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Analytical studies on stress concentration due to a rectangular small hole in thin plate under bending loads

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Abstract. In general means, the stress concentration problem of elastic plate with a rectangular hole can be investigated by numerical methods, and only approximative results are derived. This paper deduces an analytical study of the stress concentration due to a rectangular hole in an elastic plate under bending loads. Base on classical elasticity theory and FEM applying the U-transformation technique, the uncoupled governing equations with 3-DOF are established, and the analytical displacement solutions of the finite element equations are derived in series form or double integral form. Therefore, the stress concentration factor can then be discussed easily and conveniently. For the plate subjected to unidirectional bending loads, the non-conforming plate bending element with four nodes and 12-DOF is taken as examples to demonstrate the application of the proposed method. The inner force distribution is obtained. The solutions are adequate for the condition when the hole is far away from the edges and the thin plate subjected to any transverse loadings.

Keywords: U-transformation; stress concentration; rectangular hole; analytical solution; bending loads.

1. Introduction

The stress concentration problem in plate structure with a circular or elliptic hole can be analyzed by simple mathematical means, and achieves in analytical solutions. For the problem with a rectangular hole, it can be solved by some numerical methods. Savin (1958) investigated the stress concentration of finite plate by the conformal mapping technique and complex variable method, by which the right angle of the hole can be transformed to a round angle, the straight lines to curves. Timoshenko (1951), Pilkey (1997), Young (2002) and Troyani *et al.* (2002) studied the stress concentration problem of bending plates by elasticity theory, and exhaustive stress concentration factor figures and tables were proposed. With the help of computers, the finite element method was developed rapidly and shown its advantages in solving mechanics problems. By using FEM, scientists can study the stress concentration problem of plate with any shape holes and under any loadings (Zienkiewicz *et al.* 2000, Jain *et al.* 2008, Kubair *et al.* 2008). Toubal *et al.* (2005) and

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Younis (2006) studied experimentally for stress concentration around the holes in a plate.

But, by the above methods, only approximative results are derived. When the size of the hole changes, the problem must be recalculated, and furthermore, when the hole is too small to the plate, it must spend too much computation. This paper aims to deduce an analytical study of stress concentration due to a rectangular hole in an elastic plate under bending loads.

The U-transformation method is an exact analytical method, which can be used to analyze the structures with periodicity (Cai *et al.* 2002, Chen *et al.* 1998). Liu *et al*. combined the U-transformation technique with the finite difference method and the finite element method, respectively, to analyze the static and dynamic problem of rectangular plates and the simply supported beams, and the exact error expressions and convergence rates of finite difference solutions and finite element solutions are obtained (Yang *et al.* 2007, 2009). Recently, Yang *et al.* applied the U-transformation method to investigate the stress concentration problem of a rectangular hole in an infinite plate, and the analytical stress concentration factors under biaxial tension and shearing loading are derived (Yang *et al.* 2008). The present paper extends the method to study the stress concentration due to a rectangular hole in elastic plate under bending loads. For the plate subjected to unidirectional bending loads, a 12-DOF plate bending element with four nodes is taken as examples to demonstrate the application of the proposed method.

2. Bending plate with four edges free

Consider a thin plate with completely free boundary. The plate subjected to bending loads, and with a small rectangular hole in the structure, as shown in Fig. 1. m_x, m_y are the bending moments per length the two ends subjected to respectively. The thickness of the plate is supposed to be an unit length and the size of the small hole is much less than the distance from the hole to the edges of the plate.

The original loading condition can be replaced by the superposition of the two loading condition, shown in Fig. 2(a) and Fig. 2(b). The exact solution of the system shown in Fig. 2(a) is clear, i.e.

$$M_x = m_x,\ M_y = m_y,\ M_{xy} = 0 \tag{1}$$

So the original problem is changed to seek the solution of the system shown in Fig. 2(b). This solution can be applied to the bending moment concentration problem under other boundary and loading conditions, so long as the hole is small enough and far from the edges.

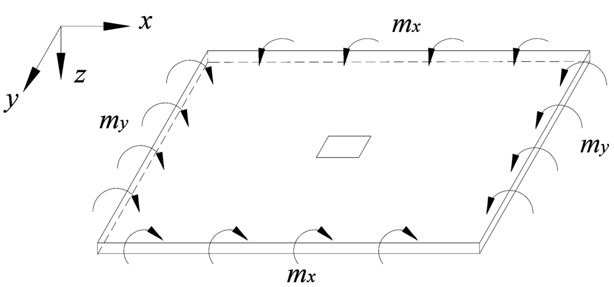
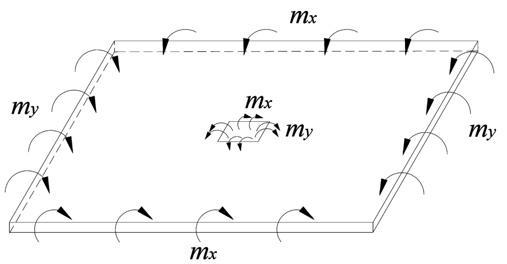
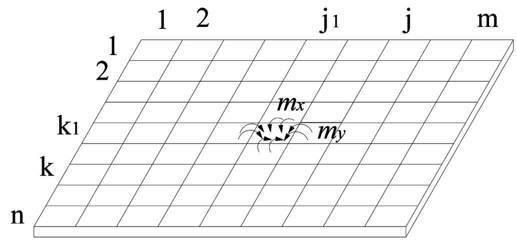


Fig. 1 Bending thin plate with a small rectangular hole



(a) Adding reversed bending loads at the hole



(b) Minus the bending loads added to the hole

Fig. 2 Equivalent system

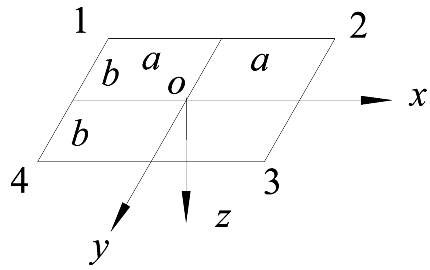


Fig. 3 Rectangular plate element

3. Governing equation

Consider the infinite plate with rectangular hole shown in Fig. 2(b). The plate is divided into $m \times n$ equal elements and then the plate may be regarded as having cyclic periodicities in two directions without influence on the result. The size of every element is the same as that of the hole, i.e., $2a \times 2b$, where $2a$ and $2b$ denote the lengths of the rectangular element in x and y direction respectively, shown in Fig. 3. (j, k) denotes the number of element and (j_1, k_1) is the hole number. Now, we can use the U-transformation method and FEM to analyze this cyclic periodic structure (Cai *et al.* 2002).

The element displacement vector can be expressed as

$$\delta_{(j,k)} = \{\delta_1^T \ \delta_2^T \ \delta_3^T \ \delta_4^T\}_{(j,k)}^T, \quad j = 1, 2, \ldots, m, \ k = 1, 2, \ldots, n$$

$$\delta_i = \{w_i \ \ \theta_{xi} \ \ \theta_{yi}\}^T, \quad i = 1, 2, 3, 4, \quad \theta_{xi} = \left(\frac{\partial w}{\partial y}\right)_i, \quad \theta_{yi} = -\left(\frac{\partial w}{\partial x}\right)_i \tag{2}$$

in which $w_i, \theta_{xi}, \theta_{yi}$ denote, respectively, the deflection and two angular rotations of node i.

In order to bring periodicity to the finite element equation, the interior disfigurement made by the rectangular small hole may be replaced by an additional loading (Yang *et al.* 2008). It can be expressed as

$$\mathbf{F}_{(j,k)}^0 = \begin{cases} \mathbf{K}^e \delta_{(j_1,k_1)}, & (j,k) = (j_1,k_1) \\ 0, & (j,k) \neq (j_1,k_1) \end{cases} \tag{3}$$

in which $\delta_{(j_1,k_1)}$ is the displacement vector of hole element (j_1, k_1) and is unknown, and $\mathbf{K}^e$ denotes the stiffness matrix of the rectangular element. $\mathbf{F}^0_{(j,k)}$ is just a loading vector in form.

For expressing conveniently, the bending loads at the edges of the small hole are regarded as applying at the suppositional hole element. The external loading vector shown in Fig. 2(b) may be expressed as

$$\mathbf{F}'_{(j_1,k_1)} = \{0\ \ m_y a\ \ -m_x b\ \ 0\ \ m_y a\ \ m_x b\ \ 0\ \ -m_y a\ \ m_x b\ \ 0\ \ -m_y a\ \ -m_x b\}^T$$
$$\mathbf{F}'_{(j,k)} = 0, \quad (j,k) \neq (j_1,k_1) \tag{4}$$

where m_x, m_y denote the bending moment per unit length in x, y direction respectively. Because of the symmetry of the loading condition and the cyclic periodicity of the structure, the hole's displacements must satisfy the following symmetric condition

$$\begin{aligned} w_{1(j_1,k_1)} &= w_{2(j_1,k_1)} = w_{3(j_1,k_1)} = w_{4(j_1,k_1)} \\ \theta_{x1(j_1,k_1)} &= \theta_{x2(j_1,k_1)} = -\theta_{x3(j_1,k_1)} = -\theta_{x4(j_1,k_1)} \\ \theta_{y1(j_1,k_1)} &= -\theta_{y2(j_1,k_1)} = -\theta_{y3(j_1,k_1)} = \theta_{y4(j_1,k_1)} \end{aligned} \tag{5}$$

So the total potential energy for the considered system may be defined as

$$\Pi = \sum_{j=1}^{m}\sum_{k=1}^{n} \pi_{(j,k)} \tag{6}$$

in which $\pi_{(j,k)}$ is the potential energy of the element (j,k), and it may be defined as

$$\pi_{(j,k)} = \frac{1}{2}\bar{\delta}^T_{(j,k)}\mathbf{K}^e\delta_{(j,k)} - \frac{1}{2}(\bar{\delta}^T_{(j,k)}\mathbf{F}_{(j,k)} + \delta^T_{(j,k)}\bar{\mathbf{F}}_{(j,k)}) \tag{7}$$

The superior bar in Eq. (7) denotes complex conjugation. $\mathbf{F}_{(j,k)}$ denotes the loading vector for the element (j,k) and including the external loading and the additional loading, i.e.

$$\mathbf{F}_{(j,k)} = \mathbf{F}^0_{(j,k)} + \mathbf{F}'_{(j,k)} \tag{8}$$

The displacement vectors must satisfy the following continuity condition

$$\delta_{2(j,k)} = \delta_{1(j+1,k)}, \quad \delta_{3(j,k)} = \delta_{1(j+1,k+1)}, \quad \delta_{4(j,k)} = \delta_{1(j,k+1)}$$
$$j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n \tag{9}$$

and cyclic symmetric condition

$$\delta_1(m+1,k) \equiv \delta_1(1,k), \quad \delta_1(j,n+1) \equiv \delta_1(j,1), \quad \delta_1(m+1,n+1) \equiv \delta_1(1,1) \tag{10}$$

Apply the double U-transformation (Yang *et al.* 2009) to Eq. (7), i.e., let

$$\mathbf{q}_{(r,s)} = \frac{1}{\sqrt{mn}}\sum_{j=1}^{m}\sum_{k=1}^{n} e^{-i(j-1)r\varphi_1} e^{-i(k-1)s\varphi_2}\delta_{(j,k)}, \quad r = 1, 2, \ldots, m;\ s = 1, 2, \ldots, n \tag{11}$$

in which $\varphi_1 = 2\pi/m$, $\varphi_2 = 2\pi/n$, and $i = \sqrt{-1}$. $\mathbf{q}_{(r,s)}$ is the generalized displacement vector which can be expressed as

$$\mathbf{q}_{(r,s)} \equiv \{\mathbf{q}_1^T \ \mathbf{q}_2^T \ \mathbf{q}_3^T \ \mathbf{q}_4^T\}_{(r,s)}^T, \quad \mathbf{q}_{i(r,s)} \equiv \begin{Bmatrix} q_i \\ q_{i,x} \\ q_{i,y} \end{Bmatrix}_{(r,s)}, \quad i = 1, 2, 3, 4 \tag{12}$$

Now, the total potential energy Eq. (7) may be rewritten as

$$\Pi = \frac{1}{2}\sum_{r=1}^{m}\sum_{s=1}^{n} \bar{\mathbf{q}}_{(r,s)}^T \mathbf{K}^e \mathbf{q}_{(r,s)} - \frac{1}{2}\sum_{r=1}^{m}\sum_{s=1}^{n} (\bar{\mathbf{q}}_{(r,s)}^T \mathbf{f}_{(r,s)} + \mathbf{q}_{(r,s)}^T \bar{\mathbf{f}}_{(r,s)}) \tag{13}$$

in which

$$\mathbf{f}_{(r,s)} = \frac{1}{\sqrt{mn}} e^{-i(j_1-1)r\varphi_1} e^{-i(k_1-1)s\varphi_2} (\mathbf{F}'_{(j_1,k_1)} + \mathbf{K}^e \boldsymbol{\delta}_{(j_1,k_1)}) \tag{14}$$

The continuity condition (9) becomes

$$\mathbf{q}_{(r,s)} = \mathbf{T}_{(r,s)} \mathbf{q}_{1(r,s)} \tag{15}$$

where

$$\mathbf{T}_{(r,s)} = [\mathbf{I}_3 \ \ e^{ir\varphi_1}\mathbf{I}_3 \ \ e^{i(r\varphi_1+s\varphi_2)}\mathbf{I}_3 \ \ e^{is\varphi_2}\mathbf{I}_3]^T \tag{16}$$

is a transform matrix, and $\mathbf{I}_3$ is an unit matrix of order three.

Applying Eq. (15) to the potential energy Eq. (13), yields

$$\Pi = \frac{1}{2}\sum_{r=1}^{m}\sum_{s=1}^{n} \bar{\mathbf{q}}_{1(r,s)}^T \mathbf{K}_{(r,s)}^* \mathbf{q}_{1(r,s)} - \frac{1}{2}\sum_{r=1}^{m}\sum_{s=1}^{n} (\bar{\mathbf{q}}_{1(r,s)}^T \mathbf{f}_{(r,s)}^* + \mathbf{q}_{1(r,s)}^T \bar{\mathbf{f}}_{(r,s)}^*) \tag{17}$$

in which

$$\mathbf{K}_{(r,s)}^* = \bar{\mathbf{T}}_{(r,s)}^T \mathbf{K}^e \mathbf{T}_{(r,s)} \tag{18}$$

$$\mathbf{f}_{(r,s)}^* = \bar{\mathbf{T}}_{(r,s)}^T \mathbf{f}_{(r,s)} \tag{19}$$

Substituting Eq. (17) into the variational equation $\delta\Pi = 0$, results in

$$\mathbf{K}_{(r,s)}^* \mathbf{q}_{1(r,s)} = \mathbf{f}_{(r,s)}^*, \quad r = 1, 2, \ldots, m, \ s = 1, 2, \ldots, n \tag{20}$$

The vector $\mathbf{f}_{(r,s)}^*$ including the nodal displacement of the hole element, so Eq. (20) is still not uncoupled. When one consider the specific element stiffness matrix and loading, the hole's nodal displacement vector $\boldsymbol{\delta}_{1(j_1,k_1)}$ must be worked out firstly, and then other elements' displacements can be solved by Eqs. (11) and (20).

4. Example

Consider now the 12-DOF non-conforming plate bending element with four nodes as shown in Fig. 3, and the thin plate subjects to unidirectional bending loads, i.e., $m_y = 0$. Introducing the stiffness matrix (East China 1978) and the transform matrix (16) into Eq. (18), and noting Eqs. (3), (4) and (19), Eq. (20) becomes

$$\frac{D}{30ab}\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}\begin{Bmatrix} q_1 \\ q_{1,x} \\ q_{1,y} \end{Bmatrix}_{(r,s)} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}^*_{(r,s)} \tag{21}$$

in which

$$k_{11} = -\frac{3B}{a^2b^2}[5a^4C_1(t_1)C_2(t_2) + 5b^4C_1(t_1)C_2(t_1) + a^2b^2C_2(t_1)C_2(t_2)(-7+2\mu)]$$

$$k_{12} = P_1(a,b,t_1,t_2) = \frac{3B}{b}C_3(t_2)[5a^2C_1(t_1) + b^2C_2(t_1)(-1+\mu)]$$

$$k_{13} = -P_1(b,a,t_2,t_1)$$

$$k_{22} = P_2(a,b,t_1,t_2) = 2B[5a^4C_1(t_1)C_1(t_2) - b^2C_2(t_1)C_4(t_2)(-1+\mu)]$$

$$k_{33} = P_2(b,a,t_2,t_1)$$

$$k_{21} = -k_{12}, \quad k_{23} = k_{32} = 0, \quad k_{31} = -k_{13}$$

$$B = e^{-i(t_1+t_2)}, \quad t_1 = r\varphi_1, \quad t_2 = s\varphi_2, \quad C_1(x) = (1+4e^{ix}+e^{2ix})$$

$$C_2(x) = (-1+e^{ix})^2, \quad C_3(x) = (-1+e^{2ix}), \quad C_4(x) = 1-8e^{ix}+e^{2ix}$$

$$f^*_{1(r,s)} = 0$$

$$f^*_{2(r,s)} = DA(1+e^{-ir\varphi_1})(1-e^{-is\varphi_2})\left[\theta_{x1(j_1,k_1)} - \frac{b}{a}\mu\theta_{y1(j_1,k_1)}\right]$$

$$f^*_{3(r,s)} = DA(1-e^{-ir\varphi_1})(1+e^{-is\varphi_2})\left[-\frac{b}{a}\mu\theta_{x1(j_1,k_1)} + \frac{b^2}{a^2}\theta_{y1(j_1,k_1)} - \frac{m_x b}{D}\right]$$

$$A = \frac{1}{\sqrt{mn}}e^{-i(j_1-1)r\varphi_1}e^{-i(k_1-1)s\varphi_2}, \quad D = \frac{Et^3}{12(1-\mu^2)} \tag{22}$$

The component of $\mathbf{q}_{1(r,s)}$ may be derived from Eq. (22)

$$\mathbf{q}_{1(r,s)} = \frac{30ab}{D} \times \frac{-k_{12}k_{33}f^*_{2(r,s)} - k_{13}k_{22}f^*_{3(r,s)}}{k_{13}^2k_{22} + k_{12}^2k_{33} + k_{11}k_{22}k_{33}}$$

$$q_{1,x(r,s)} = \frac{30ab}{D} \times \frac{(k_{13}^2 + k_{11}k_{33})f_{2(r,s)}^* - k_{12}k_{13}f_{3(r,s)}^*}{k_{13}^2 k_{22} + k_{12}^2 k_{33} + k_{11}k_{22}k_{33}}$$

$$q_{1,y(r,s)} = \frac{30ab}{D} \times \frac{-k_{12}k_{13}f_{2(r,s)}^* + (k_{12}^2 + k_{11}k_{22})f_{3(r,s)}^*}{k_{13}^2 k_{22} + k_{12}^2 k_{33} + k_{11}k_{22}k_{33}} \tag{23}$$

In Eq. (23), when $(r,s) = (m,n)$, $\mathbf{q}_1(m,n)$ denotes the rigid body displacement. So it can by supposed that $\mathbf{q}_1(m,n) = 0$.

Substituting Eqs. (22) and (23) into the U-transformation (11) results in

$$\begin{Bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \end{Bmatrix}_{(j,k)} = \frac{1}{\sqrt{mn}} \sum_{r=1}^{m} \sum_{s=1}^{n} e^{i(j-1)r\varphi_1} e^{i(k-1)s\varphi_2} \begin{Bmatrix} q_1 \\ q_{1,x} \\ q_{1,y} \end{Bmatrix}_{(r,s)} \tag{24}$$

5. Solutions

It can be proved that the right side of Eq. (24) is real number vector, so we can pick-up the real part from every item of the series. Let $(j,k) = (j_1,k_1)$, the displacement vector of node 1 in the square hole may be expressed as

$$\begin{Bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \end{Bmatrix}_{(j_1,k_1)} = \frac{1}{\sqrt{mn}} \sum_{r=1}^{m} \sum_{s=1}^{n} \mathrm{Re} \left\{ e^{i(j_1-1)r\varphi_1} e^{i(k_1-1)s\varphi_2} \begin{Bmatrix} q_1 \\ q_{1,x} \\ q_{1,y} \end{Bmatrix}_{(r,s)} \right\} \tag{25}$$

It can be proved that when m and n approach to infinity the series at the right-hand side of Eq. (25) are convergent.

For this example, the case when the hole is square and the plate subjected to unidirectional bending loads is discussed, i.e.

$$a = b,\ m = n,\ \varphi_1 = \varphi_2 = \varphi = \frac{2\pi}{n} \tag{26}$$

Substituting Eqs. (22), (23) into (25), and letting $n = 1000$, the displacement $\delta_{1(j_1,k_1)}$ can be obtained as

$$\begin{Bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \end{Bmatrix}_{(j_1,k_1)} = \frac{m_x a}{D} \begin{Bmatrix} 4.2373a \\ 0.1972 \\ -0.5686 \end{Bmatrix} \tag{27}$$

Applying solution (27) to Eq. (24), the nodal displacement for every element may be found.

After the displacements of the hole and other elements are found, we can calculate the internal force in the plate. Base on the element displacement functions, the internal force equation may be expressed as

Table 1 Some internal force results of node 1

	(j_1, k_1+1)	(j_1+1, k_1+1)	(j_1+2, k_1+1)	(j_1+3, k_1+1)	(j_1+4, k_1+1)
M_x	0.5705	-0.3167	-0.0623	-0.0250	-0.0132
M_y	0.1769	-0.0892	-0.0123	-0.0100	-0.0055
M_{xy}	0.1710	-0.1785	-0.0089	-0.0005	0.0001
Multiplier			m_x		

Table 2 Internal force of the original structure at node 1

	(j_1, k_1+1)	(j_1+1, k_1+1)	(j_1+2, k_1+1)	(j_1+3, k_1+1)	(j_1+4, k_1+1)
M_x	1.5705	0.6833	0.9377	0.9750	0.9868
M_y	0.1769	-0.0892	-0.0123	-0.0100	-0.0055
M_{xy}	0.1710	-0.1785	-0.0089	-0.0005	0.0001
Multiplier			m_x		

$$\mathbf{M}_{1(j,k)} = \mathbf{S}_1 \boldsymbol{\delta}_{(j,k)} \tag{28}$$

in which $\mathbf{M}_{1(j,k)}$, $\mathbf{S}_1$ denote the internal force vector and internal force matrix of node 1 (East China 1978) respectively, and $\mathbf{M}_{1(j,k)} = \{M_x \;\; M_y \;\; M_{xy}\}^T_{(j,k)}$.

Applying the results for nodal displacement to Eq. (27), the internal force at node 1 for every element can be found. The internal force M_x, M_y and M_{xy} at node 1 of the first 5 elements on the right side of the square hole are given in Table 1.

Adding the results in Table 1 to the solutions (1), results in the internal force of the original system shown in Fig. 1, which is given in Table 2.

Obviously, the maximum value of M_x occurs at node 1 of the element (j_1+1, k_1), which equals to $1.5705m_x$. It can be worked out easily that the maximum moment is $M_{\max} = 1.5911m_x$ and makes an angle of 21.66° with x-axis. So the stress concentration factor is 1.5911. Moreover, M_x of the element that at the right side of the hole is less than $1.0m_x$, and then increases along the direction far from the hole, tends to $1.0m_x$. The difference between the moment M_x of the fifth element at the right side of the square hole and the average moment is just 1%, which shows that the stress concentration attenuates rapidly along the direction far from the hole.

The finite element solution of the stress concentration factor on the above problem is 1.800. Comparing the studies developed in this work with that simulated by finite element method, the relative error is only 13%.

6. Conclusions

This paper presents analytical studies on the stress concentration problem of the infinite plate due to a rectangular hole under bending loads. The interior disfigurement made by the rectangular hole may be replaced by an additional loading. After the finite element governing equation with cyclic periodicity in two directions is established, the double U-transformation technique is applied to the FEM equation and the analytical nodal displacement solutions are derived. With a 12-DOF plate bending element with 4 nodes, the internal force of the elements near the square hole are obtained. The results show that the stress concentration factor of the square hole under unidirectional bending loads is approximately equal to 1.5911.

In this paper, the hole is divided into only one element. Since the stress nearby the hole changes acutely, it is not accurate enough. If the hole divided into four elements, more accurate results may be obtained. But at the same time, it leads to solving a linear equations set with four unknowns, and the analytical solution expressions will be prolix.

The proposed method and results can also be applied to analyze the bending problem of finite thin plate with normal small rectangular hole, but the hole must be far away from the plate's edges. At first, a group of internal force at the mid-point of the hole place when the plate without any hole should be worked out. Then applying another group of internal force with opposite directions to the hole's boundary, the solutions can be studied by the method provided in the present paper. Adding the later solutions to the previous ones, results in the internal force of the original problem. For the case a rectangular small hole near to the plate's edges, the stress concentration is affected by the edge and it would be investigated in the authors' next paper.

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