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Nonlinear vibration of hybrid composite plates on elastic foundations

Wei-Ren Chen*[1], Chun-Sheng Chen[2] and Szu-Ying Yu[3]

[1]*Department of Mechanical Engineering, Chinese Culture University, Taipei 11114, Taiwan*
[2]*Department of Mechanical Engineering, Lunghwa University of Science and Technology, Guishan Shiang 33306, Taiwan*
[3]*Department of Electrical Engineering, Lee Ming Institute of Technology, Taishan 24305, Taiwan*

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Abstract. In this paper, nonlinear partial differential equations of motion for a hybrid composite plate subjected to initial stresses on elastic foundations are established to investigate its nonlinear vibration behavior. Pasternak foundation and Winkler foundations are used to represent the plate-foundation interaction. The initial stress is taken to be a combination of pure bending stress plus an extensional stress in the example problems. The governing equations of motion are reduced to the time-dependent ordinary differential equations by the Galerkin's method. Then, the Runge-Kutta method is used to evaluate the nonlinear vibration frequency and frequency ratio of hybrid composite plates. The nonlinear vibration behavior is affected by foundation stiffness, initial stress, vibration amplitude and the thickness ratio of layer. The effects of various parameters on the nonlinear vibration of hybrid laminated plate are investigated and discussed.

Keywords: initial stress; Winkler foundation; Pasternak foundation; hybrid composite plate.

1. Introduction

Composite laminated structures are being increasingly used in aeronautical and aerospace constructions. The study of nonlinear vibration of plates has attracted many researchers' attention because it plays a key role in designing resonant-free structural components. A considerable amount of work dealing with nonlinear vibration of composite plates is available in the published literature and few of them (Sinfh 2000, Polit and Touratier 2000, Harras *et al*. 2002, Onkar and Yadav 2005, Ye *et al*. 2005, Kazanci and Mecitoglu 2006, Singha and Daripa 2007, Lal *et al*. 2008, Amabili and Farhadi 2009) are reported in the references. However, they merely investigated the vibration of plates laminated with a single material.

Fiber-reinforced polyester laminates have been successfully applied in many engineering applications. Similarly, the hybrid composite plates laminated with various materials are widely used in different fields. The vibration of hybrid laminated plates was studied by Barai and Durvasula (1992) to investigate the effects of stacking sequence and ply-orientation. The vibration frequencies

*Corresponding author, Ph.D., E-mail: wrchen@faculty.pccu.edu.tw

of hybrid laminate panels lie in between the values for laminates made of all plies of higher strength and lower strength fibers, respectively. Xu *et al*. (1997) dealt with the vibration frequencies of hybrid piezothermoelastic plates. The effects of variation in the plate thickness and the location of the hybrid layers on the vibration frequencies and their sensitivity coefficients were presented. An analytical solution of simply supported piezoelectric adaptive plates was presented by Benjeddou *et al*. (2002). The vibration of laminated composite plates with embedded or surface-bonded piezoelectric layers was analyzed by Topdar *et al*. (2007). An efficient plate theory was applied to model the variation of displacements across the thickness to ensures inter-laminar shear stress continuity and stress free condition at the plate top and bottom surfaces. The finite element technique was used to approximate the in-plane variation of displacement parameters at the reference plane and electric potential at the different interfaces. Dumir *et al*. (2009) developed a modified plate theory for hybrid piezoelectric plates with geometric nonlinearity to obtain the vibration response of laminated hybrid panels. The coupled nonlinear equations of motion and the boundary conditions were derived by using the extended Hamilton's principle. Ibrahim *et al*. (2009) studied the vibration behavior of shape-memory alloy hybrid composite panels under the combined effect of thermal and aerodynamic loads. The Newton-Raphson method was employed to calculate the nonlinear deflections, while an eigenvalue problem was solved at each temperature step and aerodynamic load to predict the vibration frequencies about the deflected equilibrium position. A lot of references in the specialized monographs by Ossadzow and Touratier (2003), Huang and Shen (2005), Kapuria and Achary (2005), Chen *et al*. (2006) fully attest this statement.

When a composite material is subjected to high temperature or corrosive, moisture surrounding, its mechanical properties may degrade (Patel *et al*. 2002). To enhance the mechanical properties of composites under such environment, it is possible to combine metal and fiber reinforced composites to form a hybrid laminate material by covering the composite material with a layer of metallic material. The nonlinear vibration analysis of hybrid plates laminated with aluminum and fiber reinforced composite was presented by Lee and Kim (1996). The Lagrangian equation was used to analyze the nonlinear vibration of laminated hybrid composite plates. The effects of stacking sequences, aspect ratios, number of modes, number of layers and elastic properties on the vibration were investigated and discussed. Harras *et al*. (2002) presented a theoretical model based on Hamilton's principle to study the nonlinear free vibration of a glare3 hybrid composite plate made up of alternating layers of metal and fiber reinforced composites. Various types of residual stress in structures might be induced after a manufacturing or an assembly process. Such a stress is considered as the initial stress in a structure before an external force is applied to it. The author and coworkers developed an approach for analyzing the linear (Chen *et al*. 2009, Chen *et al*. 2009) and nonlinear vibration (Chen *et al*. 2005, Chen and Fung 2004) of initially stressed hybrid laminated plates.

In many structural engineering applications, plates placed on an elastic medium are frequently encountered (Lal *et al*. 2007, Ayvaz and Oguzhan 2008, Darilmaz 2009). And the vibration of plates under various initial stresses had attracted some researchers' attention (Cheung *et al*. 1998, Muthurajan *et al*. 2005, Garg 2007, Kapuria and Achary 2008, Lu and Li 2009). Thus, the study of the nonlinear vibration analysis of laminated hybrid composite plates resting on elastic foundations is of importance in the optimum design of hybrid composite structures. Due to the complexity of nonlinear vibration, not much literature has been found on the study of the nonlinear vibration of initially stressed hybrid composite plates resting on elastic foundations. In the present paper, the nonlinear governing equations of an initially stressed hybrid composite plate resting on the elastic foundation are derived by the Hamilton's energy principle. The elastic foundation is represented as a Pasternak model that is characterized by two parameters, the vertical spring modulus of foundation (k) and the shear modulus of foundation (g_x, g_y). The Winkler model is obtained by neglecting the shear modulus of foundation in the Pasternak model. The initial stress is taken to be a combination of a pure bending stress and an extensional stress. The Galerkin method is employed to yield ordinary differential equations from the governing partial differential equations. The ordinary differential equations are then solved by the Runge-Kutta method to study the effects of the foundation stiffness, initial conditions and thickness ratio of layer on the behavior of nonlinear vibration. Two types of simply supported hybrid plates staked with laminates of aluminum and CFRP (or AFRP) layers subjected to an initial stress and rested on Winkler foundation and Pasternak foundation are investigated. The effects of various parameters on the nonlinear vibration frequency and frequency ratio are discussed.

2. Governing equations

Following a similar technique described by Brunelle and Robertson (1976) and Chen *et al.* (2001), Hamilton's principle is used to derive the nonlinear governing equations of the hybrid plate. For an initially stressed plate which is in static equilibrium and subjected to a time-varying incremental deformation, the Hamilton's energy principle can be expressed as

$$\delta\int_{t_0}^{t_1}(U_s+U_f-U_k-W_e-W_i)dt = 0 \tag{1}$$

where

$$U_s = \int_{V_0}\sigma_{ij}\varepsilon_{ij}dV$$

$$U_f = \int_{s_0}f_e v_z dS$$

$$U_k = \frac{1}{2}\int_{V_0}\rho\dot{v}_i\dot{v}_i dV$$

$$W_e = \int_{S_0}p_i v_i dS$$

$$W_i = \int_{v_0}X_i v_i dV$$

Here U_s, U_f, U_k, W_e, and W_i are the strain energy, strain energy of the foundation, kinetic energy, work due to external and internal forces, respectively. σ_{ij} and ε_{ij} are the stress and strain referred to the material coordinates (Brunelle and Robertson 1976); f_e is the density of reaction force of foundation; v_i is the displacement referred to the spatial frame; X_i is the body force per unit initial volume and p_i is is the surface force per unit initial surface area. Assume that the stresses and applied forces are constants. Then by introducing the integral forms of U_s, U_f, K_t, W_e and W_i into Eq. (1), carrying out the variation, integrating the kinetic energy term by parts with respect to time and using the assumption that δv_i vanishes at time t_0 and t_1, Eq. (1) becomes

$$\int_{t_0}^{t_1}\left[\int_{V_0}(\sigma_{ij}\delta\varepsilon_{ij}-X_i\delta v_i-\rho\ddot{v}\delta v_i)dV+\int_{S_0}(f_e\delta v_z-p_i\delta v_i)dS\right]dt = 0 \tag{2}$$

In this study, the rectangular plate is considered so the equations and boundary conditions would be rephrased in x, y coordinates. The constitutive relations for a kth lamina with respect to the laminate coordinate axes may be written in the following form

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix}^{(k)} = \begin{bmatrix}C_{11} & C_{12} & C_{16}\\ C_{12} & C_{22} & C_{26}\\ C_{16} & C_{26} & C_{66}\end{bmatrix}^{(k)}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{bmatrix}^{(k)} \tag{3}$$

where C_{ij} are the stiffness coefficients. The incremental displacements are assumed to be the following forms

$$v_x(x,y,z,t) = u_x(x,y,t)-zw_{,x}(x,y,t)$$
$$v_y(x,y,z,t) = u_y(x,y,t)-zw_{,y}(x,y,t)$$
$$v_z(x,y,z,t) = w(x,y,t) \tag{4}$$

In this study, von Karman's assumptions are used. Only those nonlinear terms that depend on $v_{z,x}, v_{z,y}$ are to be retained in the strain-displacement relations. All other nonlinear terms are to be neglected. Hence, kinematic relations can be expressed as

$$\varepsilon_{xx} = v_{x,x}+\frac{1}{2}v_{z,x}^2 = u_{x,x}-zw_{,xx}+w_{,x}^2/2$$

$$\varepsilon_{yy} = v_{y,y}+\frac{1}{2}v_{z,y}^2 = u_{y,y}+zw_{,yy}+w_{,y}^2/2$$

$$\varepsilon_{xy} = v_{x,y}+v_{y,x}+v_{z,x}v_{z,y} = u_{x,y}+zw_{,xy}+u_{y,x}+zw_{,yx}+w_{,x}w_{,y} \tag{5}$$

The Winkler and Pasternak foundation models are used to describe the plate-foundation interaction in this study. Their respective load-displacement relationships are expressed as follows

$$f_e = kw(x,y)-g_xw(x,y)_{,xx}-g_yw(x,y)_{,yy} \tag{6}$$

where f_e is the force per unit area, k is the modulus of subgrade reaction (elastic foundation stiffness) and g_x, g_y are the shear module of the subgrade (shear layer foundation stiffness) in x, y coordinates, as shown in Fig. 1. When the elastic foundation is represented by a Pasternak model, it can be characterized by two module, the vertical spring modulus of foundation (k) and the shear modulus of foundation (g_x, g_y), respectively. By neglecting the shear modulus of foundation, the elastic foundation is considered as Winkler model. If foundation is homogeneous and isotropic, one can let $g_x = g_y = g$. Basically, the one-parameter model of Winkler foundation can be considered as a system of closely spaced linear springs. The two-parameter model of Pasternak foundation can be thought as a system of closely spaced linear springs coupled with a shear force proportional to the slope of the foundation surface.

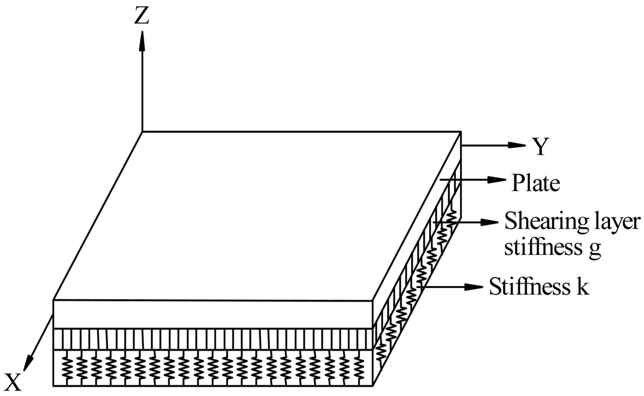

Fig. 1 Pasternak elastic foundation

Substitute Eqs. (3)-(6) into Eq. (2) and perform all necessary partial integrations and group the terms together by the displacement variation to yield the nonlinear governing equations of motion as

$$\begin{aligned}&[A_{11}(u_{x,x}+w_{,x}^2/2)+A_{12}(u_{y,y}+w_{,y}^2/2)+A_{16}(u_{x,y}+u_{y,x}+w_{,x}w_{,y})+B_{11}\,w_{,xx}\\&+B_{12}\,w_{,yy}+2B_{16}w_{,xy}+N_{xx}u_{x,x}-M_{xx}w_{,xx}+N_{xy}u_{x,y}-M_{xy}w_{,xy}+N_{xz}u_{z,x}]_{,x}\\&+[A_{12}(u_{x,x}+w_{,x}^2/2)+A_{22}(u_{y,y}+w_{,y}^2/2))+A_{66}(u_{x,y}+u_{y,x}+w_{,x}w_{,y})\\&+B_{12}\,w_{,xx}+B_{22}\,w_{,yy}+2B_{66}\,w_{,xy}+N_{yy}u_{x,y}-M_{yy}w_{,xy}+N_{xy}u_{x,x}+M_{xy}w_{,xx}\\&+N_{yz}u_{z,x}]_{,y}+f_x=I_1\ddot{u}_x\end{aligned} \tag{7}$$

$$\begin{aligned}&[A_{16}(u_{x,x}+w_{,x}^2/2)+A_{26}(u_{y,y}+w_{,y}^2/2)+A_{66}(u_{x,y}+u_{y,x}+w_{,x}w_{,y})+B_{16}\,w_{,xx}\\&+B_{26}\,w_{,yy}+2B_{66}\,w_{,xy}+N_{xx}u_{y,x}-M_{xx}w_{,yx}+N_{xy}u_{y,y}-M_{xy}w_{,yy}+N_{xz}u_{z,y}]_{,x}\\&+[A_{12}(u_{x,x}+w_{,x}^2/2)+A_{22}(u_{y,y}+w_{,y}^2/2)+A_{26}(u_{x,y}+u_{y,x}+w_{,x}w_{,y})\\&+B_{12}\,w_{,xx}+B_{22}\,w_{,yy}+2B_{26}w_{,xy}+N_{yy}u_{y,y}-M_{yy}w_{,yy}+N_{xy}u_{y,x}-M_{xy}w_{,yx}\\&+N_{xz}u_{z,y}]_{,y}+f_y=I_1\ddot{u}_y\end{aligned} \tag{8}$$

$$\begin{aligned}&\{[B_{11}u_{x,x}+B_{12}\,w_{,yy}-D_{11}w_{,xx}-D_{12}\,w_{,yy}+M_{xx}u_{x,x}-M_{xx}^*w_{,xx}+M_{xy}u_{x,y}+M_{xz}u_{z,x}\\&-M_{xy}^*w_{,xy}]_{,x}+[B_{66}(u_{x,y}+u_{y,x})+B_{66}\,w_{,x}w_{,y}-2D_{66}\,w_{,xy}+M_{yy}u_{x,y}-M_{xy}^*w_{,xy}\\&+M_{xy}u_{x,x}-M_{xy}^*w_{,xx}+M_{yz}u_{z,x}]_{,y}-(N_{xz}u_{x,x}-M_{xz}w_{,xx}-N_{zz}w_{,xx}+N_{zy}u_{x,y}\end{aligned}$$

$$-M_{zy}w_{,xy})\ +[A_{11}(u_{x,x}+w_{,x}^2/2)+A_{12}(u_{y,y}+w_{,y}^2/2)-B_{11}w_{,xx}-B_{12}w_{,yy}$$
$$+A_{66}(u_{x,y}+u_{y,x}+w_{,x}w_{,y})-2B_{66}w_{,xy}]w_{,x}+N_{xx}w_{,x}+N_{xy}w_{,y}\}_{,x}$$
$$+\{[B_{66}(u_{x,y}+u_{y,x})\ +B66\,w_{,x}w_{,y}-2D66\,w_{,xy}+M_{xx}u_{y,x}+M_{xy}u_{y,y}+M_{xz}u_{z,y}$$
$$-M_{xx}^{*}w_{,yx}-M_{xy}^{*}w_{,yy}]_{,x}+[B_{12}u_{x,x}+B_{22}u_{y,y}+B_{11}w_{,x}^2/2+B_{12}w_{,y}^2/2$$
$$+D_{12}w_{,xx}-D_{22}w_{,yy}+M_{yy}u_{y,y}-M_{yy}^{*}w_{,yy}-M_{xy}w_{,yx}+M_{xy}^{*}u_{y,x}+M_{xz}u_{z,y}]_{,y}$$
$$-(N_{xz}u_{y,x}-M_{xz}w_{,yx}-N_{zz}w_{,yx}+N_{zy}u_{y,y}-M_{zy}w_{,yy})+[A_{12}(u_{x,x}+w_{,x}^2/2)$$
$$+A_{22}(u_{y,y}+w_{,y}^2/2)-B_{12}w_{,xx}-B_{22}w_{,yy}+A_{66}(u_{x,y}+u_{y,x}+w_{,x}w_{,y})$$
$$-2B_{66}w_{,xy}]w_{,y}+N_{xy}w_{,x}+N_{yy}w_{,y}\}_{,y}+f_e+f_z=\ I_1\ddot{w}-I_3\ddot{w}_{,xx}-I_3\ddot{w}_{,yy} \tag{9}$$

where

$$(A_{ij},B_{ij},D_{ij})=\int C_{ij}(l,z,z^2)dz \quad (i,j=1,2,6)$$
$$(N_{ij},M_{ij},M_{ij}^{*})=\int \sigma_{ij}(l,z,z^2)dz \quad (i,j=x,y,z)$$
$$(I_1,I_3)=\int \rho(l,z^2)dz \tag{10}$$

Here A_{ij}, B_{ij} and D_{ij} are the laminate stiffness coefficients; N_{ij}, M_{ij} and M_{ij}^{*} are initial stress resultants; f_x, f_y, f_z are the lateral loading and body force terms. Meanwhile, all the integrations are performed through the thickness of the plate from $-h/2$ to $h/2$.

3. Numerical examples

In the present study, the case to be concerned is simply-supported rectangular hybrid laminated plates of length a, width b and uniform thickness h (Fig. 2) subjected to the spatially uniform initial stress system and resting on an elastic foundation. Lateral loads and body forces are taken to be zero. The state of initial stresses, as shown in Fig. 2, is

$$\sigma_{xx}=\sigma_n+2z\sigma_m/h \tag{11}$$

which comprises an extensional normal stress σ_n and a bending stress σ_m. σ_m and σ_n are constants and other initial stresses are assumed to be zero.

For the cross-ply FRP, core of hybrid laminated plates, the stiffness coefficients C_{16} and C_{26} will be equal to zero in Eq. (3). The displacement fields with one-term fundamental mode shape satisfying the simply-supported boundary conditions along the x-constant and y-constant edges can be given as

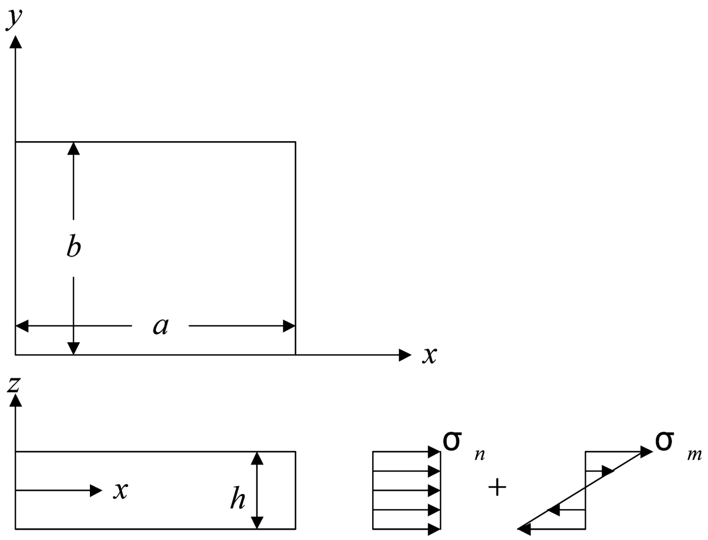

Fig. 2 The rectangular plate and the applied stress field

$$u_x = hU(t)\cos(\pi x/a)\sin(\pi y/b)$$
$$u_y = hV(t)\sin(\pi x/a)\cos(\pi y/b)$$
$$w = hW(t)\sin(\pi x/a)\sin(\pi y/b) \tag{12}$$

The details of boundary conditions are listed in previous study (Chen *et al*. 2001). Substituting the assumed displacement fields into the nonlinear partial differential Eqs. (7)-(9) and applying the Galerkin method, one can obtain the following nonlinear ordinary differential equations with time as the independent variable

$$L_1U+L_2V+L_3W+N_1W^2 = M_1\ddot{U} \tag{13}$$

$$L_2U+L_4V+L_5W+N_2W^2 = M_2\ddot{V} \tag{14}$$

$$L_3U+L_5V+L_6W+N_3UW+N_4VW+N_5W^2+N_6W^3 = M_3\ddot{W} \tag{15}$$

Here the coefficients L_i and N_i represent the linear and nonlinear terms in the strain-displacement equation, respectively. The coefficients of above equations are given as

$$L_1 = -(12 + A_{66}h^2R^2 / D_{11} + K_f)$$

$$L_2 = -(A_{12}h^2 + A_{66}h^2)R / D_{11}$$

$$L_3 = -(B_{11} + B_{66}R^2)h / D_{11} - P\beta K_f / 6 - (B_{12} + B_{66})hR / D_{11}$$

$$L_4 = -(A_{66} + A_{22}R^2)h^2 / D_{11}$$

$$L_5 = (B_{11} + B_{66}R^2)h / D_{11} - (B_{12} + B_{66})hR / D_{11} - Q\beta K / 6,$$

$$L_6 = -P^2(1 + D_{66}R^2 / D_{11}) - 2PQ(D_{12} + D_{66})R / D_{11} - K(12 + P^2 + Q^2 + 2Q\beta)/12$$

$$- Q^2(D_{66} / D_{11} + D_{22}R^2 / D_{11}) + F_k K + F_{gx}G_x + F_{gy}G_y$$

$$N_1 = -16(-2A_{11}h^2 + A_{12}h^2R^2 - A_{66}h^2R^2)/(9\pi^2 rD_{11})$$

$$N_2 = -16(-A_{66}h^2R + A_{12}h^2R - 2A_{22}h^2R^3)/(9\pi^2 rD_{11})$$

$$N_3 = 32(A_{11}h^2 - A_{66}h^2R^2 + A_{12}h^2R^2)/(9\pi^2 rD_{11})$$

$$\mathrm{N}_4 = 32(A_{12} - A_{66} + A_{22}R^2)/(9\pi^2 rD_{11})$$

$$N_5 = -16(-2B_{11}h^2 + B_{12}h^2R^2 - B_{66}h^2R^2)/(9\pi^2 rD_{11})$$

$$-16(-B_{66}h^2R + B_{12}h^2R - 2B_{22}h^2R^3)/(9\pi^2 rD_{11})$$

$$N_6 = -(9A_{11}h^2 + 2A_{12}h^2R^2 + 4A_{66}h^2R^4 + 9A_{22}h^2R^4)/(rD_{11})$$

$$M_1 = M_2 = I,\ M_3 = I(1 - P^2 - Q^2),\ K_f = \pi^2R^2N_{xx}b^2 / hr^2D_{11}$$

$$\tau = t(\pi^2D_{11} / \rho h^3a^2)^{1/2},\ F_{gx} = 1,\ F_{gy} = R^2,\ G = r^2k / \pi^2D_{11}G_x = r^2k / \pi^2D_{11}G_y$$

$$F_k = r^2 / \pi^2,\ K = r^4k / \pi^4D_{11},\ R = a / b,\ r = a / h,\ \beta = \sigma_m / \sigma_n,\ P = \pi / a,\ Q = \pi / b$$

The initial in-plane compressive (tensile) stress σ_n is included in the buckling parameter K_f. If K_f is positive, then the initial stress is tensile. β is the ratio of bending stress to normal stress, $\beta = \sigma_m/\sigma_n$. K and G are the non-dimensional foundation stiffness of k and g, respectively. τ is the non-dimensional time. The ordinary differential equations are solved by using a fourth order Runge-Kutta method with a non-dimensional time step of 0.001. Herein, the dimensionless nonlinear period for one full cycle of nonlinear vibration is measured and denoted as τ_{nl}, and the dimensionless nonlinear frequency is computed as $\omega_{nl} = 1/\tau_{nl}$. By neglecting the nonlinear terms in Eqs. (13)-(15), the dimensionless linear period τ_l and frequency ω_l can be calculated.

4. Results and discussions

In the present study, the nonlinear vibration of hybrid laminated plates under an arbitrary state of initial stress and on elastic foundations is investigated. The parameters, the nonlinear frequency (ω_{nl}) and ratio of nonlinear frequency to linear frequency (ω_{nl}/ω_l), are used to describe the behavior of nonlinear vibration which depends on the material properties, initial stresses and foundation stiffness. The nonlinear vibration behavior of hybrid laminated plate has been analyzed by using the procedure described in the previous section. This level of modeling has been used in previous studies (Chien and Chen 2006, Chen 2007, Chen *et al*. 2007) for predicting linear and nonlinear vibration of laminated plates and has been proved to be accurate. Moreover, the nonlinear vibration of an orthotropic beam resting on Winkler foundation can be analyzed by simplifying the present plate model. The results of non-dimensional frequency ratio (ω_{nl}/ω_l) versus non-dimensional vibration amplitude (w/ρ) obtained by the present model and Patel *et al*. (1999) are shown in Fig. 3. ρ is the radius of gyration of beam cross-section. It can be seen that present results are very close to Patel's at both low and high foundation stiffness.

There are so many parameters will affect the nonlinear vibration behavior of the hybrid composite plate that it would be difficult to present results for all cases. Hence, only a few typical hybrid composite rectangular plates will be selected for discussions. We consider the simply-supported hybrid composite plates consisting of fiber reinforced polymer (GFRP or CFRP) and aluminum (Al), which is similar to a sandwich structure of laminates with two Al surfaces and a FRP core. The lay-up of middle layer laminates is the cross-ply FRP. The total thickness of hybrid Al/FRP/Al

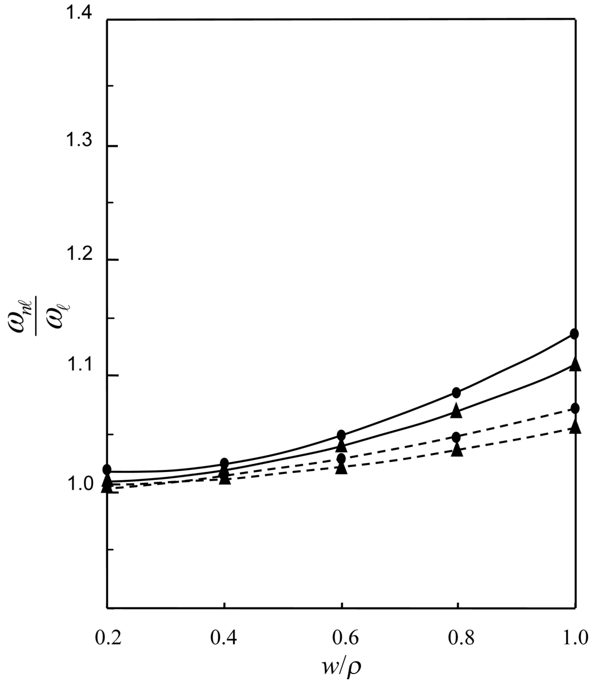

Fig. 3 Comparison of frequency ratios of a beam resting on Winkler foundation with $K_1 = 10$ (solid lines) and $K_1 = 100$ (dash lines) (▲: Patel's results (1999), ●: present results)

Table 1 Material properties of ingredients of laminated hybrid composite plates

	E_x (GPa)	E_y (GPa)	G_{xy} (GPa)	υ_{xy}
Aluminum (Al)	72.0	72.0	28.0	0.33
Carbon fiber reinforced polymer (CFRP)	181.0	10.3	7.17	0.28
Glass fiber reinforced polymer (GFRP)	38.6	8.27	4.14	0.26

Table 2 Comparison of nonlinear vibration of hybrid laminated plates on Winkler foundations with various foundation stiffness (a/b = 1, a/h = 20, $\gamma = 10$, G = 0, K_f = 0, $\beta = 0$)

	K		W				
			0.2	0.4	0.6	0.8	1.0
Al/CFRP/Al	0	ω_{nl}	0.993	1.140	1.348	1.594	1.873
		ω_{nl}/ω_l	1.059	1.216	1.437	1.700	1.998
	1	ω_{nl}	1.153	1.282	1.471	1.705	1.961
		ω_{nl}/ω_l	1.044	1.160	1.332	1.543	1.776
	5	ω_{nl}	1.641	1.737	1.885	2.072	2.292
		ω_{nl}/ω_l	1.021	1.080	1.173	1.289	1.426
	10	ω_{nl}	2.097	2.175	2.295	2.455	2.645
		ω_{nl}/ω_l	1.013	1.050	1.108	1.186	1.277
Al/GFRP/Al	0	ω_{nl}	0.873	0.935	1.029	1.147	1.284
		ω_{nl}/ω_l	1.025	1.098	1.209	1.348	1.509
	1	ω_{nl}	1.051	1.103	1.184	1.290	1.414
		ω_{nl}/ω_l	1.018	1.068	1.147	1.249	1.369
	5	ω_{nl}	1.572	1.608	1.666	1.743	1.839
		ω_{nl}/ω_l	1.008	1.031	1.068	1.117	1.179
	10	ω_{nl}	2.045	2.073	2.118	2.181	2.261
		ω_{nl}/ω_l	1.005	1.018	1.041	1.072	1.110

plate is h and the thickness of individual layer for Al, GFRP (CFRP) and Al is h_1, h_2 and h_3, respectively. The layer thickness ratio $\gamma = h_2/h_1 = h_2/h_3$ ($\gamma = t_{\mathrm{FRP}}/t_{\mathrm{Al}}$) so the increase of the geometric parameter indicates the increase of the core layer thickness of the hybrid plate. Hence, the hybrid plate is a pure Al plate as $\gamma = 0$; it is a laminated CFRP (or GFRP) plate when γ is infinite. Table 1 lists the material properties of the ingredients of laminated hybrid composite plates investigated in this study. The effects of various variables on nonlinear vibration frequency and frequency ratio of hybrid laminated plates are discussed as follows.

The effects of elastic foundation stiffness K on the nonlinear vibration of hybrid laminated plates resting on Winkler foundations are shown in Table 2. It can be observed that the vibration frequency increases with the increasing elastic foundation stiffness and vibration amplitude. In virtue of the difference of modulus ratio, the Al/CFRP/Al plate will access an obvious increment of frequency than Al/GFRP/Al plate at larger vibration amplitude. As can be seen, the frequency ratio increases with the increase of the vibration amplitude but decreases with the increasing elastic foundation stiffness. With higher elastic foundation stiffness, the plate becomes much stiffer and its nonlinear vibration frequency is increased and plate frequency ratio is reduced sharply. It can also be found that the frequency ratios of the Al/CFRP/Al plate with high modulus ratio (E_x/E_y) are larger than those of the Al/GFRP/Al plate with low modulus ratio. At very high elastic foundation stiffness ($K \cong 100$) the frequency ratios of Al/GFRP/Al plate will not vary with the vibration amplitude. Therefore, the larger the frequency ratio is, the more significant the contribution to the

Table 3 Comparison of nonlinear vibration of hybrid laminated plates on Pasternak foundations with various foundation stiffness (a/b = 1, a/h = 20, γ = 10, K = 1, K_f = 0, β = 0)

	G		W				
			0.2	0.4	0.6	0.8	1.0
Al/CFRP/Al	0	ω_{nl}	1.153	1.282	1.471	1.705	1.961
		ω_{nl}/ω_l	1.044	1.160	1.332	1.543	1.776
	1	ω_{nl}	1.418	1.527	1.692	1.899	2.138
		ω_{nl}/ω_l	1.027	1.106	1.226	1.376	1.549
	5	ω_{nl}	2.179	2.252	2.368	2.524	2.712
		ω_{nl}/ω_l	1.012	1.046	1.100	1.172	1.260
	10	ω_{nl}	2.855	2.913	3.006	3.132	3.293
		ω_{nl}/ω_l	1.007	1.027	1.060	1.104	1.161
Al/GFRP/Al	0	ω_{nl}	1.051	1.103	1.184	1.290	1.414
		ω_{nl}/ω_l	1.018	1.068	1.147	1.249	1.369
	1	ω_{nl}	1.337	1.379	1.446	1.533	1.641
		ω_{nl}/ω_l	1.011	1.042	1.093	1.159	1.240
	5	ω_{nl}	2.127	2.154	2.199	2.260	2.335
		ω_{nl}/ω_l	1.004	1.017	1.038	1.067	1.102
	10	ω_{nl}	2.819	2.840	2.874	2.922	2.981
		ω_{nl}/ω_l	1.002	1.010	1.022	1.039	1.060

vibration is from the nonlinear terms N_i of the governing Eqs. (13)-(15). In the other words, the larger the frequency ratio is the greater the difference between the nonlinear frequency and the linear frequency is. As the frequency ratio approaches unity, the value of nonlinear frequency is close to the linear frequency and the effect of nonlinear terms in the vibration is fairly small and thus negligible.

Table 3 presents the effects of the shear layer foundation stiffness G on the nonlinear vibration of hybrid plates resting on Pasternak foundations. As can be observed, the vibration frequency and frequency ratio are affected by the shear layer foundation stiffness of Pasternak foundation in a similar way as the elastic foundation stiffness of Winkler foundation does. When the shear layer foundation stiffness is increased, the plate frequency ratio is decreased and vibration frequency is increased, but the variation decreases for the plate with low modulus ratio and resting on the foundation with higher shear layer foundation stiffness. It can also be seen that the plate with higher modulus ratio (Al/CFRP/Al), null shear layer foundation stiffness and largest vibration amplitude possesses the largest plate frequency ratio.

Figs. 4 and 5 depict the effect of layer thickness ratio γ on the nonlinear vibration frequency of various hybrid composite plates resting on Winkler and Pasternak foundations, respectively. As seen in Figs. 4 and 5, the nonlinear vibration frequency of the hybrid plate with a stiff core of layer is increased as γ is increased but that of the hybrid plate with a soft one has the opposite tendency. The vibration frequency for all hybrid plates is always increased with the increasing foundation

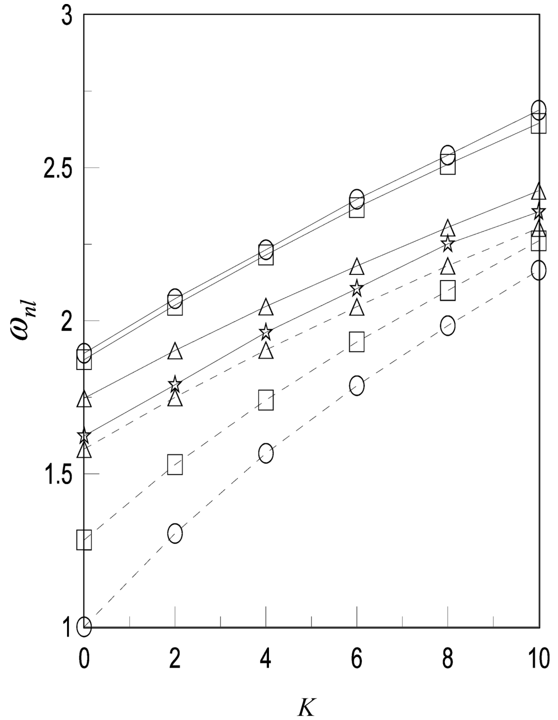

Fig. 4 Nonlinear frequencies of hybrid laminated plates with various layer thickness ratios on Winkler foundations. (☆ pure Al, Al/CFRP/Al (solid lines), Al/GFRP/Al (dash lines), △ γ= 1, □ γ= 10, ○ γ= = ∞, a/b = 1, a/h = 20)

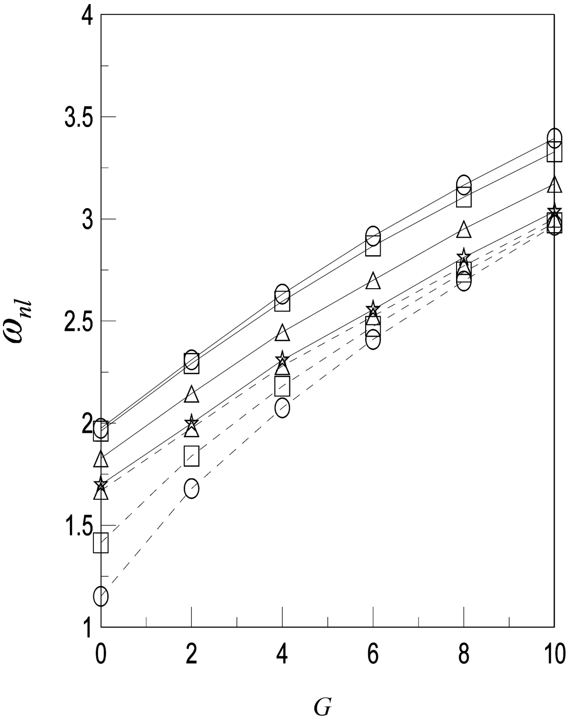

Fig. 5 Nonlinear frequencies of hybrid laminated plates with various layer thickness ratios on Pasternak foundations. (☆ pure Al, Al/CFRP/Al (solid lines), Al/GFRP/Al (dash lines), △ γ = 1, □ γ = 10,○ γ = ∞, K = 1, a/b = 1, a/h = 20)

stiffness. Hence, the Al/CFRP/Al plate having a largest γ and resting on the foundation with the highest stiffness will yield a highest vibration frequency; the Al/GFRP/Al plate with a largest γ and on a softest foundation will possess a lowest vibration frequency. The results reveal that the γ will weaken the effect of the foundation stiffness for plates with lower modulus ratio (E_x/E_y) of GFRP core. In other words, the increase of thickness of core layer has more conspicuous effects on vibration frequency and frequency ratio.

Fig. 6 shows the influence of γ and foundation stiffness K on the frequency ratio of hybrid laminated plate resting on the Winkler foundation. Apparently, with the increase of elastic foundation stiffness K, the vibration ratio of hybrid Al/CFRP/Al plate drops significantly. While the layers of GFRP in the middle of hybrid plates increase, there is slight decrease of frequency ratio. It can be found that the increase of the layer thickness of core CFRP of the hybrid plates, namely, the increase of layer thickness ratio, increases its frequency ratio but, the increase of that of core GFRP has a contrary tendency. Therefore, the frequency ratio will be the maximum for the pure CFRP plate, but will be the minimum for the pure GFRP plate. Thus, for a hybrid plate with the higher modulus ratio (CFRP) and larger thickness ratio resting on elastic foundation of lower foundation stiffness, a higher frequency ratio can be seen and the effect of nonlinear terms is notable. The

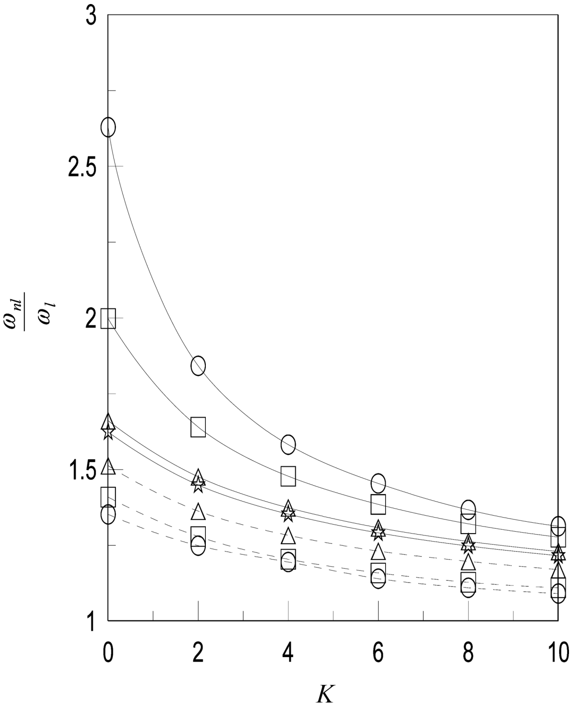

Fig. 6 Frequency ratios of hybrid laminated plates with various layer thickness ratios on Winkler foundations. (☆ pure Al, Al/CFRP/Al (solid lines), Al/GFRP/Al (dash lines), △ $\gamma = 1$, □ $\gamma = 10$, ○ $\gamma = \infty$, $a/b = 1$, $a/h = 20$)

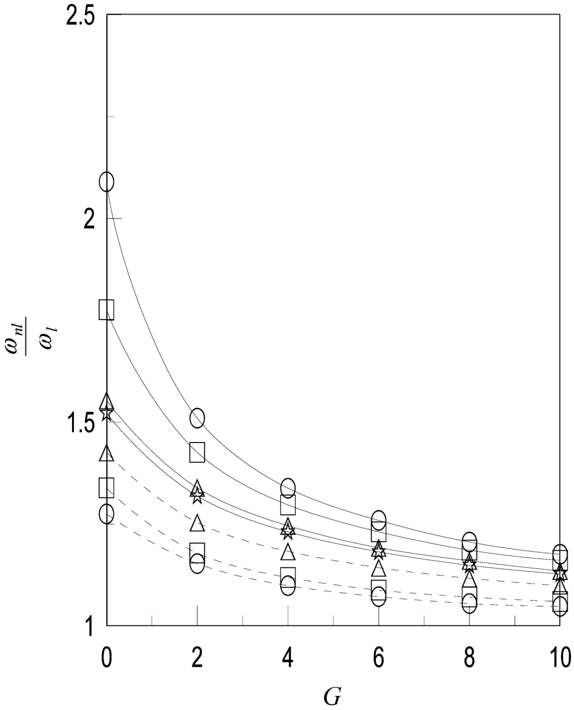

Fig. 7 Frequency ratios of hybrid laminated plates with various layer thickness ratios on Pasternak foundations. (☆ pure Al, Al/CFRP/Al (solid lines), Al/GFRP/Al (dash lines), △ $\gamma = 1$, □ $\gamma = 10$, ○ $\gamma = \infty$, $K = 1$, $a/b = 1$, $a/h = 20$)

effect of γ and G on nonlinear vibration of hybrid laminate plates is given in Fig. 7. Likewise, the results show that frequency ratio is decreased with the increase of G but might be increased or decreased with the increasing γ depending on the core material used. The pure laminated CFRP plate on lowest foundation stiffness with largest layer thickness ratio γ has a larger frequency ratio and the GFRP plate resting on highest foundation stiffness with largest layer thickness ratio has a lowest frequency ratio.

The effects of an initial in-plane compressive/tensile stress on nonlinear frequency of hybrid plates resting on the Winkler foundations are presented in Fig. 8. It can be seen that the vibration frequency of hybrid plates with an initial tensile stress is larger than that of plates without an initial stress. The initial in-plane tensile stress enlarges the nonlinear vibration frequency of hybrid plates while the compressive stress gives a reverse result. The vibration frequency of Al/CFRP/Al plates is still larger that of Al/GFRP/Al plates since the core layer CFRP has a higher elastic modulus than GFRP. Fig. 9 shows the influence of initial stress and shear layer foundation stiffness on the nonlinear vibration frequency of hybrid plates resting on Pasternak foundation. The results reveal that the variation of vibration frequency has a similar trend as that in Fig. 8. As can be observed, the vibration frequency is increased with the increasing initial stress, foundation stiffness and elastic modulus of core layer. Thus, an initially tensile stressed Al/CFRP/Al plate with the highest foundation stiffness has the largest frequency. The initial stress still has a similar influence on

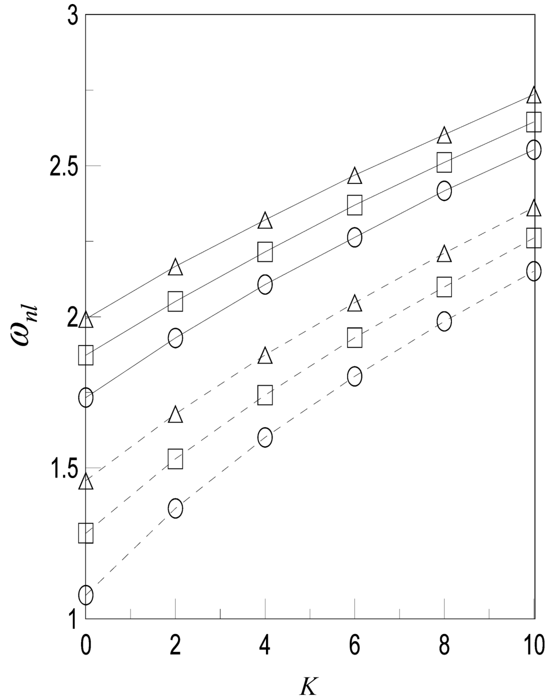

Fig. 8 Nonlinear frequencies of hybrid laminated plates with an initial stress on Winkler foundations. (Al/CFRP/Al (solid lines), Al/GFRP/Al (dash lines), △ $K_f = 2$, □ $K_f = 0$, ○ $K_f = -2$, $\gamma = 10$, $a/b = 1$, $a/h = 20$)

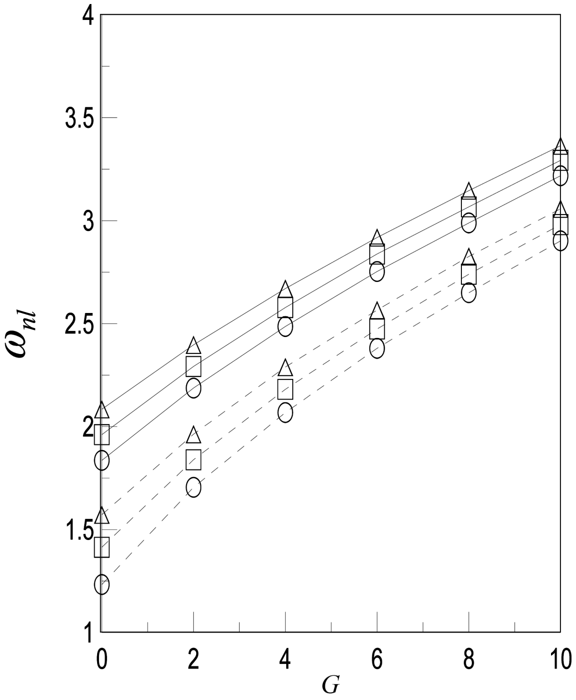

Fig. 9 Nonlinear frequencies of hybrid laminated plates with an initial stress on Pasternak foundations. (Al/CFRP/Al (solid lines), Al/GFRP/Al (dash lines), △ $K_f = 2$, □ $K_f = 0$, ○ $K_f = -2$, $\gamma = 10$, $K = 1$, $a/b = 1$, $a/h = 20$)

vibration frequency for hybrid plates on Winkler foundations and Pasternak foundations, respectively.

The effects of initial stress on frequency ratio are shown in Figs. 10-11 for hybrid plates on two types of elastic foundations. It can be seen that the initial stresses have the opposite influence on the frequency ratio. The compressive stress ($K_f > 0$) develops a hardening effect on the frequency ratio but the tensile stress has a softening effect. The high modulus ratio and low elastic foundation stiffness K and shear layer foundation stiffness G have an intensifying effect on the frequency ratio of the hybrid plates so they having a high modulus ratio resting on a softer foundation and subjected to a compressive initial stress will produce a high frequency ratio.

Table 4 present the effects of initial bending stress on frequency ratio of hybrid laminated plates resting on Winkler foundation and Pasternak foundation, respectively. The frequency ratio of Al/CFRP/Al plates is increased when the bending stress is increased for all the foundation conditions. However, the influence of the bending stress becomes insignificant as the foundation stiffness is slightly increased. The reason is that the foundation stiffness makes the plate much stiffer and diminishes the effect of bending stress. As can be observed, the initial bending stress has an unapparent effect on the frequency ratio for the Al/GFRP/Al plates regardless of foundation conditions. Thus, the increase of bending stress has much less influence on the frequency ratio of the Al/GFRP/Al plates under initial stress.

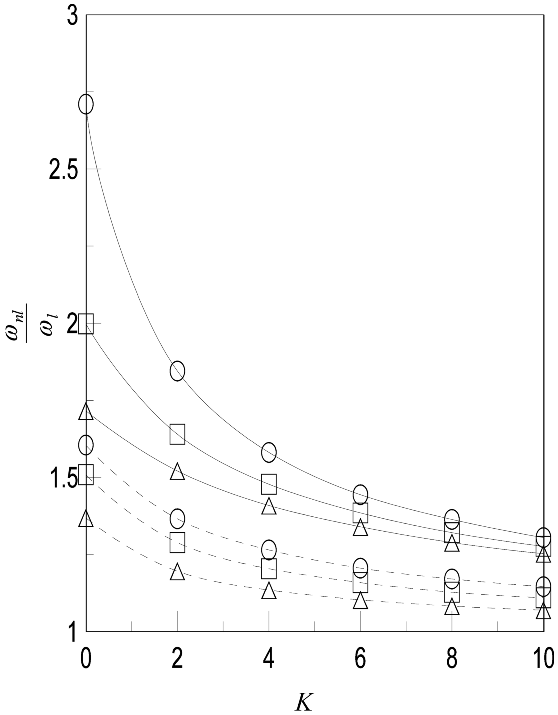

Fig. 10 Frequency ratios of hybrid laminated plates with an initial stress on Winkler foundations. (Al/CFRP/Al (solid lines), Al/GFRP/Al (dash lines), △ K_f = 2, □ K_f = 0, ○ K_f = −2, γ = 10, a/b = 1, a/h = 20)

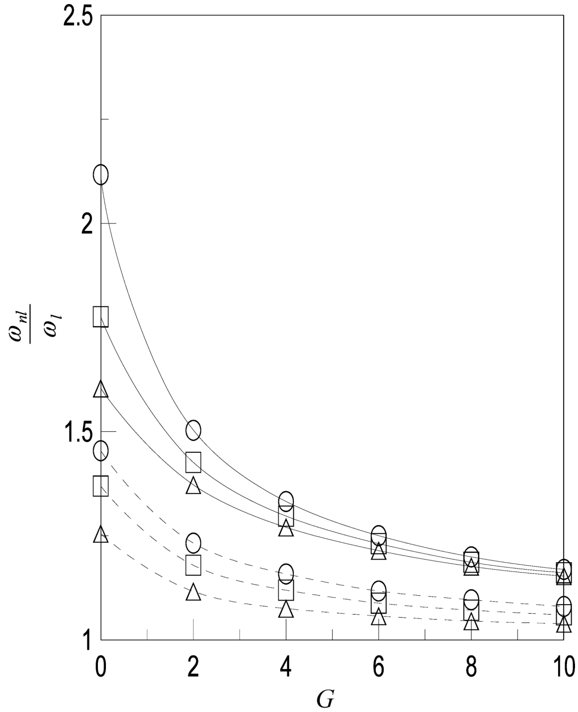

Fig. 11 Frequency ratios of hybrid laminated plates with an initial stress on Pasternak foundations. (Al/CFRP/Al (solid lines), Al/GFRP/Al (dash lines), △ K_f = 2, □ K_f = 0, ○ K_f = −2, γ = 10, K = 1, a/b = 1, a/h = 20)

Table 4 Comparison of nonlinear vibration of hybrid laminated plates with various bending ratios resting on elastic foundations. (a/b = 1, a/h = 20, γ = 100, W = 1, K_f = −2)

	K	G	β					
			0	10	20	30	40	50
	0	0	7.545	7.554	7.576	7.637	7.698	7.798
	0.5	0	3.632	3.634	3.638	3.644	3.653	3.663
Al/CFRP/Al	0.5	0.5	2.412	2.413	2.414	2.415	2.417	2.420
	0.5	1	1.994	1.995	1.996	1.997	1.998	1.999
	1	1	1.876	1.876	1.877	1.877	1.878	1.879
	0	0	1.585	1.587	1.588	1.587	1.585	1.582
	0.5	0	1.354	1.355	1.355	1.354	1.353	1.350
Al/GFRP/Al	0.5	0.5	1.205	1.205	1.205	1.205	1.204	1.202
	0.5	1	1.147	1.147	1.147	1.146	1.146	1.145
	1	1	1.126	1.127	1.126	1.126	1.125	1.124

5. Conclusions

Nonlinear vibration behaviors of initially stressed hybrid composite plate resting on elastic foundations have been investigated and discussed. The preliminary results are summarized as follows:

1. Nonlinear vibration frequencies and frequency ratios of various hybrid laminate plates have been influenced by the vibration amplitude, elastic foundation stiffness, shear layer foundation stiffness, layer thickness ratio, initial stress and modulus ratio.
2. The nonlinear vibration frequency increases with the vibration amplitude, elastic foundation stiffness, shear layer foundation stiffness and initial tensile stress. The frequency ratio increases with large vibration amplitude, initial compressive stress, decreasing elastic foundation stiffness and shear layer foundation stiffness.
3. The vibration frequency and frequency ratio of hybrid AL/CFRP/Al plates are larger than those of corresponding hybrid AL/GFRP/Al plates at various conditions.
4. The vibration frequency and frequency ratio increase with the increasing layer thickness ratio for AL/CFRP/Al plates. However, they increase with the decreasing layer thickness ratio for AL/GFRP/Al plates.

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