Structural Engineering and Mechanics, Vol. 36, No. 2 (2010) 149-163 149
DOI: http://dx.doi.org/10.12989/sem.2010.36.2.149

Nonlinear vibrations of axially moving beams with multiple concentrated masses Part I: primary resonance

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(Received February 22, 2010, Accepted May 19, 2010)

Abstract. Transverse vibrations of axially moving beams with multiple concentrated masses have been investigated. It is assumed that the beam is of Euler-Bernoulli type, and both ends of it have simply supports. Concentrated masses are equally distributed on the beam. This system is formulated mathematically and then sought to find out approximately solutions of the problem. Method of multiple scales has been used. It is assumed that axial velocity of the beam is harmonically varying around a mean-constant velocity. In case of primary resonance, an analytical solution is derived. Then, the effects of both magnitude and number of the concentrated masses on nonlinear vibrations are investigated numerically in detail.

Keywords: axially moving beam; concentrated mass; method of multiple scales; nonlinear vibrations.

1. Introduction

Axially moving beams are used in many engineering device (Ulsoy *et al.* 1978, Wickert and Mote 1988, Abrate 1992). Moving beams used for transport and printing purposes in some devices, involve various projecting parts. These projecting parts can be modeled as concentrated mass. Thus, a conclusion about vibration characteristics of real system can be made by means of the model.

Large transverse vibrations are one of the most important problems in moving continua systems. This problem is resulted from beam pre-stress, beam axial velocity change, etc. According to these parameters, many different linear and nonlinear continua models have been constructed. By means of these models, many studies on vibration characteristics of the real systems were carried out; Pasin (1972), studied the stability of transverse vibrations of beams with periodically reciprocating motion in axial direction. Simpson (1973), studied the unstressed moving beam, while their models did not account for the effect of tension. Wickert and Mote (1990) presented a complex modal method for axially moving continua including beams where natural frequencies and modes associated with free vibrations serve as a basis for analysis. Wickert and Mote (1991) observed that Rayleigh's quotient provide a variation method in determining eigenfunctions of axially moving

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continua for gyroscopic systems. Öz and Pakdemirli (1999), and Öz (2001) obtained mode shapes of an axially moving beam under pinned-pinned and clamped-clamped supported cases. To find natural frequencies of an axially moving beam pinned-pinned supported, Özkaya and Öz (2002) treated artificial neural networks. Öz (2003) calculated natural frequencies of an axially moving beam which was in touch with a small mass under pinned-pinned and clamped-clamped boundary conditions. Öz *et al.* (1998), studied dynamic stability of an axially accelerating beam with small bending stiffness by means of multiple scale method. To construct non-resonant boundary layer solutions for an axially accelerating beam with small bending stiffness, Özkaya and Pakdemirli (2000) combined the method of multiple scales and the method of matched asymptotic expansions. Öz *et al.* (2001) and Öz (2001) applied the method of multiple scales to calculate analytically the stability boundaries of an axially accelerating beam. To determine stability boundary of an axially accelerating beam, Özkaya and Öz (2002) used an artificial neural network algorithm. Based on one-term Galerkin discretization, Ravindra and Zhu (1998) analyzed chaotic behavior of axially accelerating beam. Pellicano *et al.* (2001), studied on vibrations of power transmission belts, and developed a theoretical model. Pellicano and Vestroni (2002) investigated axially moving beam subjected to a transverse force at super-critical velocity range. To study dynamic stability of an axially accelerating beam subjected to a tension fluctuation, Parker and Lin (2001) adopted one-term Galerkin discretization and the perturbation method. Chen and Yang (2007), analyzed equations of motion of axially moving beams as two models; a partial differential equation and an integro-partial differential equation, and obtained natural frequencies. Based on Timoshenko model, Tang *et al.* (2008) analyzed natural frequencies, mode shapes, and critical velocity of the axially moving beam for different end conditions. Lee and Jang (2007), investigated the effects of the continuously incoming and outgoing semi-infinite beam parts on the dynamic characteristics and stability of an axially moving beam by using the spectral element method. Finally, nonlinear transverse vibrations of a slightly curved Euler Bernoulli beam carrying a concentrated mass has been studied by Ozkaya *et al.* (2009).

The case considered at all studies about axially moving beams is that the beam is homogeneous. Aim of this assumption is to ease the calculations in engineering works. In case of nonhomogeneity such as trapezoid and square shapes etc. that may occur on the beam, to calculate solution as approximate as possible to real system, they must be considered as concentrated masses. Thus, this study has been built on vibrations and stability of axially moving beam with multiple concentrated masses. For solutions to such systems, firstly mathematical model of the system has been derived by means of Hamilton's Principle. After model has been formed as dimensionless, system was solved approximately by means of the method of multiple scales. Under primary resonance assumption, analytical solution has been made and amplitude-phase modulation equations have been derived. Natural frequencies of the system have been obtained from modal functions. For steady-state solutions of the system, free-undamped and forced-damped vibrations have been investigated. Effects of concentrated masses on vibrations have been searched with regard to both magnitude and number of masses.

2. Problem formulation

We consider a simply supported axially moving beam-mass system shown in Fig. 1. The beam which is prestressed with initial tensile load P moves with harmonically varying average transport

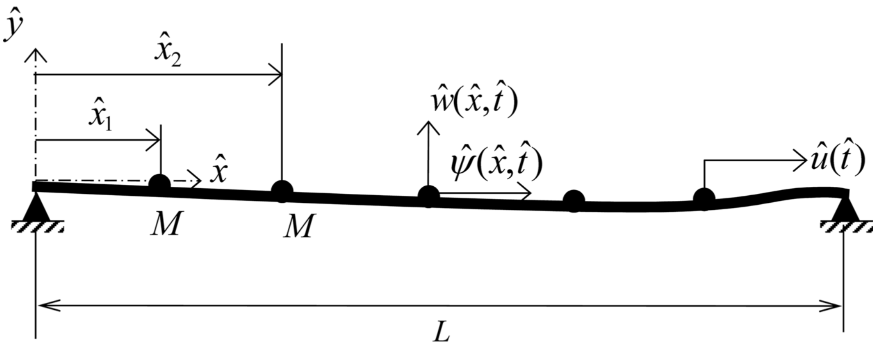

Fig. 1 Axially moving beam with multiple concentrated masses

velocity $u(t)$. M concentrated masses are placed on the beam with equal span along L distance. The model with s masses is made of $s+1$ parts, and to formulate the model mathematically energy of the system has been used. Total energy of the system consists of kinetic (T) and potential (U) as shown below

$$T = \frac{1}{2}.\sum_{r=0}^{s}\rho.A.\int_{\hat{x}_r}^{\hat{x}_{r+1}}(\hat{v}_{r+1})^2.d\hat{x}+\frac{1}{2}.\sum_{r=1}^{s}M.\left(\hat{v}_r\Big|_{\substack{\hat{x}=\hat{x}_r\\ \hat{t}=\hat{t}}}\right)^2$$

$$\hat{v} = \sqrt{(\dot{\hat{w}}_{r+1}+\hat{w}'_{r+1}.\hat{u})^2+(\dot{\hat{\psi}}_{r+1}+\hat{\psi}'_{r+1}.\hat{u}+\hat{u})^2} \tag{1}$$

$$U = \frac{1}{2}.\sum_{r=0}^{s}E.A.\int_{\hat{x}_r}^{\hat{x}_{r+1}}\left(\hat{\psi}'_{r+1}+\frac{1}{2}.\hat{w}'^2_{r+1}\right)^2 d\hat{x}+\frac{1}{2}.\sum_{r=0}^{s}E.I.\int_{\hat{x}_r}^{\hat{x}_{r+1}}\hat{w}''^2_{r+1}.d\hat{x}$$

$$+P.\sum_{r=0}^{s}\int_{\hat{x}_r}^{\hat{x}_{r+1}}\left(\hat{\psi}'_{r+1}+\frac{1}{2}.\hat{w}'^2_{r+1}\right).d\hat{x} \tag{2}$$

In Eqs. (1)-(2), the terms $\hat{w}$ and $\hat{\psi}$ are defined as transversal and longitudinal displacements, respectively. Other properties of the beam is such that; A is the cross sectional area, ρ is the density, I is the moment of inertia of the beam cross-section with respect to the neutral axis and E is the Young's Modulus. $(\cdot)$ and $(\,)'$ denote differentiations with respect to the time t^* and the spatial variable x^*, respectively. Terms in Eq. (1) are kinetic energies of the beam and concentrated masses respectively. Potential energy terms in Eq. (2) are due to stretching, bending and prestressing of the beam, respectively. Using these terms of energy and invoking Hamilton's principle

$$\delta\int_{t_1}^{t_2}(T-U).dt = 0 \tag{3}$$

and substituting Eqs. (1) and (2) into Eq. (3), and performing necessary calculations, it is observed that the axial displacement $\hat{\psi}$ can be eliminated from the equations (see Öz *et al.* 2001). Finally, one obtains the following equations of motion

$$E.I.\hat{w}_{m+1}^{iv}+\rho.A.\left(\ddot{\hat{w}}_{m+1}+2.\hat{u}.\dot{\hat{w}}_{m+1}'+\dot{\hat{u}}.\hat{w}_{m+1}'+\hat{u}^2.\hat{w}_{m+1}''\right)-P.\hat{w}_{m+1}''$$

$$-\frac{1}{2}.E.A.\hat{w}_{m+1}''.\left[\sum_{r=0}^{s}\int_{\hat{x}_r}^{\hat{x}_{r+1}}\hat{w}_{r+1}'^2dx\right]=0 \tag{4}$$

where $\hat{x}_0 = 0$ and $\hat{x}_{s+1} = L$, $m = 0,1,2,\ldots,s$.

There are $(s+1)$ equations in Eq. (4). For simply supported ends, boundary and continuity conditions can be written as follows

$$\hat{w}_1\big|_{x=\hat{x}_0}=\hat{w}_1''\big|_{x=\hat{x}_0}=\hat{w}_{s+1}\big|_{x=\hat{x}_{s+1}}=\hat{w}_{s+1}''\big|_{x=\hat{x}_{s+1}}=0, \quad \hat{w}_p\big|_{x=\hat{x}_p}=\hat{w}_{p+1}\big|_{x=\hat{x}_p}, \quad \hat{w}_p'\big|_{x=\hat{x}_p}=\hat{w}_{p+1}'\big|_{x=\hat{x}_p}$$

$$E.I.(\hat{w}_p''-\hat{w}_{p+1}'')\Big|_{x=\hat{x}_p}=M.\left(\hat{u}.\dot{\hat{w}}_p+\hat{u}^2.\hat{w}_p'\right)\times\Big|_{x=\hat{x}_p}$$

$$E.I.(\hat{w}_p'''-\hat{w}_{p+1}''')\Big|_{x=\hat{x}_p}=M\left(\ddot{\hat{w}}_p+2.\hat{u}.\dot{\hat{w}}_p'+\dot{\hat{u}}.\hat{w}_p'\right)\Big|_{x=\hat{x}_p} \tag{5}$$

The nondimensional quantities are defined as follows

$$w=\hat{w}/L\,,\ x=\hat{x}/L\,,\ t=\sqrt{P/\rho.A.L^2}.\hat{t},\ u=\sqrt{\rho.A/P}.\hat{u}.,\ \alpha=\frac{M}{\rho.A.L},\ v_f^2=\frac{E.I}{P.L^2}$$

$$v_k^2=E.A/P,\ I=\upsilon^2.A \tag{6}$$

where υ is the radius of gyration, v_f and v_k are stiffness and slenderness coefficients of the beam, respectively. After some manipulation, it is obtained that there is a relation between rigidity and slenderness as $v_f = v_k \cdot L/\upsilon$.

Substituting Eq. (6) into Eqs. (4) and (5), and adding damping (μ) and harmonical transverse forcing (F) to these equations (see Chakraborty *et al.* 1999, Ozkaya 2009), one obtains dimensionless equation of motion as follows

$$\ddot{w}_{m+1}+2.u.\dot{w}_{m+1}'+\dot{u}.w_{m+1}'+(u^2-1).w_{m+1}''+v_f^2.w_{m+1}^{iv}-\frac{1}{2}.v_k^2.w_{m+1}''.\left[\sum_{r=0}^{s}\int_{x_r}^{x_{r+1}}w_{r+1}'^2dx\right]=$$

$$F_{m+1}.\cos(\Omega.t)-\mu.\dot{w}_{m+1} \tag{7}$$

and boundary conditions, which provide the equation of motion, are

$$w_1\big|_{x=x_0}=w_1''\big|_{x=x_0}=w_{s+1}\big|_{x=x_{s+1}}=w_{s+1}''\big|_{x=x_{s+1}}=0, \quad w_p\big|_{x=x_p}=w_{p+1}\big|_{x=x_p}, \ w_p'\big|_{x=x_p}=w_{p+1}'\big|_{x=x_p}$$

$$v_f^2.\left(w_p''-w_{p+1}''\right)\Big|_{x=x_p}=\alpha.\left(u.\dot{w}_p+u^2.w_p'\right).\Big|_{x=x_p}$$

$$v_f^2.\left(w_p'''-w_{p+1}'''\right)\Big|_{x=x_p}=\alpha.\left(\ddot{w}_{p+1}+2.u.\dot{w}_{p+1}'.\dot{u}.w_{p+1}'\right)\Big|_{x=x_p} \tag{8}$$

where they have been nondimensionalized as $x_0 = 0$, $x_{s+1} = 1$.

3. Analytical solutions

3.1 Perturbation analysis

First, lets us consider the velocity of the beam varying harmonically about a constant velocity u_0 as follows

$$u = u_0 + \varepsilon . u_1 . \sin(\Omega t) \tag{9}$$

where ε is the book-keeping parameter. $\varepsilon \cdot u_1$ and Ω is the magnitude and frequency of the harmonical variation respectively.

We assume approximate expansions of the form

$$w_{m+1}(x, t; \varepsilon) = \sum_{g=1}^{3} \varepsilon^{g-1} . w_{(m+1)g}(x, T_0, T_1, T_2) \tag{10}$$

where $T_0 = t$ is the fast time scale, $T_1 = \varepsilon \cdot t$ and $T_2 = \varepsilon^2 \cdot t$ are slow time scales.

The slenderness, forcing and damping terms are treated as follows

$$v_k = \varepsilon . v_{k1}, \quad \mu = \varepsilon^2 . \mu_1, \quad F_{m+1} = \varepsilon^2 . F_{(m+1)1} \tag{11}$$

Thus, problem becomes of a weakly nonlinear system.

For the method of multiple scales, time derivatives can be written as follows

$$\frac{d}{dt} = D_0 + \varepsilon . D_1 + \varepsilon^2 . D_2 + \dots, \quad \frac{d^2}{dt^2} = D_0^2 + 2 . \varepsilon . D_0 . D_1 + \varepsilon^2 . (D_1^2 + 2 . D_0 . D_2) + \dots \tag{12,13}$$

where D_0, D_1, and D_2 indicate the time derivatives with respect to T_0, T_1, and T_2, respectively.

Inserting Eqs. (9)-(12) into Eqs. (7)-(8), and equating like powers of ε, and omitting the higher terms of ε in the resulting equations, one obtains a general form for any power of ε as follows

$$\begin{aligned}
D_0^2 \cdot w_{(m+1)g} + 2 \cdot D_0 \cdot w'_{(m+1)g} \cdot u_0 + (u_0^2 - 1) \cdot w''_{(m+1)g} + v_f^2 \cdot w^{iv}_{(m+1)g} = F_{(m+1)(g-2)} \cdot \cos\Omega t - \mu_1 \cdot D_0 \cdot w_{(m+1)(g-2)} \\
-(2 \cdot D_0 \cdot D_1 \cdot w_{(m+1)(g-1)} + 2 \cdot u_0 \cdot D_1 \cdot w'_{(m+1)(g-1)} + 2 \cdot \mu_1 \cdot \sin\Omega t \cdot D_0 \cdot w'_{(m+1)(g-1)} \\
+ u_1 \cdot \Omega \cdot \cos\Omega t \cdot w'_{(m+1)(g-1)} + 2 \cdot u_0 \cdot u_1 \cdot \sin\Omega t \cdot w''_{(m+1)(g-1)} + 2 \cdot u_0 \cdot D_2 \cdot w'_{(m+1)(g-2)} \\
+ 2 \cdot u_1 \cdot \sin\Omega t \cdot D_1 \cdot w'_{(m+1)(g-2)} + u_1^2 \cdot \sin^2\Omega t \cdot w''_{(m+1)(g-2)} \\
+ (D_1^2 + 2 \cdot D_0 \cdot D_2) \cdot w_{(m+1)(g-2)} - \frac{1}{2} \cdot v_{k1}^2 \cdot w''_{(m+1)(g-2)} \cdot \sum_{r=0}^{s} \int_{x_r}^{x_{r+1}} w'^2_{(r+1)(g-2)} \cdot dx \Bigg)
\end{aligned} \tag{14}$$

and necessary boundary conditions for this equation

$$w_{1g}\big|_{x=x_0} = w''_{1g}\Big|_{x=x_0} = w_{(s+1)g}\big|_{x=x_{s+1}} = w''_{(s+1)g}\Big|_{x=x_{s+1}} = 0, \; w_{pg}\big|_{x=x_p} = w_{(p+1)g}\big|_{x=x_p}$$

$$w'_{pg}\Big|_{x=x_p} = w'_{(p+1)g}\Big|_{x=x_p}$$

$$v_f^2 \cdot (w''_{pg} - w''_{(p+1)g})\Big|_{x=x_p} = \alpha \cdot (u_0 \cdot D_0 \cdot w_{pg} + u_0^2 \cdot w'_{pg}$$

$$+ u_0 \cdot D_1 \cdot w_{p(g-1)} + u_1 \cdot \sin\Omega t \cdot D_0 \cdot w_{p(g-1)} + 2 \cdot u_0 \cdot u_1 \cdot \sin\Omega t \cdot w'_{p(g-1)}$$

$$+ u_0 \cdot D_2 \cdot w_{p(g-2)} + u_1 \cdot \sin\Omega t \cdot D_1 \cdot w_{p(g-2)} + u_1^2 \cdot \sin^2\Omega t \cdot w'_{p(g-2)})\Big|_{x=x_p}$$

$$v_f^2 \cdot (w'''_{pg} - w'''_{(p+1)g})\Big|_{x=x_p} = \alpha \cdot (D_0^2 \cdot w_{pg} + 2 \cdot u_0 \cdot D_0 \cdot w'_{pg}$$

$$+ 2 \cdot u_0 \cdot D_1 \cdot w'_{p(g-1)} + 2 \cdot D_0 \cdot D_1 \cdot w_{p(g-1)} + 2 \cdot u_1 \cdot \sin\Omega t \cdot D_0 \cdot w'_{p(g-1)} + u_1 \cdot \Omega \cdot \cos\Omega t \cdot w'_{p(g-1)}$$

$$+ 2 \cdot u_0 \cdot D_2 \cdot w'_{p(g-2)} + (D_1^2 + 2 \cdot D_0 \cdot D_2) \cdot w_{p(g-2)} + 2 \cdot u_1 \cdot \sin\Omega t \cdot (D_1 \cdot w'_{p(g-2)})\Big|_{x=x_p} \tag{15}$$

where $p = 1, 2, \ldots s$, $m = s$ and $x_0 = 0$, $x_{s+1} = 1$. If values of g is equal to 1, 2, and 3, Eqs. (14)-(15) correspond to order ε^0, ε^1, and ε^2 respectively (see Eq. (10)).

3.2 Primary resonance

Taking into consideration the primary resonance case, we assume the frequency of the harmonical forcing term which defined at Eq. (9) as follows

$$\Omega = \omega + \varepsilon^2 \cdot \sigma \tag{16}$$

where ω is the corresponding natural frequency, and σ is the detuning parameter.

Taking g as 1 at Eq. (14), one obtains order $\varepsilon(0)$ which correspond to linear problem of the system. We assume solution of this problem as follows

$$w_{(m+1)1}(x, T_0, T_1, T_2) = A(T_1, T_2) \cdot e^{i\omega T_0} \cdot Y_{(m+1)}(x) + \overline{A}(T_1, T_2) \cdot e^{-i\omega T_0} \cdot \overline{Y}_{(m+1)}(x) \tag{17}$$

where overbar denotes the complex conjugate of the preceding terms, and $Y_{(m+1)}$ is the eigenfunction.

Inserting Eq. (17) into Eq. (14), one has the following differential equation which satisfies mode shapes.

$$v_f^2 \cdot Y^{iv}_{(m+1)} + (u_0^2 - 1) \cdot Y''_{(m+1)} + 2 \cdot i \cdot \omega \cdot u_0 \cdot Y'_{(m+1)} - \omega^2 \cdot Y_{(m+1)} = 0 \tag{18}$$

Solution of the eigenfunction $Y_{(m+1)}(x)$ can be written as follows

$$Y_{(m+1)} = c_{(m+1)1} \cdot e^{i\beta_1 x} + c_{(m+1)2} \cdot e^{i\beta_2 x} + c_{(m+1)3} \cdot e^{i\beta_3 x} + c_{(m+1)4} \cdot e^{i\beta_4 x} \tag{19}$$

where $c_{(p+1)j}$ ($j = 1..4$) are constants, and β_j are four roots of the following equation obtained from Eq. (18)

$$v_f^2 \cdot \beta^4 + (1 - u_0^2) \cdot \beta^2 - 2 \cdot u_0 \cdot \omega \cdot \beta - \omega^2 = 0 \tag{20}$$

Substituting Eq. (17) into Eq. (15), conditions which provide Eq. (18) are obtained

$$Y_1\big|_{x=x_0} = Y_1''\big|_{x=x_0} = 0\,,\; Y_{(s+1)}\big|_{x=x_{s+1}} = Y_{(s+1)}''\big|_{x=x_{s+1}} = 0\,,\; Y_p\big|_{x=x_p} = Y_{(p+1)}\big|_{x=x_p}\,,\; Y_p'\big|_{x=x_p} = Y_{(p+1)}'\big|_{x=x_p}$$

$$v_f^2\cdot(Y_p''-Y_{(p+1)}'')\big|_{x=x_p} = \alpha\cdot(u_0\cdot i\cdot\omega\cdot Y_p+u_0^2\cdot Y_p')\big|_{x=x_p}$$

$$v_f^2\cdot(Y_p'''-Y_{(p+1)}''')\big|_{x=x_p} = \alpha\cdot(-\omega^2\cdot Y_p+2\cdot i\cdot u_0\cdot\omega\cdot Y_p')\big|_{x=x_p} \tag{21}$$

Taking g as 2 at Eq. (14), order $\varepsilon(1)$ of Perturbation series is obtained. Substituting Eq. (17) into Eq. (14) and performing necessary calculations, it is observed that $D_1\cdot A(T_1,\, T_2)$ must be equal to zero in order to determine a solution to this order. Thus, it must be provided that $A = A(T_2)$. Considering this case and substituting Eq. (17) into Eq. (14), we obtain following solution of the form

$$w_{(m+1)2} = A\cdot e^{i(\Omega+\omega)T_0}\cdot\phi_{(m+1)1}(x)+A\cdot e^{i(\omega-\Omega)T_0}\cdot\phi_{(m+1)2}(x)+cc \tag{22}$$

where cc stands for the complex conjugate of the preceding terms. Inserting this solution into Eq. (14), and performing necessary calculations, one has the following differential equations

$$v_f^2\cdot\phi_{(m+1)1}^{iv}+(u_0^2-1)\cdot\phi_{(m+1)1}''+4\cdot u_0\cdot i\cdot\omega\cdot\phi_{(m+1)1}'-4\cdot\omega^2\cdot\phi_{(m+1)1}$$
$$= -u_1\cdot\left(i\cdot\omega+\frac{\omega}{2}\right)\cdot Y_{(m+1)}'-u_0\cdot u_1\cdot Y_{(m+1)}'' \tag{23}$$

$$v_f^2\cdot\phi_{(m+1)2}^{iv}\cdot(u_0^2-1)\cdot\phi_{(m+1)2}'' = u_1\cdot\left(i\cdot\omega-\frac{\omega}{2}\right)\cdot Y_{(m+1)}'+u_0\cdot u_1\cdot Y_{(m+1)}'' \tag{24}$$

Solutions to Eqs. (23)-(24) may be expressed in the form

$$\phi_{(m+1)1} = f_{(m+1)1}\cdot e^{i\kappa_1 x}+f_{(m+1)2}\cdot e^{i\kappa_2 x}+f_{(m+1)3}\cdot e^{i\kappa_3 x}+f_{(m+1)4}\cdot e^{i\kappa_4 x}$$
$$+f_{(m+1)5}\cdot e^{i\beta_1 x}+f_{(m+1)6}\cdot e^{i\beta_2 x}+f_{(m+1)7}\cdot e^{i\beta_3 x}+f_{(m+1)8}\cdot e^{i\beta_4 x} \tag{25}$$

$$\phi_{(m+1)2} = g_{(m+1)1}\cdot e^{i\tau_1 x}+g_{(m+1)2}\cdot x\cdot e^{i\tau_2 x}+g_{(m+1)3}\cdot e^{i\tau_3 x}+g_{(m+1)4}\cdot e^{i\tau_4 x}$$
$$+g_{(m+1)5}\cdot e^{i\beta_1 x}+g_{(m+1)6}\cdot e^{i\beta_2 x}+g_{(m+1)7}\cdot e^{i\beta_3 x}+g_{(m+1)8}\cdot e^{i\beta_4 x} \tag{26}$$

Using Eqs. (25)-(26) in Eqs. (23)-(24), one has following equations

$$v_f^2\cdot\kappa^4+(1-u_0^2)\cdot\kappa^2-4\cdot u_0\cdot\omega\cdot\kappa-4\cdot\omega^2 = 0\,,\; v_f^2\cdot\tau^4+(1-u_0^2)\cdot\tau^2 = 0 \tag{27-28}$$

Substituting Eq. (22) into Eq. (15), conditions which provide Eqs. (25)-(26) are obtained as follows

$$\phi_1\big|_{x=x_0} = \phi_1''\big|_{x=x_0} = 0\,,\ \phi_{(s+1)1}\big|_{x=x_{s+1}} = \phi_{(s+1)1}''\big|_{x=x_{s+1}} = 0\,,\ \phi_{p1}\big|_{x=x_p} = \phi_{(p+1)1}\big|_{x=x_p}\,,\ \phi_{p1}'\big|_{x=x_p} = \phi_{(p+1)1}'\big|_{x=x_p}$$

$$v_f^2\cdot(\phi_{p1}''-\phi_{(p+1)1}'')\Big|_{x=x_p} = \alpha\cdot\Big(2\cdot u_0\cdot i\cdot\omega\cdot\phi_{p1}+u_0^2\cdot\phi_{p1}'+\frac{u_1}{2}\cdot i\cdot\omega\cdot Y_p+u_0\cdot u_1\cdot Y_p'\Big)\Big|_{x=x_p}$$

$$v_f^2\cdot(\phi_{p1}'''-\phi_{(p+1)1}''')\Big|_{x=x_p} = \alpha\cdot\Big(-4\cdot\omega^2\cdot\phi_{p1}+4\cdot u_0\cdot i\cdot\omega\cdot\phi_{p1}'+u_1\cdot\Big(i\cdot\omega+\frac{\omega}{2}\Big)\cdot Y_p'\Big)\Big|_{x=x_p} \tag{29}$$

$$\phi_2\big|_{x=x_0} = \phi_2''\big|_{x=x_0} = 0\,,\ \phi_{(s+1)2}\big|_{x=x_{s+1}} = \phi_{(s+1)2}''\big|_{x=x_{s+1}} = 0\,,\ \phi_{p2}\big|_{x=x_p} = \phi_{(p+1)2}\big|_{x=x_p}\,,\ \phi_{p2}'\big|_{x=x_p} = \phi_{(p+1)2}'\big|_{x=x_p}$$

$$v_f^2\cdot(\phi_{p2}''-\phi_{(p+1)2}'')\Big|_{x=x_p} = \alpha\cdot\Big(u_0^2\cdot\phi_{p2}'-\frac{u_1}{2}\cdot i\cdot\omega\cdot Y_p-u_0\cdot u_1\cdot Y_p'\Big)\Big|_{x=x_p}$$

$$v_f^2\cdot(\phi_{p2}'''-\phi_{(p+1)2}''')\Big|_{x=x_p} = -\alpha\cdot u_1\cdot\Big(i\cdot\omega-\frac{\omega}{2}\Big)\cdot Y_p'\Big|_{x=x_p} \tag{30}$$

Taking g as 3 at Eq. (14), problem at order $\varepsilon(2)$ is obtained, and to determine a solution to this problem we assume a solution of the form

$$w_{(m+1)3}(x,T_0,T_2) = \varphi_{(m+1)}(x,T_2)\cdot e^{i\omega T_0}+W_{(m+1)}(x,T_2)+cc \tag{31}$$

where φ and W denote secular and nonsecular terms, respectively.

Substituting Eq. (31) into Eq. (14), and eliminating nonsecular terms from equations, one obtains following differential equation

$$\begin{aligned}
&v_f^2\cdot\varphi_{(m+1)}^{iv}+(u_0^2-1)\cdot\varphi_{(m+1)}''+2\cdot u_0\cdot i\cdot\omega\cdot\varphi_{(m+1)}'-\omega^2\cdot\varphi_{(m+1)} = \frac{1}{2}\cdot F_{(m+1)1}\cdot e^{i\sigma T_2}-\mu_1\cdot i\cdot\omega\cdot Y_{(m+1)}\cdot A\\
&+\Big[2\cdot u_1\cdot i\cdot\omega\cdot\Big(\phi_{(m+1)1}'-\frac{u_1\cdot\omega}{2}\cdot\phi_{(m+1)1}'+\phi_{(m+1)2}'+u_0\cdot u_1\cdot\Big(\phi_{(m+1)1}''-\phi_{(m+1)2}''\Big)+\frac{u_1^2}{2}\cdot Y_{(p+1)}''\Big)\Big]\cdot A\\
&+\frac{1}{2}\cdot v_{k1}^2\cdot\left[Y_{(m+1)}''\cdot\sum_{r=0}^{s}\int_{x_r}^{x_{r+1}}2\cdot Y_{(r+1)}'\cdot\overline{Y}_{(r+1)}'\cdot dx+\overline{Y}_{(m+1)}''\cdot\sum_{r=0}^{s}\int_{x_r}^{x_{r+1}}Y_{(r+1)}'^2\cdot dx\right]\cdot A^2\cdot\overline{A}\\
&-\Big[\frac{u_1\cdot\omega}{2}\cdot\overline{\phi}_{(m+1)2}'+u_0\cdot u_1\cdot\overline{\phi}_{(m+1)2}''+\frac{u_1^2}{4}\cdot Y_{(m+1)}''\Big]\cdot\overline{A}\cdot e^{2i\sigma T_2}-\Big[2\cdot u_0\cdot Y_{(m+1)}'+2\cdot i\cdot\omega\cdot Y_{(m+1)}\Big]\cdot\dot{A}
\end{aligned} \tag{32}$$

Conditions providing Eq. (32) may be obtained from Eq. (15) as follows

$$\varphi_1\big|_{x=x_0} = \varphi_1''\big|_{x=x_0} = 0\,,\ \varphi_{(s+1)}\big|_{x=x_{s+1}} = \varphi_{(s+1)}''\big|_{x=x_{s+1}} = 0\,,\ \varphi_p\big|_{x=x_p} = \varphi_{(p+1)}\big|_{x=x_p}\,,\ \varphi_p'\big|_{x=x_p} = \varphi_{(p+1)}'\big|_{x=x_p}$$

$$\begin{aligned}
v_f^2\cdot(\varphi_p''-\varphi_{(p+1)}'')\Big|_{x=x_p} &= \alpha\cdot\Big(u_0\cdot i\cdot\omega\cdot\varphi_p+u_0^2\cdot\varphi_p'+u_0\cdot Y_p\cdot\dot{A}+\Big[u_0\cdot u_1\cdot\overline{\phi}_{p2}'+\frac{u_1^2}{4}\cdot\overline{Y}_p'\Big]\cdot\overline{A}\cdot e^{2i\sigma T_2}\\
&-\Big[u_1\cdot i\cdot\omega\cdot\phi_{p1}+u_0\cdot u_1(\phi_{p1}'-\phi_{p2}')+\frac{u_1^2}{2}\cdot Y_p'\Big]\cdot A\Big)\Big|_{x=x_p}
\end{aligned}$$

$$v_f^2 \cdot (\varphi_p''' - \varphi_{(p+1)}''')\Big|_{x=x_p} = \alpha \cdot (-\omega^2 \cdot \varphi_p + 2 \cdot u_0 \cdot i \cdot \omega \cdot \varphi_p' + [2 \cdot i \cdot \omega \cdot Y_p + 2 \cdot u_0 \cdot Y_{(p+1)}'] \cdot \dot{A}$$
$$+\left[-2 \cdot i \cdot u_1 \cdot \omega \cdot \phi_{p1}' + \frac{u_1 \cdot \omega}{2} \cdot (\phi_{p1}' + \phi_{p2}')\right] \cdot A + \frac{u_1 \cdot \omega}{2} \cdot \bar{\phi}_{p2}' \cdot \bar{A} \cdot e^{2i\sigma T_2}\Big)\Big|_{x=x_p} \tag{33}$$

In order to find a solution to Eq. (32), one requires a solvability condition. Applying procedures which are determined in Nayfeh (1981), the solvability conditions for Eqs. (32)-(33) are satisfied

$$K_1 \cdot \dot{A} + K_2 \cdot A + K_3 \cdot A^2 \cdot \bar{A} + K_4 \bar{A} \cdot e^{2i\sigma T_2} + \frac{1}{2} \cdot f \cdot e^{i\sigma T_2} - \mu_1 \cdot i \cdot \omega \cdot A = 0 \tag{34}$$

where f and K are defined in Appendix, and for mode shapes equations are normalized as follows

$$\sum_{r=0}^{s} \int_{x_r}^{x_{r+1}} Y_{(r+1)} \cdot \bar{Y}_{(r+1)} \cdot dx = 1 \tag{35}$$

Introducing the polar form

$$A(T_2) = \frac{1}{2} \cdot a \cdot e^{i\theta} \quad , \quad \theta = \theta(T_2) \tag{36}$$

where a_n and θ_n are real functions of time, one obtains the following equation from Eq. (34)

$$K_1 \cdot (\dot{a} + a \cdot i \cdot \dot{\theta}) + K_2 \cdot a + K_3 \cdot \frac{1}{4} \cdot a^3 + K_4 \cdot a \cdot (\cos 2\gamma + i \sin 2\gamma) \cdot f \cdot (\cos\gamma + i \sin\gamma) - \mu_1 \cdot i \cdot \omega \cdot a = 0 \tag{37}$$

where the phase is defined as

$$\gamma = \sigma \cdot T_2 - \theta \tag{38}$$

From the Appendix, one observes that K has both real and imaginary values. In this case, separating Eq. (38) into the real and imaginary parts, finally, one obtains the following amplitude and phase modulation equations

$$K_1^{re} \cdot \dot{a} + K_1^{im} \cdot a \cdot \dot{\gamma} = -\left[K_2^{re} \cdot a + \frac{1}{4} \cdot K_3^{re} \cdot a^3 + K_4^{re} \cdot a \cdot \cos 2\gamma - K_4^{im} \cdot a \cdot \sin 2\gamma + f \cdot \cos\gamma - K_1^{im} \cdot a \cdot \sigma\right] \tag{39}$$

$$K_1^{im} \cdot \dot{a} - K_1^{re} \cdot a \cdot \dot{\gamma} = -\left[K_2^{im} \cdot a + \left(\frac{1}{4} \cdot K_3^{im} \cdot a^3 + K_4^{im} \cdot a\right) \cdot \cos 2\gamma + K_4^{re} \cdot a \cdot \sin 2\gamma + f \cdot \sin\gamma \cdot K_1^{re} \cdot a \cdot \sigma - \mu_1 \cdot \omega \cdot a\right] \tag{40}$$

where *im* and *re* indices indicate imaginary and real parts of K, respectively.

4. Numerical analysis

From Eqs. (17)-(20), natural frequencies can be calculated numerically. In order to understand well the effects of the axial velocity and mass ratio on the natural frequencies, these parameter have been drawn on the same graph. In Fig. 2 and Fig. 3, natural frequency-mean axial velocity curves of beam with 5 and 10 concentrated masses shown, respectively. These figures present first and

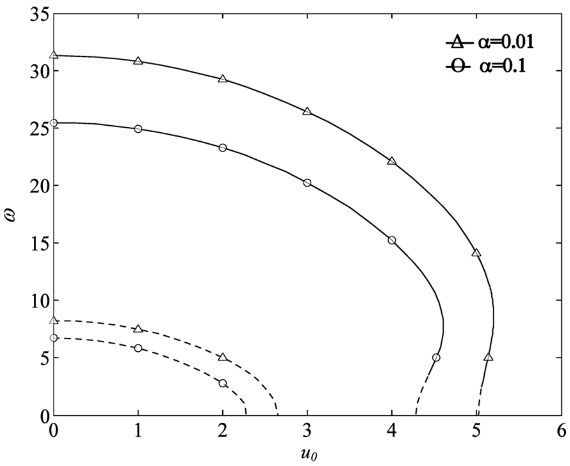

Fig. 2 Axial velocity versus natural frequency of the beam having 5 masses for different α values, $v_f = 0.8$ (dashed line: first mode, solid line: second mode)

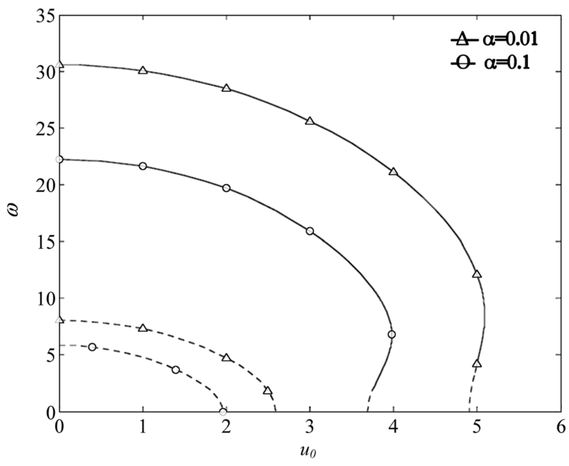

Fig. 3 Axial velocity versus natural frequency of the beam having 10 masses for different α values, $v_f = 0.8$ (dashed line: first mode, solid line: second mode)

second modes. They have been found that the natural frequencies decrease with increasing both the axial velocity and the ratio and number of masses. Once, frequency of the first mod decreases with increasing velocity and then becomes zero, but after a certain velocity, first mode is again activated and reaches to the second mode. Thus, mode transition occurs and this transition zone shown as mode shapes in the figures in Öz (2001). While holding the mass ratio very small (0.01) natural frequencies given at Fig. 2 is compatible with the ones given Öz *et al.* (2001).

We assume that axially moving beam-mass system has undamped-free vibrations. Taking f, σ and μ_1 as zero, how behavior system would exhibit can be investigated from non-linear frequency-response curves. From Eq. (36), one obtains the following non-linear frequency equation

$$\omega_{nl} = \omega + \dot{\theta} \tag{41}$$

In steady state case, $\dot{a}$ vanishes and may be taken as zero. Taking σ as zero in Eq. (38), one obtains that $\dot{\theta} = -\dot{\gamma}$. Eliminating γ from Eqs. (39)-(40) yields the non-linear frequency term as follows

$$\dot{\theta} = \frac{\left(K_1^{im}\cdot K_2^{re} - K_1^{re}\cdot K_2^{im} + \frac{1}{4}\cdot(K_1^{im}\cdot K_3^{re} - K_1^{re}\cdot K_3^{im})\cdot a^2 \pm \sqrt{\left(\left[K_1^{im}\cdot K_2^{re} - K_1^{re}\cdot K_2^{im} + \frac{1}{4}\cdot(K_1^{im}\cdot K_3^{re} - K_1^{re}\cdot K_3^{im})\cdot a^2\right]^2 - (K_1^{im^2} + K_1^{re^2})\cdot\left(K_2^{re^2} + K_2^{im^2} - K_4^{im^2} - K_4^{re^2} + \frac{1}{2}\cdot(K_2^{re}\cdot K_3^{re} + K_2^{im}\cdot K_3^{im})\cdot a^2 + \frac{1}{16}\cdot((K_3^{re^2} + K_3^{im^2})\cdot a^4)\right)\right)}\right)}{K_1^{im^2} + K_1^{re^2}} \tag{42}$$

The non-linear frequency-response curves are plotted for the steady state case in Figs. 4-7. In these figures, effects of the ratio and number of the concentrated masses as well as effects of the mean axial velocity and slenderness coefficient on the non-linear frequency are studied. In Fig. 4, the effects of different slenderness coefficients, which correspond to radius of gyration, on curves

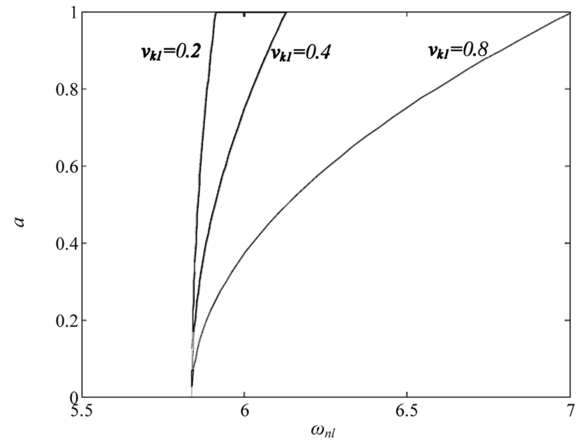

Fig. 4 Non-linear frequency-response curves for different slenderness coefficients. $v_f = 0.8$, $u_0 = 1$, $u_1 = 0.1$, 5 masses, $\alpha = 0.1$

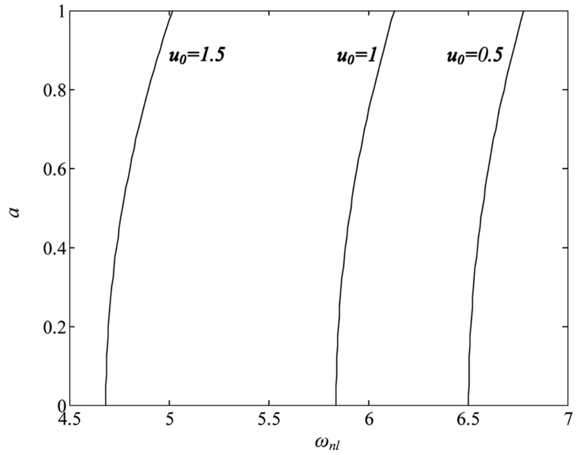

Fig. 5 Non-linear frequency-response curves for different mean axial velocities. $v_f = 0.8$, $v_{k1} = 0.4$ ($\nu = L/2$), $u_1 = 0.1$, 5 masses, $\alpha = 0.1$

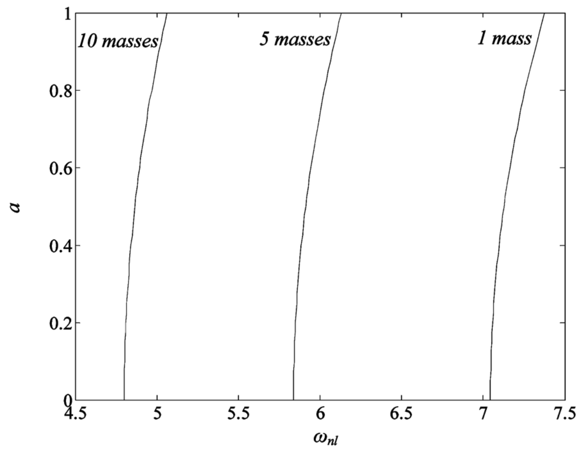

Fig. 6 Non-linear frequency-response curves for different mass numbers. $v_f = 0.8$, $v_{k1} = 0.4$ ($\nu = L/2$), $u_0 = 1$, $u_1 = 0.1$, $\alpha = 0.1$

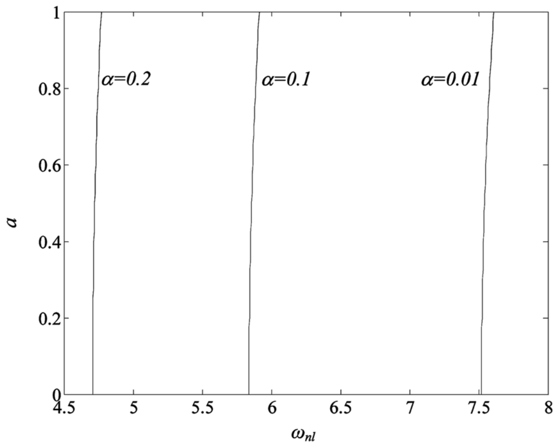

Fig. 7 Non-linear frequency-response curves for different mass ratios. $v_f = 0.8$, $v_{k1} = 0.2$ ($\nu = L/4$), $u_0 = 1$, $u_1 = 0.1$, $m = 5$ masses

determined. According to these results, the non-linear frequency increases with the increasing radius of gyration at any amplitude of vibration. In Fig. 5, non-linear frequency-response curves are plotted for different mean axial velocities. Non-linear frequency decreases with the increasing axial velocity. In Fig. 6, the non-linear frequency-response curves are plotted for different numbers of concentrated masses with equal ratio and their effects on the non-linear frequency are searched. It is seen that the non-linear frequencies decrease with the increasing numbers of masses. In Fig. 7 the non-linear frequency-response curves are plotted for different ratios of the concentrated masses with equal number, and it is seen that the non-linear frequencies decrease with the increasing ratios of the masses. From this point, it is reached as a result that; the ratio and number of the masses have same the effect, and the non-linear frequency decreases with increasing these parameters.

We assume that axially moving beam-mass system has damped-forced vibrations. In Eqs. (39)-(40), $\dot{a}$ and $\dot{\gamma}$ vanish for the steady-state case. Taking these parameters as zero, fixed-points can be found. Stability of these fixed-points are sought by means of the following Jacobian matrix

$$\left.\begin{bmatrix} \dfrac{\partial G_1}{\partial a} & \dfrac{\partial G_1}{\partial \gamma} \\ \dfrac{\partial G_2}{\partial a} & \dfrac{\partial G_2}{\partial \gamma} \end{bmatrix}\right|_{\substack{a = a_0 \\ \gamma = \gamma_0}} \tag{43}$$

where it is assumed that $\dot{a} = G_1$ and $\dot{\gamma} = G_2$, and terms with 0 indices define fixed points of the steady state. If eigenvalues of the Jacobian matrix have negative real parts, these fixed points are stable.

In Figs. 8-11 force-response curves are drawn to investigate effects of the ratio and number of the concentrated masses as well as those of the axial mean velocity and slenderness coefficient of the beam on vibration response of the system. In these curves, it assumed that forcing is $f = 1$ and damping is $\mu_1 = 0.1$, and solid lines denote stable and dashed lines denote unstable solutions. In Fig. 8, force-response curves are plotted for different slenderness coefficients. In these curves, as slenderness ratio increases, hardening behavior increases, multi-valued region expands, but maximum magnitudes have no notable variation. In Fig. 9, force-response curves are drawn for different mean axial velocities. According to this figure as axial velocity increases, multi-valued region expands and maximum magnitudes increase. In Fig. 10, different numbers of masses with the same ratios are considered, and it is seen that maximum magnitudes increase, and multi-valued region expands with the increasing of number of the masses. In Fig. 11, different mass ratios are considered for the same number of the masses, and it is seen that multi-valued region expands and maximum magnitudes increase with increasing mass ratios.

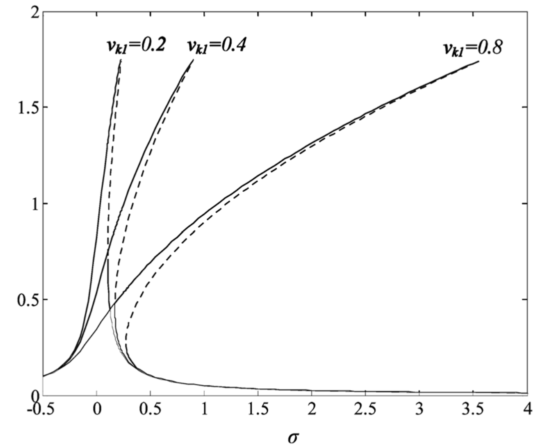

Fig. 8 Force-response curves for different slenderness coefficients. $v_f = 0.8$, $u_0 = 1$, $u_1 = 0.1$, $m = 5$ masses, $\alpha = 0.1$

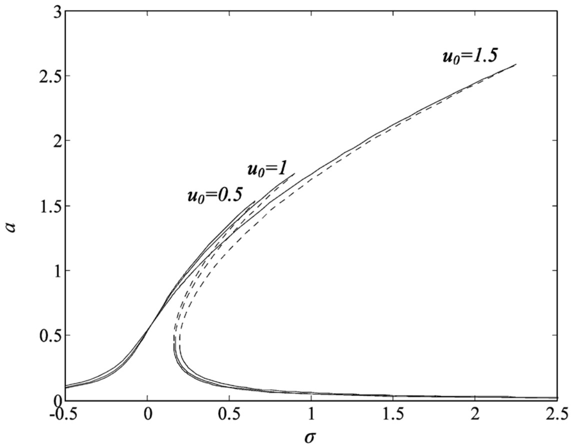

Fig. 9 Force-response curves for different mean axial velocities. $v_f = 0.8$, $v_{k1} = 0.4$ ($v = L/2$), $u_1 = 0.1$, $m = 5$ masses, $\alpha = 0.1$

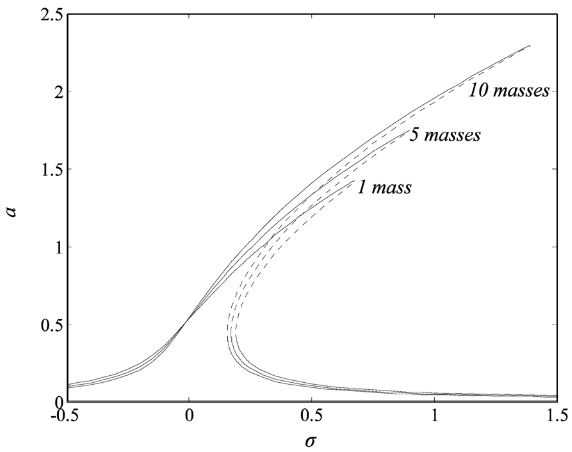

Fig. 10 Force-response curves for different mass numbers. $v_f = 0.8$, $v_{k1} = 0.4$ ($\nu = L/2$), $u_0 = 1$, $u_1 = 0.1$, $\alpha = 0.1$

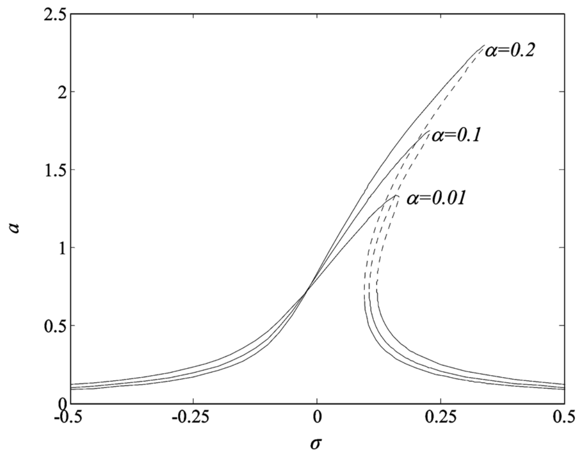

Fig. 11 Force-response curves for different mass ratios. $v_f = 0.8$, $v_{k1} = 0.2$ ($\nu = L/4$), $u_0 = 1$, $u_1 = 0.1$, $m = 5$ masses

5. Conclusions

Aim of this study is to measure vibration behaviors of axially moving beams with multiple concentrated masses in primary resonance case. First, beam is assumed as Euler-Bernoulli type, and beam's both ends are simply supported. Then mathematical model is built assuming that concentrated masses are placed on the beam with equal spans. Obtained differential equations are solved by means of the method of multiple scales (a perturbation method).

From the first order in perturbation expansions, natural frequencies of the system are calculated. It is observed that natural frequencies decrease with increasing both the mass numbers and the mass ratios. Frequency of the first mode becomes zero with the increasing mean axial velocity, after that if velocity continues to increase first mode is activated again and transition to the second mode occurs.

By means of the second and third orders of the perturbation expansions, the amplitude and phase modulation equations are obtained. For steady state case, behaviors of undamped-free and forced-damped vibrations are investigated by means of these equations. Investigations on both mass numbers and mass ratios result the same behavior. According to this result; system has hardening type behavior and non-linear frequency increases with increasing frequency. It is observed that with increasing either mass numbers or mass ratios, the non-linear frequency decreases, the multi-valued region expands, and maximum amplitude increases.

According to the results obtained from this study, writers are investigating 3:1 internal resonances that may occur in the system. It will be presented effects of the numbers and ratios of the concentrated masses as well as the effects of the slenderness ratios on the 3:1 internal resonances.

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Appendix

$$f=\sum_{r=0}^{s}\int_{x_r}^{x_{r+1}}\overline{Y}_{r+1}\,.\,F_{(r+1)1}\,.\,dx$$

$$K_1=-\sum_{r=0}^{s}\int_{x_r}^{x_{r+1}}\overline{Y}_{(r+1)}\,.\,(2\,.\,u_0\,.\,Y'_{(r+1)}+2\,.\,i\,.\,\omega\,.\,Y_{(r+1)})\,.\,dx$$
$$+\sum_{r=1}^{s}\alpha\,.\left(u_0\,.\,Y_r\,.\,\overline{Y}'_r-2\,.\,i\,.\,\omega\,.\,Y_r\,.\,\overline{Y}_r-2\,.\,u_0\,.\,Y'_r\,.\,\overline{Y}_r\right)\Big|_{x=x_r}$$

$$K_2=\sum_{r=0}^{s}\int_{x_r}^{x_{r+1}}\overline{Y}_{(r+1)}\,.\left(2\,.\,i\,.\,u_1\,.\,\omega\,.\,\phi'_{(r+1)1}-\frac{u_1\,.\,\omega}{2}\,.\,(\phi'_{(r+1)1}+\phi'_{(r+1)2})\right)+u_0\,.\,u_1\,.\,(\phi''_{(r+1)1}-\phi''_{(r+1)2})+\frac{u_1^2}{2}\,.\,Y''_{(r+1)}\Big)\,.\,dx$$

$$+\sum_{r=1}^{s}\alpha\,.\left(-\left[u_1\,.\,i\,.\,\omega\,.\,\phi_{r1}+u_0\,.\,u_1\,.\,(\phi'_{r1}-\phi'_{r2})+\frac{u_1^2}{2}\,.\,Y'_r\right].\,Y'_r-\left[-2\,.\,i\,.\,u_1\,.\,\omega\,.\,\phi'_{r1}+\frac{u_1\,.\,\omega}{2}\,.\,(\phi'_{r1}+\phi'_{r2})\right].\,\overline{Y}_r\right)\Big|_{x=x_r}$$

$$K_3=\frac{1}{2}\,.\,v_{k1}^2\,.\,\sum_{r=0}^{s}\int_{x_r}^{x_{r+1}}\overline{Y}_{(r+1)}\,.\left(Y''_{(r+1)}\,.\,\sum_{r=0}^{s}\int_{x_r}^{x_{r+1}}2\,.\,Y'_{(r+1)}\,.\,\overline{Y}'_{(r+1)}\,.\,dx+\overline{Y}''_{(r+1)}\,.\,\sum_{r=0}^{s}\int_{x_r}^{x_{r+1}}Y'^2_{(r+1)}\,.\,dx\right).\,dx$$

$$K_4=-\sum_{r=0}^{s}\int_{x_r}^{x_{r+1}}\overline{Y}_{(r+1)}\,.\left(\frac{u_1\,.\,\omega}{2}\,.\,\overline{\phi}'_{(r+1)2}+u_0\,.\,u_1\,.\,\overline{\phi}''_{(r+1)2}+\frac{u_1^2}{2}\,.\,\overline{Y}''_{(r+1)}\right).\,dx$$
$$+\sum_{r=1}^{s}\alpha\,.\left(\left(u_0\,.\,u_1\,.\,\overline{\phi}'_{r2}+\frac{u_1^2}{4}\,.\,\overline{Y}'_r\right).\,\overline{Y}'_r-\left(\frac{u_1\,.\,\omega}{2}\,.\,\overline{\phi}'_{r2}\right).\,\overline{Y}_r\right)\Big|_{x=x_r}$$