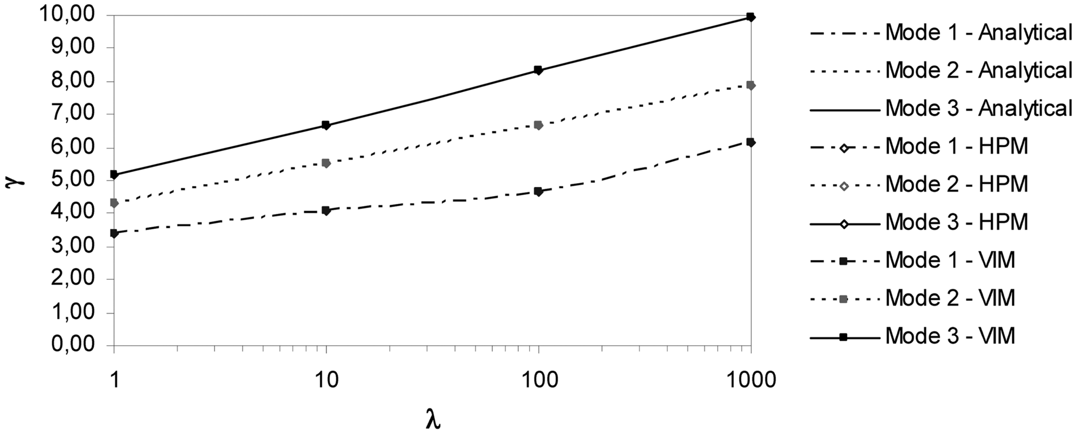
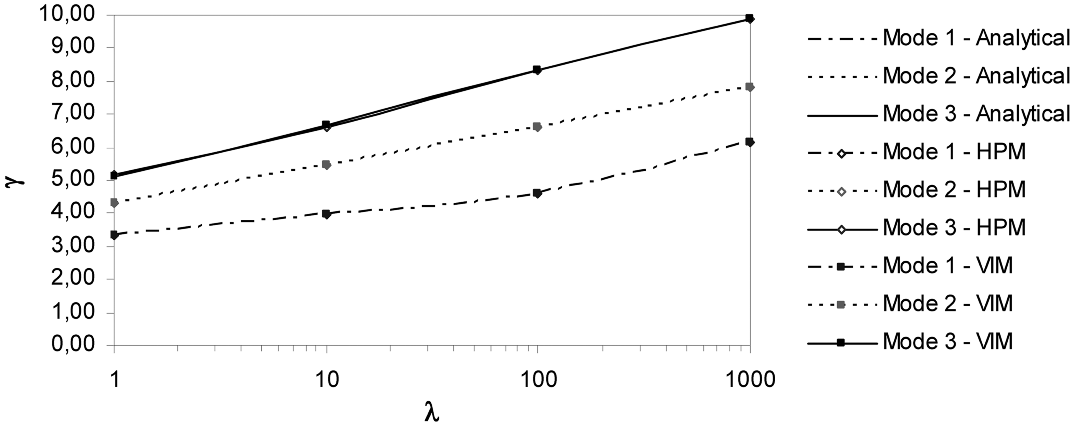


(c)

Fig. 3 Variation of frequency factors due to relative stiffness for one end fixed and other simply supported beam on elastic foundation: (a) for $N_r = 0.25$, (b) for $N_r = 0.5$, (c) for $N_r = 0.75$



(a)



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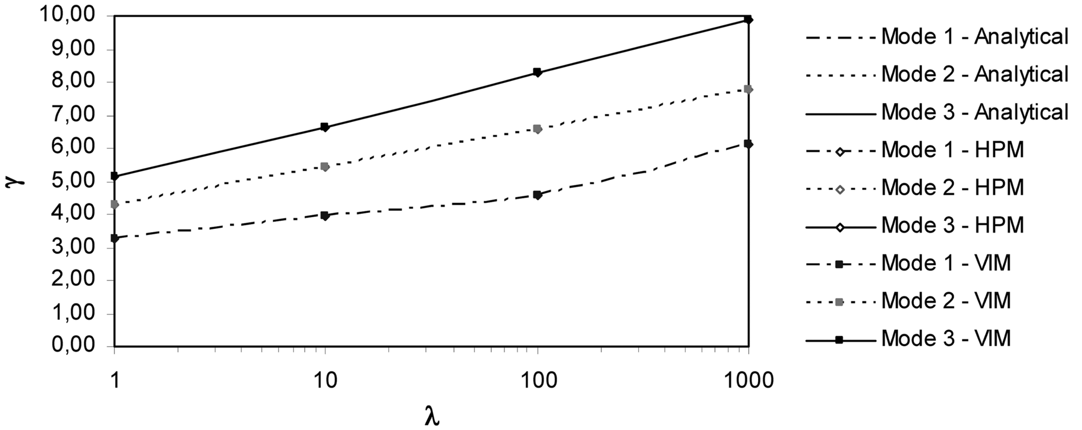
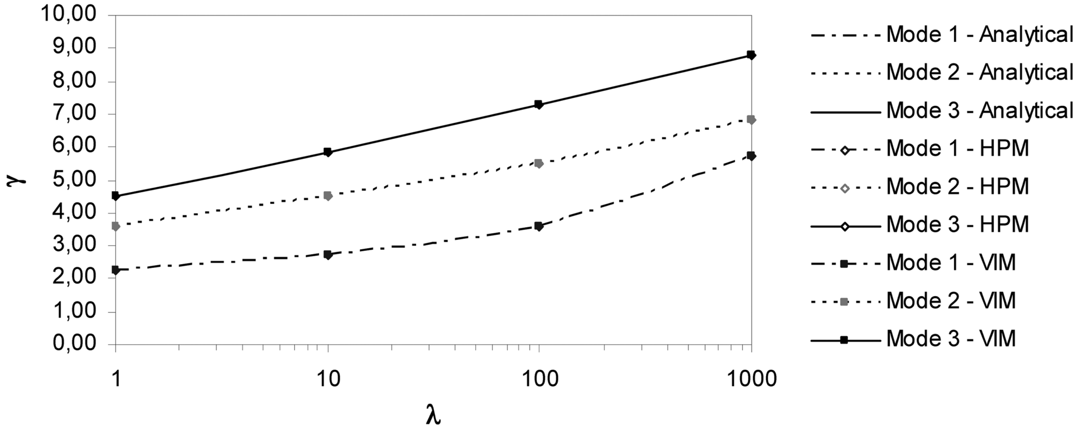
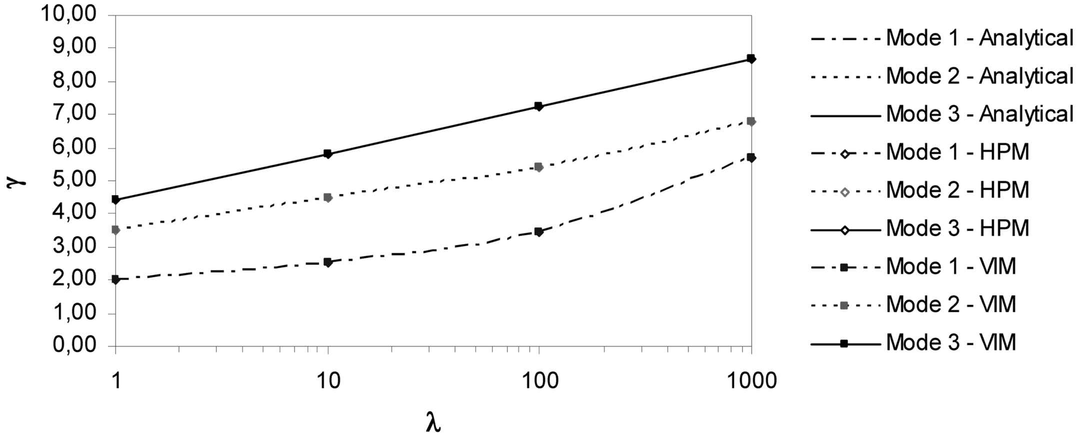


(c)

Fig. 4 Variation of frequency factors due to relative stiffness for both ends fixed beam on elastic foundation: (a) for $N_r = 0.25$, (b) for $N_r = 0.5$, (c) for $N_r = 0.75$



(a)



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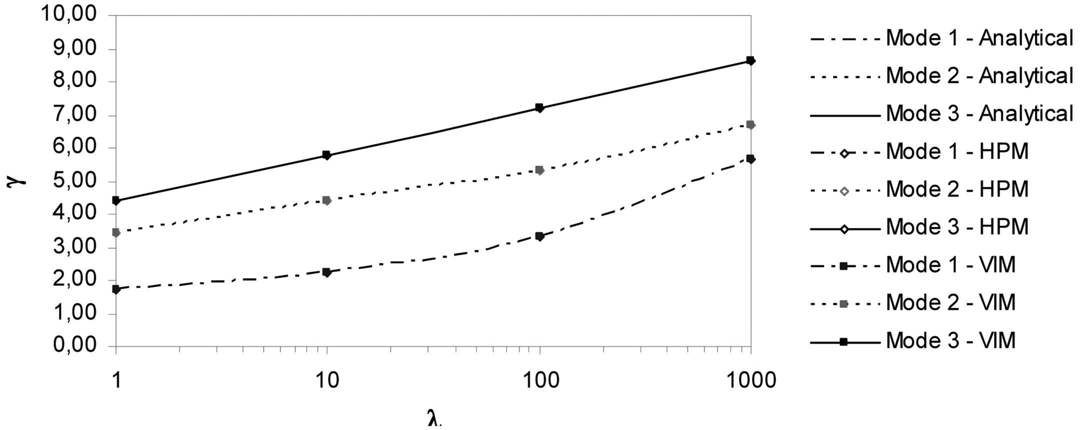

(c)

Fig. 5 Variation of frequency factors due to relative stiffness for both ends pinned beam on elastic foundation: (a) for $N_r = 0.25$, (b) for $N_r = 0.5$, (c) for $N_r = 0.75$

4. Conclusions

In this study, the homotopy perturbation method (HPM) is used for the free vibration analysis of beam on elastic foundation. The HPM is applied on three different axially loaded cases of beam on elastic foundation which are namely one end fixed, the other end simply supported; both ends fixed and both ends simply supported cases. Fifteen iterations are conducted for each case considered. This analytical study revealed that application of the HPM for the particular problem in this study gives results which are in excellent agreement with both analytical solutions and the VIM results for all the cases considered, and the differential transform method (DTM) results available in the literature for the fixed-pinned case. The HPM offers a faster and more accurate solution for the particular problem compared to other approximation techniques referred above and the analytical solution approach.

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