Structural Engineering and Mechanics, Vol. 37, No. 2 (2011) 149-162
DOI: http://dx.doi.org/10.12989/sem.2011.37.2.149

A comparative study for beams on elastic foundation models to analysis of mode-I delamination in DCB specimens

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(Received January 16, 2010, Accepted September 1, 2010)

Abstract. The aim of this research is a comprehensive review and evaluation of beam theories resting on elastic foundations that used to model mode-I delamination in multidirectional laminated composite by DCB specimen. A compliance based approach is used to calculate critical strain energy release rate (SERR). Two well-known beam theories, i.e. Euler-Bernoulli (EB) and Timoshenko beams (TB), on Winkler and Pasternak elastic foundations (WEF and PEF) are considered. In each case, a closed-form solution is presented for compliance versus crack length, effective material properties and geometrical dimensions. Effective flexural modulus (E_{fx}) and out-of-plane extensional stiffness (E_z) are used in all models instead of transversely isotropic assumption in composite laminates. Eventually, the analytical solutions are compared with experimental results available in the literature for unidirectional ($[0^o]_6$) and antisymmetric angle-ply ($[\pm 30^o]_5$, and $[\pm 45^o]_5$) lay-ups. TB on WEF is a simple model that predicts more accurate results for compliance and SERR in unidirectional laminates in comparison to other models. TB on PEF, in accordance with Williams (1989) assumptions, is too stiff for unidirectional DCB specimens, whereas in angle-ply DCB specimens it gives more reliable results. That it shows the effects of transverse shear deformation and root rotation on SERR value in composite DCB specimens.

Keywords: delamination; compliance; strain energy release rate; beam theory; elastic foundation.

1. Introduction

Delamination is one of the common forms of failure modes in laminated composites that occurs due to the lack of reinforcement in the thickness direction and existence of interlaminar stresses likely to debond layers. It is very important to consider delamination failure during the composite structures design because it reduces mechanical properties and limits the safe life of a component. So, a designer needs to know the interlaminar fracture toughness of composite laminates in order to design damage tolerant structures. One typical procedure for assessing the propensity for a delamination to grow is to compare the strain energy release rate (SERR), G, to its critical value or toughness, G_c. Double Cantilever Beam (DCB) specimen is widely used for determining critical strain energy release rate, G_{Ic} under mode-I delamination loading (Benzeggagh *et al.* 1991). Testing

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to determine the fracture toughness for each possible laminate configuration would be expensive and time-consuming work. In recent years, extensive research has been conducted on many aspects of delamination failure by analytical solutions for DCB specimen. One of the privilege ideas, because of its simplicity, is using classical beam theory (CBT) or Euler-Bernoulli beam (EB) to predict SERR. In early research the cracked part of DCB specimen is considered as a cantilever beam that fixed at crack tip. Bathias and Laksimi (1985) proposed a modified model for G_I, which takes into account the shearing force, owing to specimen rotation at the crack tip. Weatherby (1982) included a rotational spring at crack tip and gave an expression for compliance. The mentioned models have high offset from experimental results and far from reality. Thus, to increase the accuracy, Kanninen (1973) modeled one arm of a DCB specimen of isotropic materials as an EB on Winkler elastic foundation (WEF). This method was extended by Williams (1989) to the transversely isotropic DCB specimen. William's model is only evaluated for unidirectional laminates. Ozdil and Carlsson (1999) then used Kanninen's model (1973) for angle-ply laminates by estimating out-of-plane stiffness with laminate homogenization method instead of transversely isotropic assumption. In classical beam theory, straight lines normal to the neutral beam axis are assumed remain straight and normal after deformation. This implies that the effect of transverse shear deformations are neglected, a condition that is justified for slender beams. Two sources of errors, assumption of zero slope and displacement at the beam root, are possible. Kondo (1995), therefore, analyzed the unidirectional DCB specimen by utilizing a Timoshenko beam (TB) supported by a WEF. Although the Winkler model has given satisfactory results in many practical problems, it may be considered a crude approximation of the true mechanical behavior of the foundation material. This situation gave rise to the development of the so-called two-parameter elastic foundation models (Shirima1 and Giger 1992). Williams (1989) considered the rotational stiffness of the foundation (Pasternak foundation) for unidirectional laminate and concluded that root rotation due to low shear stiffness is the main factor causing composite DCB specimens to deviate from the cantilever beam theory. Olsson (1992) reviewed the analyses of the DCB specimen based on the elementary beam models for graphite-epoxy composites, concluding that plane strain solution of Whitney (1985) is the most accurate compared with the finite element solution.

From the literature, among the factors, like shear deformation, root rotation, mode mixity, residual stresses, curved crack front, geometrical nonlinearity, etc, the two factors, shear deformation and root rotation, seem to have significant effect on SERR of mode-I delamination. Therefore, the main goal of the present study is investigating the contributions of transverse shear deformation, elastic foundations or root rotation on various laminate lay-ups by using beam theories. To assess the accuracy of the available models in predicting compliance and SERR of DCB specimen, Euler-Bernoulli beam on Winkler elastic foundation (EB on WEF), Timoshenko beam as a first-order shear deformable theory on WEF, and Timoshenko beam on Pasternak elastic foundation (TB on PEF) are considered. In multidirectional laminates, the idea of effective properties i.e., effective flexural stiffness (E_{fx}) and out-of-plane extensional stiffness (E_z) are taken into account. The analytical results are then compared with available experimental data of glass/polyester unidirectional and angle-ply DCB specimens in Ozdil and Carlsson (1999). As a result, comparisons among the beam models resting on elastic foundation allow an understanding to be developed for the relative roles of shear deformations, and elastic foundations or root rotation on the fracture behavior of unidirectional and multidirectional composite laminates.

2. Problem statement

A convenient approach for determining the fracture toughness of mode-I delamination in unidirectional laminated composites is DCB specimen test according to ASTM D5528 (Fig. 1) (Ozdil and Carlsson 1999, ASTM D5528). Various analytical models are proposed in the literature to estimate SERR. Most of these models work with compliance method which is defined as the ratio of opening displacement of crack mouth (δ) to applied load at that point (P).

$$C = \frac{\delta}{P} \tag{1}$$

Having a relationship between compliance and crack length, the critical SERR, G_{Ic}, which is a measure of the fracture toughness of a bonded interface and available for propagating the crack, is obtained by differentiating the compliance with respect to crack length (Irwin and Kies 1954, Ewalds and Wanhill 1989).

$$G_{Ic} = \frac{P^2}{2b}\frac{dC}{da} \tag{2}$$

where P is the applied load, b is the specimen width, a is the crack length, and C is the compliance.

3. Review of elementary theories

3.1 Cantilever Beam Theory (CBT)

The most popular way of describing a DCB specimen is by analyzing a rigidly fixed cantilever beam in pure bending. For a symmetrical orthotropic cantilever beam having a rectangular section with uniform force applied to the end of the specimen (Fig. 2), and two assumptions: (1) the beam to be fixed at the crack tip (no end-rotation at crack tip), (2) the cracked part to be completely rigid; the compliance due to only the bending moment is expressed as

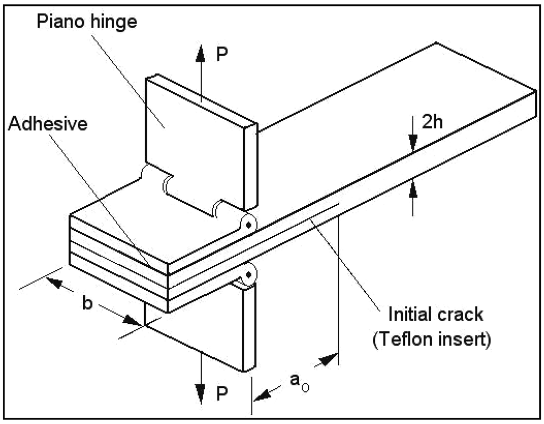


Fig. 1 Configuration of DCB specimen (a_0: Initial crack length, b: specimen width)

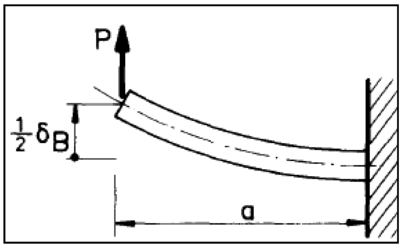


Fig. 2 Deformation of the cracked region as a rigidly cantilever beam (Olsson 1992)

$$C_0 = \frac{8}{E_{fx}b}\left(\frac{a}{h}\right)^2 \tag{3}$$

where E_{fx} is effective bending modulus of the laminate along beam axis that it is obtained from classical lamination theory (CLT) with plane stress assumption as

$$E_{fx} = \frac{12}{d_{11}h^3}, \quad [d] = ([D]-[B][A]^{-1}[B])^{-1} \tag{4}$$

where $[A], [B]$, and $[D]$ are extensional, coupling and bending stiffness of a laminate that can be found in (Tsai 1980). Thus, G_I can be obtained from Eq. (2)

$$G_I = (12P^2a^2)/(E_{fx}b^2h^3) \tag{5}$$

3.2 Modified beam theory with bending and shear at end

By considering effect of shear force at the end of beam and assuming bending displacements by δ_{BB} and shear displacements by δ_{BS}, the total displacement is given by $\delta_B = \delta_{BB} + \delta_{BS}$. From the classical beam theory, corrected for shear with a shear correction factor k that is chosen 5/6 in this study, the compliance of the cracked part is given by (Olsson 1992)

$$C = \frac{8}{E_{fx}b}\left(\frac{a}{h}\right)^3 + \frac{2}{kbG_{xz}}\left(\frac{a}{h}\right) \tag{6}$$

where G_{xz} is shear modulus. The mode-I SERR can be written as

$$G_I = \frac{12P^2a^2}{E_{fx}b^2h^3} + \frac{P^2}{kG_{xz}b^2h} \tag{7}$$

3.3 Modified beam theory with rotational spring at crack tip

Weatherby (1982) included a rotational spring stiffness, $k_r = Pa/\theta$, at the crack front and obtained the following expressions for compliance and SERR

$$C = \frac{8}{E_{fx}b}\left[\frac{E_{fx}}{4kG_{xz}}\left(\frac{a}{h}\right) + \frac{E_{fx}bh^2}{4k_r}\left(\frac{a}{h}\right)^2 + \left(\frac{a}{h}\right)^3\right] \tag{8}$$

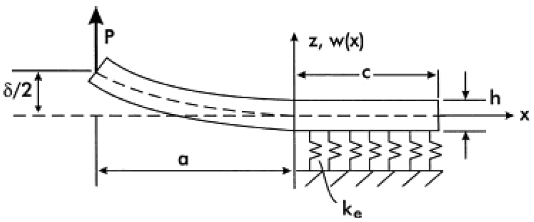


Fig. 3 Winkler elastic foundation model of the DCB Specimen (Ozdil and Carlsson 1999)

$$G_I = \frac{12P^2a^2}{E_{fx}b^2h^3} + \frac{P^2(1+4a)}{kG_{xz}b^2h} \tag{9}$$

Despite of some modifications in elementary theories, G_I has more offsets from the practical results yet. In order to predict SERR of DCB specimen with more accuracy, the idea of using beams resting on elastic foundations is proposed by Kanninen (1973). The Winkler model has its advantages in obtaining fast solutions, sometimes analytical, to more complicated structure problems. In the whole following models, by exploiting symmetry, only one half of the DCB specimen is modeled as configuration shown in Fig. 3. As a matter of fact, the beam is divided into two regions: a cracked region (I: $-a \le x \le 0$), and an uncracked region (II: $0 \le x \le c$). The cracked region is modeled as an unsupported beam under a point load at the tip end of region I, whereas the uncracked region II is modeled as a linear beam on an elastic foundation. Since the deflection of each arm at the point of load application is $\delta/2 = w_I(-a)$, the DCB specimen compliance can be written

$$C = \frac{\delta}{P} = \frac{2w_I(-a)}{P} \tag{10}$$

Thus, the displacement function, $w_I(x)$, should be calculated.

4. Euler-bernoulli beam on Winkler elastic foundation

The Euler-Bernoulli beam theory is the simplest beam theory, and it is based on the following displacement field

$$u_1(x,y,z) = -\frac{dw^E(x)}{dx}z, \quad u_2(x,y,z) = 0, \quad u_3(x,y,z) = w^E(x) \tag{11}$$

u_1, u_2, and u_3 denote the displacement components along the longitudinal axis, width, and height of the beam, respectively. The superscript "E" denotes the quantities in Euler-Bernoulli beam theory. In this theory, the transverse shear deformation is neglected. The governing bending equation of an Euler-Bernoulli beam on Winkler elastic foundation can be written as (Yavari *et al.* 2001, Hetenyi 1946)

$$\frac{d^2}{dx^2}\left(E_{fx}I\frac{d^2w^E(x)}{dx^2}\right) + k_e w^E(x) = q(x) \tag{12}$$

where $E_{fx}I$, k_e, and $q(x)$ are the flexural rigidity, foundation modulus, and transverse load density, respectively. For the beam with a constant flexural stiffness and $q(x) = 0$, Eq. (12) may be written as

$$\frac{d^4 w^E(x)}{dx^4} + 4\beta^4 w^E(x) = 0 \tag{13}$$

where $\beta^4 = k_e/4E_{fx}I = 3k_e/E_{fx}bh^3$. Since the elastic foundation incorporates the elasticity of the uncracked part of the DCB specimen, the foundation modulus may related to the out-of-plane extensional stiffness (E_z) of the laminate. In most of earlier studies on DCB specimen, E_z is equaled to E_2 by assumption of transversely isotropic material. The foundation spring stiffness per unit length, k_e, is obtained by computing the stiffness for a bar which is EA/L. Here $A = b$ times one unit length and L is the distance from the specimen symmetry plane $z = 0$ to the neutral axis of the beam at $z = h/2$. Thus, the foundation stiffness is given by

$$k_e = \frac{2E_z b}{h} \tag{14}$$

The model used in Eq. (13), the simplest model for an elastic foundation, is called Winkler model. For a Winkler foundation, it is assumed that at any point the pressure the foundation exerts on the beam is proportional to the deflection of the beam at that point and is independent of the deflection of the other parts of the foundation. This is an acceptable approximate model for many engineering applications (Yavari *et al.* 2001). The appropriate boundary conditions are those for a shear force P acting on the left-hand end of the beam and a free end condition at the right-hand end. Using the prime notation to indicate differentiation with respect to x, these can be written in accordance with Fig. 3 configuration as

$$w''(-a) = 0, \quad w'''(-a) = P/E_{fx}I = 12P/E_{fx}bh^3, \quad w''(c) = 0, \quad w'''(c) = 0 \tag{15}$$

Kanninen (1973) solved Eq. (13) by considering the two intervals ($-a$, 0) and (0, c) separately, then matching the values of deflection and it's the first three derivatives at $x = 0$. By obtaining the displacement function for cracked region, and assuming the length of the undelaminated region, c, much larger than beam thickness for simplicity, compliance is obtained from Eq. (10) as (Ozdil and Carlsson 1999)

$$C = \frac{8}{E_{fx}b}\left(\frac{a}{h}\right)^3\left[1 + 1.92\left(\frac{h}{a}\right)\left(\frac{E_{fx}}{E_z}\right)^{1/4} + 1.22\left(\frac{h}{a}\right)^2\left(\frac{E_{fx}}{E_z}\right)^{1/2} + 0.39\left(\frac{h}{a}\right)^3\left(\frac{E_{fx}}{E_z}\right)^{3/4}\right] \tag{16}$$

Using Eq. (2) results in

$$G_I = \frac{12P^2a^2}{E_{fx}b^2h^3}\left[1 + 1.28\left(\frac{h}{a}\right)\left(\frac{E_{fx}}{E_z}\right)^{1/4} + 0.41\left(\frac{h}{a}\right)^2\left(\frac{E_{fx}}{E_z}\right)^{1/2}\right] \tag{17}$$

5. Timoshenko beam on Winkler elastic foundation

In this section, the governing differential equations of Timoshenko beam on Winkler elastic foundation (TB on WEF) are reviewed. The Timoshenko beam theory (TB) is the simplest shear deformation beam theory; it has the following displacement field assumptions (Reddy 1999, Wang *et al.* 2000)

$$u_1(x,y,z) = z\psi^T(x), \quad u_2(x,y,z) = 0, \quad u_3(x,y,z) = w^T(x) \tag{18}$$

where u_1, u_2 and u_3 are displacement components along the x, y, z axes, respectively; ψ and w denote the rotation of the cross-section about the y-axis and the transverse beam deflection, respectively. The superscript "T" denotes the quantities in Timoshenko beam theory. The conventional beam equations including shear deformation are defined as

$$M(x) = E_{fx}I\frac{d\psi^T}{dx}, \quad Q(x) = kG_{xz}bh\left(\psi^T + \frac{dw^T}{dx}\right) \tag{19}$$

where $M(x)$ is the bending moment of the beam, $Q(x)$ is the transverse shear force of the beam. Equilibrium equations can be expressed in terms of the displacements as

$$\begin{cases} E_{fx}I\dfrac{d^2\psi^T}{dx^2} - kG_{xz}A\left(\psi^T + \dfrac{dw^T}{dx}\right) = 0 \\ kG_{xz}A\left(\dfrac{d\psi^T}{dx} + \dfrac{d^2w^T}{dx^2}\right) - k_e w^T = 0 \end{cases} \tag{20}$$

Two regions similar to EB on WEF model are considered:

Region I: cracked region ($-a \le x \le 0$)
For the cracked region, moment and shear force can be expressed in terms of the tip load P as

$$M(x) = -P(x+a), \quad Q(x) = -P \tag{21}$$

The generalized displacements of the cracked region can be obtained by integrating Eq. (21). For region I ($-a \le x \le 0$)

$$\psi_I(x) = -\frac{P}{E_{fx}I}\left(\frac{x^2}{2} + ax\right) + c_1 \tag{22}$$

$$w_I(x) = \frac{P}{E_{fx}I}\left(\frac{x^3}{6} + \frac{ax^2}{2}\right) - \left(\frac{P}{kGbh} + c_1\right)x + c_2 \tag{23}$$

Region II: uncracked region ($0 \le x \le c$)
For the case of the uncracked portion, the beam is supported by an elastic foundation with extensional elastic coefficients (k_e). Based on Timoshenko beam theory, the governing equations of beam deformations in region II ($0 \le x \le c$) are given with Eq. (20). They are two coupled linear second order differential equations that can be simplified in one equation. Differentiating from Eq. (20a) with respect to x yields

$$E_{fx}I\frac{d^3\psi}{dx^2} - kG_{xz}A\left(\frac{d\psi}{dx} + \frac{d^2w}{dx^2}\right) = 0 \tag{24}$$

The second derivatives of Eq. (20b) with respect to x result in

$$kG_{xz}A\left(\frac{d^3\psi}{dx^3}+\frac{d^4w^T}{dx^4}\right)-k_e\frac{d^2w}{dx^2} = 0 \tag{25}$$

Substituting Eqs. (20b) and (25) into Eq. (24) yields

$$\frac{d^4w}{dx^4}-2\alpha^2\frac{d^2w}{dx^2}+4\beta^4w = 0 \tag{26}$$

where $2\alpha^2 = \dfrac{2E_z}{kG_{xz}h^2}$, $4\beta^4 = 4\left(\dfrac{6E_z}{E_{fx}h^4}\right)$. Also, the boundary conditions (B.C.'s) for two regions are

$$M_I = 0 \quad \text{and} \quad Q_I = -P \quad \text{at} \quad x = -a \tag{27}$$

$$M_{II} = 0 \quad \text{and} \quad Q_{II} = 0 \quad \text{at} \quad x = c \tag{28}$$

Furthermore, the continuity conditions (C.C.'s) at $x = 0$ are

$$\psi_I = \psi_{II}, \quad w_I = w_{II} \tag{29}$$

The equilibrium conditions (E.C.'s) at $x = 0$ are as the following

$$M_I = M_{II}, \quad Q_I = Q_{II} \tag{30}$$

By imposing conditions and using a symbolic mathematics software (e.g., Mathematica or Maple), the explicit expressions for rotation and displacement at the crack tip are obtained. Due to relatively being lengthy of final expression for displacement, therefore, they are omitted in this paper. Kondo (1995) by assuming deflection and rotation dissipate as $x\to\infty$, reduced the unknown constants instead of using boundary conditions at $x = c$. Finally, the compliance equation versus crack length was obtained as

$$\frac{C}{C_0} = 1+3\sqrt{\left(\frac{E_{fx}}{6E_z}\right)^{1/2}+\frac{1}{12}\frac{E_{fx}}{kG_{xz}}}\left(\frac{a}{h}\right)^{-1}+3\left\{\left(\frac{E_{fx}}{6E_z}\right)^{1/2}+\frac{1}{12}\frac{E_{fx}}{kG_{xz}}\right\}\left(\frac{a}{h}\right)^{-2}$$
$$+\frac{3}{2}\left(\frac{E_{fx}}{6E_z}\right)^{1/2}\sqrt{\left(\frac{E_{fx}}{6E_z}\right)^{1/2}+\frac{1}{12}\frac{E_{fx}}{kG_{xz}}}\left(\frac{a}{h}\right)^{-3} \tag{31}$$

where C_0 is the compliance of the completely clamped Euler-Bernoulli beam in Eq. (3). Strain energy release rate was reported as

$$G_I = \frac{12P^2}{E_{fx}b^2h}\left\{\left(\frac{a}{h}\right)+\sqrt{\left(\frac{E_{fx}}{6E_z}\right)^{1/2}+\frac{1}{12}\frac{E_{fx}}{kG_{xz}}}\right\}^2 \tag{32}$$

5. Timoshenko beam on elastic foundation with rotational spring

The deformation in the crack-tip region has been described in terms of "root rotation" effects, where planes rotate in response to the applied loading and "elastic-foundation" effects, where there are strains perpendicular to the crack plane (Li *et al.* 2004). Consider a section just behind the crack tip that is initially normal to the centroidal axis of an arm in a beam-like geometry. After loading, two types of rotation can occur. First, the angle between the section and the centroidal axis can change. This is the shear strain effect at the crack tip. The other type of rotation occurs when the centroidal axis and the section rotate together. This is called "root rotation". Therefore, Williams (1989) proposed beam on an elastic foundation model for estimating the end rotation correction for unidirectional DCB specimen. The free arm of length a is loaded with P at one end and the end section beyond the crack is modeled as a beam on an elastic foundation with stiffness per unit length characteristics of k_e in extension and k_r in rotation. Thus, the governing equilibrium equations of a Timoshenko beam on a two-parameter elastic foundation, when all the mechanical properties of the beam are constant, was reported as

$$\begin{cases} E_{fx}I\dfrac{d^2\psi^T}{dx^2} - kG_{xz}A\left(\psi^T + \dfrac{dw^T}{dx}\right) - k_r\psi^T = 0 \\ kG_{xz}A\left(\dfrac{d\psi^T}{dx} + \dfrac{d^2w^T}{dx^2}\right) - k_e w^T = 0 \end{cases} \tag{33}$$

where GA and EI are shear and flexural stiffnesses, respectively; and k_e is extensional stiffness of foundation defined in Eq. (14) and k_r is the rotational moduli of the foundation with assumption bending moment Pa is applied to crack tip is defined as

$$k_r = \frac{Pa}{\theta} = \frac{kG_{xz}bh}{2} \tag{34}$$

For the free beam section ($-a \le x \le 0$), $k_e = k_r = 0$ and Eq. (33) simplifies as before. So, $\psi_I(x)$ and $w_I(x)$ are the same as Eqs. (22) and (23). For region II, combination of Eq. (33) with elimination of ψ yields

$$\frac{d^4w^T(x)}{dx^4} - \left(\beta + \frac{1}{2\alpha}\right)\frac{1}{h^2}\frac{d^2w^T(x)}{dx^2} + \left(\frac{3\beta}{2\alpha}\right)\frac{1}{h^4}w^T(x) = 0 \tag{35}$$

where $\alpha = S_{11}/12kS_{66}$, $\beta = 2S_{22}/kS_{66}$, and S_{ij} are components of compliance matrix and defined as: $S_{11} = 1/E_{11}$, $S_{22} = 1/E_{22}$, $S_{66} = 1/G_{12}$. Assuming solutions of the form $w = e^{\mu x}$, Eq. (35) becomes

$$\mu^4 - 2\lambda_1^2\mu^2 + \lambda_2^4 = 0 \tag{36}$$

where

$$\lambda_1^2 = \frac{1}{2(\beta + (1/2\alpha))}, \quad \lambda_2^4 = \frac{3\beta}{2\alpha} \tag{37}$$

Williams (1989) discussed three cases of characteristic equation roots.

<u>*Case 1*</u>: $\lambda_1^2 > \lambda_2^2$, then $\mu_{1,2}^2 = \lambda_1^2 \pm \sqrt{\lambda_1^4 - \lambda_2^4}$

In this case, all four roots of characteristic equation are real. Compliance equation was concluded as

$$\frac{C}{C_0} = 1 + 3\left(\frac{h}{a}\right)^3\left[A_1\left(1 + \mu_1\frac{a}{h}\right) + A_2\left(1 + \mu_2\frac{a}{h}\right)\right] \tag{38}$$

where

$$A_1 = \frac{\mu_1[(a/h)-\mu_2(\mu_2^2-\beta)\alpha/\beta]}{[(\mu_1^3-\mu_2^3)-(\mu_1-\mu_2)\beta]}, \quad A_2 = \frac{-\mu_2[(a/h)-\mu_1(\mu_1^2-\beta)\alpha/\beta]}{[(\mu_1^3-\mu_2^3)-(\mu_1-\mu_2)\beta]} \tag{39}$$

<u>Case 2</u>: $\lambda_1^2 < \lambda_2^2$, then $\mu = \mu_1 + i\mu_2$ where $2\mu_1^2 = \lambda_2^2 + \lambda_1^2$ and $2\mu_2^2 = \lambda_2^2 - \lambda_1^2$
In this case, roots are complex. Compliance equation was obtained as

$$\frac{C}{C_0} = 1 + 3\left(\frac{h}{a}\right)^3\left[A_2\left(1+\mu_1\frac{a}{h}\right) - A_1\mu_2\frac{a}{h}\right] \tag{40}$$

where

$$A_1 = \frac{-[\mu_1(a/h)-(\mu_1^2+\mu_2^2)-(\mu_1^2-\mu_2^2-\beta)\alpha/\beta]}{\mu_2(3\mu_1^2-\mu_2^2-\beta)}, \quad A_2 = \frac{[(a/h)+2\mu_1(\mu_1^2+\mu_2^2)\alpha/\beta]}{3\mu_1^2-\mu_2^2-\beta} \tag{41}$$

<u>Case 3</u>: $\lambda_1^2 = \lambda_2^2$, then $\mu = \lambda_1$
In this case, Compliance relation is equal to

$$\frac{C}{C_0} = 1 + 3\left(\frac{h}{a}\right)^3\left[A_2\left(1+\mu\frac{a}{h}\right) - A_1\frac{a}{h}\right] \tag{42}$$

where

$$A_1 = \frac{-[\mu(a/h)-\mu^2(\mu^2-\beta)\alpha/\beta]}{3\mu^2-\beta}, \quad A_2 = \frac{[(a/h)+2\mu^3\alpha/\beta]}{3\mu^2-\beta} \tag{43}$$

Because of the complexity of solutions, Williams (1989) simplified all three case in one equation with the assumption of $h/a \ll 1$.

6. Results and discussion

In this study, $[0^\circ]_6$ unidirectional and $[\pm\theta]_5$ ($\theta = 30^\circ$, 45°) angle-ply laminates are taken into account to assess different models. The strain energy release rates of these three lay-ups are measured experimentally by Ozdil and Carlsson (1999) and are frequently used by other researchers i.e., Hamed *et al.* (2006) and Gordnian *et al.* (2008). The presented compliance relations in previous sections depend on E_{fx}, E_z, G_{xz}, h, a, and b parameters. h, a, and b are half-thickness, crack length, and width of DCB specimen, respectively. The modulus E_z was also estimated in Ozdil and Carlsson (1999) using the laminate homogenization method of Hyer and Knott (1994). For the mechanical properties of E-glass/polyester lamina listed in Table 1, E_z, E_{fx}, and geometry parameters are shown in Table 2 for unidirectional and angle-ply lay-ups the same as those given in Ozdil and Carlsson (1999). During the calculation of effective moduli, it should be noted that for the particular laminates considered in this research, $[\pm\theta]_5$ laminate, the beams of delaminated region consist of $[+\theta/-\theta/+\theta/-\theta/+\theta]$ and $[-\theta/+\theta/-\theta/+\theta/-\theta]$ lay-ups.

To compare the accuracy of the different beam models on elastic foundation in predicting compliance and SERR, these values at crack initiation are presented in Table 3. It should be noted that for estimating SERR, load at crack initiation as well as initial crack length are needed.

Table 1 Mechanical properties of unidirectional E-glass/polyester (Ozdil and Carlsson 1999)

E_1, GPa	$E_2 = E_3$, GPa	$\nu_{12} = \nu_{13}$	ν_{23}	$G_{12} = G_{13}$, GPa	G_{23}, GPa
34.7	8.5	0.27	0.5	4.34	2.83

Table 2 Geometry and calculated values of modulus for glass/polyester laminates with b = 20 mm

Lay-up	$[0^\circ]_6$	$[\pm 30^\circ]_5$	$[\pm 45^\circ]_5$
2h, mm *	4.4	7.3	7.3
E_z, GPa**	8.5	9.37	9.85
E_{fx}, GPa***	34.7	19.22	12.32

*h is half-thickness of DCB specimen.
**E_z is effective out-of-plane extensional modulus of a laminate.
***E_{fx} is effective flexural modulus of a laminate.

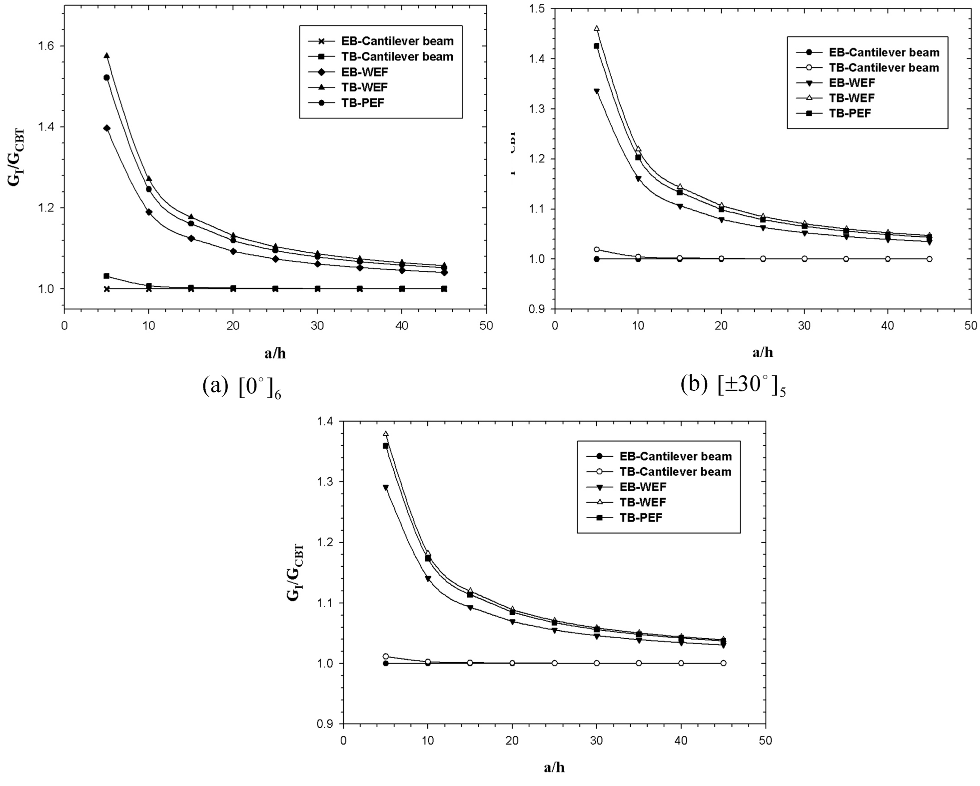
Table 3 Comparison of initial compliance (μm/N) and SERR (J/m^2) for DCB specimen

	$[0^\circ]_6$, a = 33 mm, P = 46 N		$[\pm 30^\circ]_5$, a = 30 mm, P = 64 N		$[\pm 45^\circ]_5$, a = 32 mm, P = 51 N	
	Compliance	SERR	Compliance	SERR	Compliance	SERR
Ozdil *et al.* (1999), Experiment	61	282±42	13	214±14	29	176±32
Beam Models						
Completely clamped EB	39.44	189.67	11.56	118.33	21.87	133.37
Completely clamped TB	39.86	190.34	11.78	119.1	22.12	133.87
Modified EB (rotational spring at crack tip)	39.88	190.43	11.8	119.2	22.14	133.93
EB-WEF, Ozdil (1999)	47.03	213.27	15.1	141.41	27.35	154.76
TB-WEF, Kondo (1995)	50.31	223.11	16.38	149.3	28.97	160.84
TB-PEF, Williams (1989)	43.11	201.23	12.97	124.57	26.12	148.5

Therefore, loads at each crack length are measured from the available load-displacement experimental curves in Ozdil and Carlsson (1999). In Table 3, EB, TB, WEF, and PEF stand for Euler-Bernoulli beam, Timoshenko beam, Winkler elastic foundation, and Pasternak elastic foundation, respectively. The initiation fracture toughness using experimental compliance calibration was reported for $[0^\circ]_6$, $[\pm 30^\circ]_5$, and $[\pm 45^\circ]_5$ lay-ups in Ozdil and Carlsson (1999).

Table 3 shows that predicted results by EB and TB on elastic foundations are more accurate and acceptable than predicted ones by elementary theories. Although experimental results have a wide range of deviation, but TB on WEF is predicting initial compliance and SERR very well in all laminates lay-ups. Williams solution (1989) is depended on the compliance matrix components (S_{ij}) of a laminate. As a result, the effective moduli are not used for Williams relation.

In general, compliances and SERR are underestimated in all lay-ups when beam theories (i.e., EB and TB) resting on elastic foundation are used. Among the beam models, TB on WEF is predicting SERR well than the other models. This is in contrast to previous report of Ozdil and Carlsson (1999) that claim EB on WEF was in very good agreement with those experimental compliances. This comparison shows that shear deformation is an important factor in unidirectional and, in particular, angle-ply laminates. However, errors between analytical and experimental results are negligible for unidirectional and angle-ply with $\theta = 45^{\circ}$ DCB specimens, whereas errors are significant in $[\pm 30^{\circ}]_5$ angle-ply laminate. It means that in angle-ply laminates other factors in additional to shear deformation and root rotation are effective. Mode mixity, residual stresses, non-uniformity of SERR distribution along the delamination front seem to have the most contributions. In unidirectional laminates, the stiffnesses of adjacent plies are the same, so the mode partitioning is unambiguous. But in angle-ply laminates partitioning of SERR, due to oscillatory character of stresses and displacements near the two different layers, is more likely. Ozdil and Carlsson (1999) reports that Davidson *et al.* (1996) using a 3D finite element analysis of DCB specimens shows the mode II and mode III contributions to the total SERR are negligible (<1%). Residual stresses due to thermal gradients during cooling of plies can arise in multidirectional laminates. If the two cracked regions and the uncracked region to be symmetric about their own midplanes, the residual effect can be eliminated (Andersons and König 2004). For the special DCB specimens of this study, the beams of delaminated region have $[+\theta/-\theta/+\theta/-\theta/+\theta]$ and $[-\theta/+\theta/-\theta/+\theta/-\theta]$ lay-ups that the coupling matrix ($[B]$) is equaled to zero, while the full laminate is unsymmetric ($[B] \neq 0$). So, it can be concluded that residual stresses can affect slightly on SERR in angle-ply lay-ups. The other main factor may affect on the SERR is the non-uniformity of SERR along the delamination front. It is desirable for data reduction that the SERR distribution along the crack front is uniform (Andersons and König 2004). Hence, it is suggested that the stacking sequence of the laminate should be



(a) $[0^{\circ}]_6$ (b) $[\pm 30^{\circ}]_5$

(c) $[\pm 45^{\circ}]_5$

Fig. 4 Effect of crack length on the normalized SERR on various beams on EF models designed so that longitudinal-transverse bending coupling of the delaminated legs is minimized. In angle-ply lay-ups, this coupling is more dominated than unidirectional ones. By increasing bending-bending coupling, the behavior of DCB specimen changes between plane strain and plane stress conditions. According to Olsson (1992), the crack length to specimen width ratio shows a moderate state (neither plane strain nor plane stress), whereas effective flexural modulus is obtained with plane stress assumption in all models. Also, by comparing EB on WEF and Williams solution, it can be concluded that TB on PEF compliances approach to EB on WEF ones with changing lay-up angle. It shows Williams's model, in addition to complicated expressions and complex roots, is too stiff in unidirectional DCB specimens because of considering rotational foundation stiffness.

To investigate the influence of crack length on SERR for various beam models on elastic foundation, a parametric study is carried on the normalized SERR. Fig. 4 shows normalized SERR that normalization has been done with classical beam theory (CBT) versus crack length to thickness ratio (*a/h*). It can be observed from Fig. 4 that the SERR values obtained from different beam theories resting on various elastic foundations approach those of CBT as the crack length increase. For short crack lengths (i.e., $15 < a/h < 25$), shear deformation effect is significant on SERR (Prasad and Kumar 2009). Also, by increasing laminate angle from 0^o to 45^o the shear deformation effect on SERR decrease. For example, at $a/h = 5$, the difference between normalized SERR in EB on WEF and TB on WEF models are 12.94%, 8.95%, and 6.97% in $[0^o]_6, [\pm 30^o]_5$ and $[\pm 45^o]_5$ respectively.

7. Conclusions

The accuracy of various beam theories (Euler-Bernoulli and Timoshenko beams) on different elastic foundation (Winkler and Pasternak elastic foundations) for analysis of Double cantilever beam (DCB) is taken into account. In other words, the contributions of two main factors, transverse shear deformation and root rotation, are investigated on the compliance and strain energy release rate (SERR) in unidirectional and angle-ply laminates. In each case, a closed-form solution for compliance versus crack length is presented using effective flexural modulus (E_{fx}) and out-of-plane extensional stiffness (E_z). The results have been compared with those available experimental data in Ozdil and Carlsson (1999). The predicted compliance and SERR values from TB on WEF are in good agreement with those determined experimentally. Although TB on WEF predicts initiation of SERR in all lay-ups, TB on PEF as a general case, by increasing laminate angle shows better compliances for angle-ply lay-ups, especially $[\pm 30^o]_5$. It should be noted that warping of the beam occurs owing to the shear force and to the non-uniformity of the stresses at the crack tip region. This limits the accuracy of the analytical approaches to problems involving relatively small strain.

Ozdil and Carlsson (1999) suggested EB on WEF predicts good compliances for both unidirectional and angle-ply lay-ups while this study shows contrary to this claim. Williams (1989) concluded that root rotation due to low shear stiffness is the main factor causing composite DCB specimens to deviate from the cantilever beam theory. However, in this research it is observed that other factors with the exception root rotation, e.g., residual stress, mode mixity, and non-uniformity of SERR distribution along delamination front, for delamination modeling of angle-ply laminates will definitely be effective. Also, the behavior of laminated DCB specimens is assumed to be linear elastic in a state of plane stress, whereas obtaining effective properties in terms of real state of specimen as well as considering crack tip plastic or craze/damage zone might provide a better prediction of delamination fracture toughness.

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