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Dynamic stiffness matrix of an axially loaded slender double-beam element

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Abstract. The dynamic stiffness matrix is formulated for an axially loaded slender double-beam element in which both beams are homogeneous, prismatic and of the same length by directly solving the governing differential equations of motion of the double-beam element. The Bernoulli-Euler beam theory is used to define the dynamic behaviors of the beams and the effects of the mass of springs and axial force are taken into account in the formulation. The dynamic stiffness method is used for calculation of the exact natural frequencies and mode shapes of the double-beam systems. Numerical results are given for a particular example of axially loaded double-beam system under a variety of boundary conditions, and the exact numerical solutions are shown for the natural frequencies and normal mode shapes. The effects of the axial force and boundary conditions are extensively discussed.

Keywords: double-beam system; Bernoulli-Euler beam; axial force; free vibration; dynamic stiffness matrix.

1. Introduction

Beams are basic structural elements widely used in the mechanical, aeronautical and civil engineering. The validity of the Bernoulli-Euler beam theory has been studied, and some generalizations of that useful theory have also been developed. The free and forced vibrations of the single Bernoulli-Euler and Timoshenko beams with different boundary conditions have been investigated extensively by a lot of authors. However, a relatively few works have been done on the structures built up from beams. A particular case of interest consists of two parallel slender beams of uniform properties with a set of linear elastic springs in-between. The geometry and material, as well as the boundary conditions of these two parallel beams of uniform properties, can be all different for a general case. The vibration problems of the elastically connected double-beam systems are of particular interest because of their possible use in various areas of technology. The double-beam system can be regarded as an ideal model of a complex continuous system consisting of two one-dimensional structures joined by linear elastic layer, or as an approximate model of

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sandwich beams. It is also shown that the elastically connected double-beam system can be used to study the performance of the beam-type continuous dynamic vibration absorber since a system consisting of one beam and coupling springs can be used as a dynamic vibration absorber to reduce the vibration of another beam. Due to the great practical importance of the elastically connected double-beam systems, the different problems concerning the dynamic characteristics of this interesting vibratory system have been treated in a few works.

Seelig and Hoppmann (1964) presented the development and solution of the differential equations of motion of a system of elastically connected parallel beams. Vibration experiments were performed to ascertain the degree of applicability of the theory. Kessel (1966) investigated the system consisting of two parallel, simply-supported beams that were elastically connected and subjected to a moving load that oscillated longitudinally about a fixed point along the length of one of the beams. It was shown that the resonances of an elastically connected double-beam system might result when the system was subjected to a cyclic moving load. Rao (1974) derived the differential equations governing the flexural vibrations of the systems of elastically connected parallel beams with the effects of shear deformation and rotary inertia considered. For computational purposes, the simple case of a three-beam system with identical springs and with all ends hinged was considered. Chonan (1976) investigated the dynamical behaviors of two beams connected with a set of independent springs subjected to an impulsive load where the effect of the mass of springs was taken into account. He considered the case of a two-beam system consisting of two identical beams in the analysis. Hamada *et al.* (1983) analyzed the free and forced vibrations of a system of two elastically connected parallel beams having unequal masses and unequal flexural rigidities by using a generalized method of finite integral transformation and the Laplace transformation. Aida *et al.* (1992) presented a beam-type dynamic vibration absorber which was composed of an absorbing beam under the same boundary conditions as the main beam and a set of uniformly distributed, connecting springs and dampers between the main and absorbing beams. Chen and Sheu (1994) presented an analytical theory to study the dynamic interactions between two parallel beams connected to each other by the vertical springs and dashpots uniformly distributed along the beam length. Vu *et al.* (2000) presented an exact method for solving the vibration problem of a double-beam system subject to a harmonic excitation. The Bernoulli-Euler model was used for the transverse vibrations of beams. Oniszczuk (2000) presented the free vibration analysis of two parallel simply-supported beams continuously joined by a Winkler elastic layer. The motion of the system was described by a homogeneous set of two partial differential equations, which was solved by using the classical Bernoulli-Fourier method. Oniszczuk (2003) devoted to analyze the undamped forced transverse vibrations of an elastically connected double-beam system in the case of simply-supported boundary conditions. Ritdumrongkul *et al.* (2004) developed a spectral element method, based on a wave propagation approach, to simulate the bolted joint structures. A spectral element model of a bolted joint element consisting of three Bernoulli-Euler beams connected by uniformly distributed springs and dashpots was formulated. De Rosa and Lippiello (2007) applied the differential quadrature method to find the free vibrations of parallel double-beams having the translation and rotation elastic constraints at their ends and joined by a Winkler-type homogeneous elastic foundation. Zhang *et al.* (2008) investigated the forced transverse vibrations of an elastically connected simply-supported double-beam system under compressive axial load on the basis of Bernoulli-Euler beam theory. Gurgoze *et al.* (2008) derived the characteristic equation of a combined system consisting of a cantilever Bernoulli-Euler beam with a tip mass and an in-span visco-elastic helical spring-mass, considering the mass of the helical spring. Balkaya *et al.* (2010)

carried out the free vibration analyses of a parallel placed twin pipe system simulated by simply supported-simply supported and fixed-fixed Euler-Bernoulli beams resting on Winkler elastic soil using the differential transform method.

Most of the solution methods mentioned in the above references are applicable only to the cases of the double-beam systems made of two identical parallel beams with the continuous massless springs uniformly distributed along the beam length and/or to the cases of some particular boundary conditions. For a general case, the geometric and material properties of the beams are different and the mass of springs may need to be taken into account. Only a few researchers have studied this case. Since the effect of an important parameter, namely the axial force, was seldom included in the relevant literature, this paper sets out to extend the previous development by including this effect. This paper is a consequence of need to develop a tool to assess the free vibration characteristics of the general axially loaded double-beam systems. The determination of the natural frequencies and mode shapes is a significant problem in the dynamic analysis of the double-beam systems, and it is of importance in the design of the double-beam systems subjected to dynamic loadings. It is generally considered that the structural responses of the double-beam systems are dependent on both lower and higher modes. In designing this kind of systems, therefore, it is essential that the natural frequencies and mode shapes are obtained accurately.

This investigation is partly motivated by the fact that earlier research on the free vibration characteristics of three-layered sandwich beams (Sisemore and Darvennes 2002) has shown a continuous trend of compressional vibration in the middle layer over a relatively wide range of frequencies if the middle layer is relatively soft with respect to the top and bottom layers. Compressional or extensional deformation results when the top and bottom layers move perpendicular to each other, acting to compress or stretch the middle layer. Since the middle layer is relatively soft with respect to the top and bottom layers, its displacement can be assumed to be completely defined as a linear function of the displacements of the top and bottom layers, i.e., simply the average displacement of the top and bottom layers. The potential energy from bending of the middle layer is neglected due to the softness of the middle layer in comparison with the top and bottom layers. Also, it is assumed that the middle layer is sufficiently soft that it can be modeled as a set of continuous linear springs. The classical beams are used to define the dynamic behaviors of the top and bottom layers. Thus, the idealized structural model of a three-layered sandwich beam is composed of two parallel beams connected by a set of continuous springs with mass effect, uniformly distributed along the beam length. The structural model introduced here is a relatively accurate and simple method for predicting the natural frequencies of compressional vibration in a three-layered sandwich beam in the design applications.

In the present paper, the dynamic stiffness matrix is established to compute the natural frequencies and mode shapes of the axially loaded slender double-beam systems with the mass of springs included. The dynamic stiffness method employs the closed-form exact solutions of the governing differential equations under harmonic nodal excitations as shape functions to formulate the frequency-dependent stiffness matrix called the dynamic stiffness matrix. The history of the dynamic stiffness method and early development and applications can be found in Leung's monograph (Leung 1993). A comprehensive literature review on the dynamic stiffness method and some recent development of the method can be found in the works of Doyle (1997), Lee (2004) and Gopalakrishnan *et al.* (2008), Kim and Kim (2005) and the references cited therein in which the development and applications of the dynamic stiffness method to various kinds of elements and structures, including conservative and non-conservative problems, uniform and non-uniform components, straight and curved members, are presented in detail. The dynamic stiffness method is based on the closed-form analytical solutions of the governing differential equations of the element. It eliminates the spatial discretization error and can produce the exact modal solutions in the vibration analysis of structures. Therefore, the dynamic stiffness method provides the analyst with better model accuracy when compared to the classical finite element or other approximate methods. The usefulness of the method becomes apparent particularly when higher frequencies and better accuracies of results are required. The application of the dynamic stiffness method to the axially loaded slender double-beam systems with the mass of springs included is demonstrated and discussed by a typical example, in which the effects of the axial force and boundary conditions on the natural frequencies and mode shapes are extensively investigated.

2. Mathematical formulation

An elastically connected double-beam system under consideration is depicted in Fig. 1, which consists of two parallel beams joined by an elastic layer modeled as a set of distributed springs. Both beams subjected to the constant axial forces are homogeneous, prismatic, slender and of the same length. Small undamped vibrations of the system are discussed.

The governing equations of the axially loaded double-beam element based on the classical Bernoulli-Euler beam theory are derived by use of the extended Hamilton's principle which can be expressed in the form

$$\int_{t_1}^{t_2}(\delta U - \delta T - \delta W)dt = 0 \tag{1}$$

where $\delta T, \delta U$ and δW are the variations in the kinetic energy, potential energy and work of the axial forces of the element, respectively.

Since the classical Bernoulli-Euler beam theory is used, the kinetic energy of the axially loaded double-beam element is given by

$$T = \frac{1}{2}\int_0^L m_1\left(\frac{\partial w_1}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^L m_2\left(\frac{\partial w_2}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^L m_3\left(\frac{\partial w_3}{\partial t}\right)^2 dx \tag{2a}$$

where $w_i = w_i(x, t)$ and m_i are the transverse deflection and mass per unit length of the upper beam, lower beam and spring layer, respectively; x and t are the spatial co-ordinate and time, respectively; L is the length of the beams. Subscript i denotes the quantities concerned with the upper beam ($i = 1$), lower beam ($i = 2$) and spring layer ($i = 3$). For simplicity of the analysis, the deflection of the spring layer is simply defined as the average displacement of the upper and lower beams, i.e., $w_3 = \frac{1}{2}(w_1 + w_2)$.

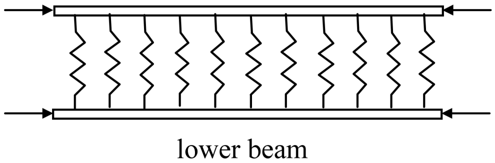

Fig. 1 The geometry model of an axially loaded double-beam element

It is assumed that both the shear deformation and rotary inertia of the beams are negligible, so the potential energy of the axially loaded double-beam element can be written as

$$U = \frac{1}{2}\int_0^L E_1 I_1 \left(\frac{\partial^2 w_1}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_0^L E_2 I_2 \left(\frac{\partial^2 w_2}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_0^L k(w_1 - w_2)^2 dx \tag{2b}$$

where E_i is the Young's modulus of elasticity; I_i is the moment of inertia of the beam cross-section; k is the spring stiffness per unit beam length.

The work done by the axial forces is given by

$$W = \frac{1}{2}\int_0^L P_1 (w_1')^2 dx + \frac{1}{2}\int_0^L P_2 (w_2')^2 dx \tag{2c}$$

where P_i is a constant compressive axial force acting through the centroid of the cross-section of the beam. P_i can be positive or negative.

Substitution of Eq. (2) into Eq. (1), and performing the variational operations yields the following governing differential equations of motion of the axially loaded double-beam element

$$m_1 \ddot{w}_1 + \frac{m_3}{4}(\ddot{w}_1 + \ddot{w}_2) + E_1 I_1 w_1'''' + k(w_1 - w_2) + P_1 w_1'' = 0 \tag{3a}$$

$$m_2 \ddot{w}_2 + \frac{m_3}{4}(\ddot{w}_1 + \ddot{w}_2) + E_2 I_2 w_2'''' - k(w_1 - w_2) + P_2 w_2'' = 0 \tag{3b}$$

where superscripts dot and prime denote the partial derivatives with respective to t and x, respectively.

The appropriate boundary conditions at the beam ends ($x = 0, L$) of the axially loaded double-beam element are

$$[-E_1 I_1 w_1''' - P_1 w_1']\delta w_1 = 0 \tag{4a}$$

$$[-E_2 I_2 w_2''' - P_2 w_2']\delta w_2 = 0 \tag{4b}$$

$$[E_1 I_1 w_1'']\delta w_1' = 0 \tag{4c}$$

$$[E_2 I_2 w_2'']\delta w_2' = 0 \tag{4d}$$

3. Dynamic stiffness matrix

It can be shown that Eq. (3) have solutions that are separable in time and space, and that the time dependence is harmonic. Letting

$$w_1(x, t) = W_1(x) e^{i\omega t} \tag{5a}$$

$$w_2(x, t) = W_2(x) e^{i\omega t} \tag{5b}$$

where ω is the circular frequency, $W_1(x)$ and $W_2(x)$ are the amplitudes of the sinusoidally varying transverse deflections.

Using Eq. (5), the system of partial differential Eq. (3) reduces to the following system of ordinary differential equations

$$-m_1\omega^2 W_1 - \frac{m_3}{4}\omega^2(W_1 + W_2) + E_1 I_1 W_1'''' + k(W_1 - W_2) + P_1 W_1'' = 0 \tag{6a}$$

$$-m_2\omega^2 W_2 - \frac{m_3}{4}\omega^2(W_1 + W_2) + E_2 I_2 W_2'''' - k(W_1 - W_2) + P_2 W_2'' = 0 \tag{6b}$$

The solutions to Eq. (6) can be expressed as

$$W_1(x) = \tilde{A}e^{\kappa x} \tag{7a}$$

$$W_2(x) = \tilde{B}e^{\kappa x} \tag{7b}$$

Substitution of Eq. (7) into Eq. (6), the equivalent algebraic eigenvalue equations are obtained and the equations have nontrivial solutions when the determinant of the coefficient matrix of $\tilde{A}$ and $\tilde{B}$ vanishes. Setting the determinant equal to zero yields an eighth order polynomial characteristics equation in κ

$$\eta_4\kappa^8 + \eta_3\kappa^6 + \eta_2\kappa^4 + \eta_1\kappa^2 + \eta_0 = 0 \tag{8}$$

where

$$\eta_4 = E_1 I_1 E_2 I_2$$

$$\eta_3 = E_2 I_2 P_1 + E_1 I_1 P_2$$

$$\eta_2 = (E_1 I_1 + E_2 I_2)k + P_1 P_2 - \frac{1}{4}(E_2 I_2(4m_1 + m_3) + E_1 I_1(4m_2 + m_3))\omega^2$$

$$\eta_1 = k(P_1 + P_2) - \frac{1}{4}(4m_2 P_1 + 4m_1 P_2 + m_3(P_1 + P_2))\omega^2$$

$$\eta_0 = -k(m_1 + m_2 + m_3)\omega^2 + \frac{1}{4}(4m_1 m_2 + (m_1 + m_2)m_3)\omega^4$$

The solutions to Eq. (8) can be found in closed-form as follows. First, Eq. (8) can be rewritten as

$$\chi^4 + a_1\chi^3 + a_2\chi^2 + a_3\chi + a_4 = 0 \tag{9}$$

where

$$\chi = \kappa^2 \quad a_1 = \eta_3/\eta_4 \quad a_2 = \eta_2/\eta_4 \quad a_3 = \eta_1/\eta_4 \quad a_4 = \eta_0/\eta_4$$

Then, the fourth order Eq. (9) can be factorized as

$$(\chi^2 + p_1\chi + q_1)(\chi^2 + p_2\chi + q_2) = 0$$

where

$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \frac{1}{2}\left[a_1 \pm \sqrt{a_1^2 - 4a_2 + 4\lambda_1}\right] \qquad \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \frac{1}{2}\left[\lambda_1 \pm \frac{a_1\lambda_1 - 2a_3}{\sqrt{a_1^2 - 4a_2 + 4\lambda_1}}\right]$$

and λ_1 is a real root of the following cubic equation

$$\lambda^3 - a_2\lambda^2 + (a_1 a_3 - 4a_4)\lambda + (4a_2 a_4 - a_3^2 - a_1^2 a_4) = 0 \tag{10}$$

Finally, the four roots of Eq. (9) can be written as

$$\begin{Bmatrix}\chi_1\\ \chi_2\end{Bmatrix} = -\frac{p_1}{2}\pm\sqrt{\frac{p_1^2}{4}-q_1} \qquad \begin{Bmatrix}\chi_3\\ \chi_4\end{Bmatrix} = -\frac{p_2}{2}\pm\sqrt{\frac{p_2^2}{4}-q_2} \tag{11}$$

The general solutions to Eq. (6) are given by

$$\begin{aligned} W_1(x) &= \bar{A}_1e^{\kappa_1 x}+\bar{A}_2e^{-\kappa_1 x}+\bar{A}_3e^{\kappa_2 x}+\bar{A}_4e^{-\kappa_2 x}+\bar{A}_5e^{\kappa_3 x}+\bar{A}_6e^{-\kappa_3 x}+\bar{A}_7e^{\kappa_4 x}+\bar{A}_8e^{-\kappa_4 x}\\ &= \sum_{j=1}^{4}(\bar{A}_{2j-1}e^{\kappa_j x}+\bar{A}_{2j}e^{-\kappa_j x}) \end{aligned} \tag{12a}$$

$$\begin{aligned} W_2(x) &= \bar{B}_1e^{\kappa_1 x}+\bar{B}_2e^{-\kappa_1 x}+\bar{B}_3e^{\kappa_2 x}+\bar{B}_4e^{-\kappa_2 x}+\bar{B}_5e^{\kappa_3 x}+\bar{B}_6e^{-\kappa_3 x}+\bar{B}_7e^{\kappa_4 x}+\bar{B}_8e^{-\kappa_4 x}\\ &= \sum_{j=1}^{4}(\bar{B}_{2j-1}e^{\kappa_j x}+\bar{B}_{2j}e^{-\kappa_j x}) \end{aligned} \tag{12b}$$

where $\kappa_j = \sqrt{\chi_j}$ $(j = 1-4)$, $\bar{A}_1-\bar{A}_8$ and $\bar{B}_1-\bar{B}_8$ are two sets of eight constants. In the solution of Eq. (9), if any of the χ_j's are zero or are repeated, the solutions (12) to the differential Eq. (6) will be modified according to the well-known methods for the ordinary differential equations with constant coefficients, for those particular values of χ_j.

From Eq. (6), only eight of the sixteen constants are independent. The relationship between the constants is given by

$$\bar{B}_{2j-1} = t_j\bar{A}_{2j-1} \qquad \bar{B}_{2j} = t_j\bar{A}_{2j}$$

in which

$$t_j = \frac{4k+4P_1\kappa_j^2+4E_1I_1\kappa_j^4-(4m_1+m_3)\omega^2}{4k+m_3\omega^2} \qquad (j=1-4)$$

According to the sign convention shown in Fig. 2, the expressions of shear forces $Q_1(x)$, $Q_2(x)$, and bending moments $M_1(x)$, $M_2(x)$ can be obtained from Eqs. (4) and (12) as follows

$$Q_1(x) = \sum_{j=1}^{4}[E_1I_1\kappa_j^3+P_1\kappa_j](\bar{A}_{2j-1}e^{\kappa_j x}-\bar{A}_{2j}e^{-\kappa_j x}) \tag{13a}$$

$$Q_2(x) = \sum_{j=1}^{4}[E_2I_2t_j\kappa_j^3+P_2t_j\kappa_j](\bar{A}_{2j-1}e^{\kappa_j x}-\bar{A}_{2j}e^{-\kappa_j x}) \tag{13b}$$

$$M_1(x) = \sum_{j=1}^{4}-E_1I_1\kappa_j^2(\bar{A}_{2j-1}e^{\kappa_j x}+\bar{A}_{2j}e^{-\kappa_j x}) \tag{13c}$$

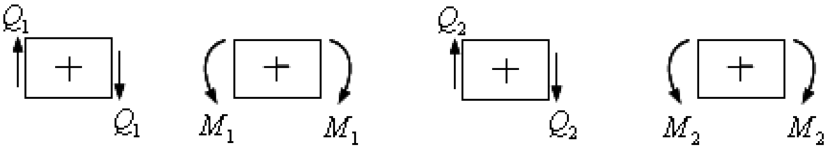

Fig. 2 The sign convention for positive shear forces $Q_1(x), Q_2(x)$ and bending moments $M_1(x), M_2(x)$

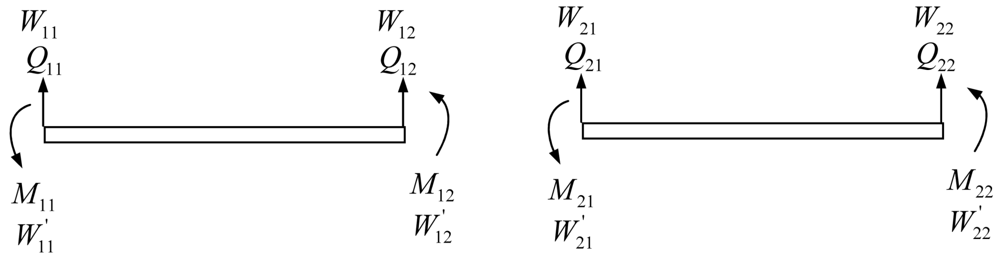

Fig. 3 The boundary conditions for the displacements and forces of the double-beam element

$$M_2(x) = \sum_{j=1}^{4} -E_2 I_2 t_j \kappa_j^2 (\overline{A}_{2j-1} e^{\kappa_j x} + \overline{A}_{2j} e^{-\kappa_j x}) \tag{13d}$$

The boundary conditions for the generalized displacements and forces of the axially loaded double-beam element (referring to Fig. 3) are, respectively

$$\begin{aligned} x = 0:&\quad W_1 = W_{11} \quad W_2 = W_{21} \quad W_1' = W_{11}' \quad W_2' = W_{21}' \\ x = L:&\quad W_1 = W_{12} \quad W_2 = W_{22} \quad W_1' = W_{12}' \quad W_2' = W_{22}' \end{aligned} \tag{14a}$$

$$\begin{aligned} x = 0:&\quad Q_1 = Q_{11} \quad Q_2 = Q_{21} \quad M_1 = M_{11} \quad M_2 = M_{21} \\ x = L:&\quad Q_1 = -Q_{12} \quad Q_2 = -Q_{22} \quad M_1 = -M_{12} \quad M_2 = -M_{22} \end{aligned} \tag{14b}$$

Substituting Eq. (14a) into Eq. (12), the nodal displacements defined in Fig. 3 can be expressed in terms of $\overline{A}_i$ as

$$\{D\} = [R]\{\overline{A}\} \tag{15}$$

where $\{D\}$ is the nodal degree-of-freedom vector defined by

$$\{D\} = \{W_{11} \;\; W_{21} \;\; W_{11}' \;\; W_{21}' \;\; W_{12} \;\; W_{22} \;\; W_{12}' \;\; W_{22}'\}^T$$

$$\{\overline{A}\} = \{\overline{A}_1 \;\; \overline{A}_3 \;\; \overline{A}_5 \;\; \overline{A}_7 \;\; \overline{A}_2 \;\; \overline{A}_4 \;\; \overline{A}_6 \;\; \overline{A}_8\}^T$$

$$[R] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 & t_1 & t_2 & t_3 & t_4 \\ \kappa_1 & \kappa_2 & \kappa_3 & \kappa_4 & -\kappa_1 & -\kappa_2 & -\kappa_3 & -\kappa_4 \\ t_1\kappa_1 & t_2\kappa_2 & t_3\kappa_3 & t_4\kappa_4 & -t_1\kappa_1 & -t_2\kappa_2 & -t_3\kappa_3 & -t_4\kappa_4 \\ e^{\kappa_1 L} & e^{\kappa_2 L} & e^{\kappa_3 L} & e^{\kappa_4 L} & e^{-\kappa_1 L} & e^{-\kappa_2 L} & e^{-\kappa_3 L} & e^{-\kappa_4 L} \\ t_1 e^{\kappa_1 L} & t_2 e^{\kappa_2 L} & t_3 e^{\kappa_3 L} & t_4 e^{\kappa_4 L} & t_1 e^{-\kappa_1 L} & t_2 e^{-\kappa_2 L} & t_3 e^{-\kappa_3 L} & t_4 e^{-\kappa_4 L} \\ \kappa_1 e^{\kappa_1 L} & \kappa_2 e^{\kappa_2 L} & \kappa_3 e^{\kappa_3 L} & \kappa_4 e^{\kappa_4 L} & -\kappa_1 e^{-\kappa_1 L} & -\kappa_2 e^{-\kappa_2 L} & -\kappa_3 e^{-\kappa_3 L} & -\kappa_4 e^{-\kappa_4 L} \\ t_1\kappa_1 e^{\kappa_1 L} & t_2\kappa_2 e^{\kappa_2 L} & t_3\kappa_3 e^{\kappa_3 L} & t_4\kappa_4 e^{\kappa_4 L} & -t_1\kappa_1 e^{-\kappa_1 L} & -t_2\kappa_2 e^{-\kappa_2 L} & -t_3\kappa_3 e^{-\kappa_3 L} & -t_4\kappa_4 e^{-\kappa_4 L} \end{bmatrix}$$

Substituting Eq. (14b) into Eq. (13), the nodal forces defined in Fig. 3 can also be expressed in terms of $\overline{A}_i$ as

$$\{F\} = [H]\{\bar{A}\} \tag{16}$$

where $\{F\}$ is the nodal force vector defined by

$$\{F\} = \{Q_{11}\ \ Q_{21}\ \ M_{11}\ \ M_{21}\ \ Q_{12}\ \ Q_{22}\ \ M_{12}\ \ M_{22}\}^T$$

$$[H] = \begin{bmatrix} \hat{\hat{t}}_1 & \hat{\hat{t}}_2 & \hat{\hat{t}}_3 & \hat{\hat{t}}_4 & -\hat{\hat{t}}_1 & -\hat{\hat{t}}_2 & -\hat{\hat{t}}_3 & -\hat{\hat{t}}_4 \\ \tilde{\tilde{t}}_1 & \tilde{\tilde{t}}_2 & \tilde{\tilde{t}}_3 & \tilde{\tilde{t}}_4 & -\tilde{\tilde{t}}_1 & -\tilde{\tilde{t}}_2 & -\tilde{\tilde{t}}_3 & -\tilde{\tilde{t}}_4 \\ \tilde{\bar{t}}_1 & \tilde{\bar{t}}_2 & \tilde{\bar{t}}_3 & \tilde{\bar{t}}_4 & \tilde{\bar{t}}_1 & \tilde{\bar{t}}_2 & \tilde{\bar{t}}_3 & \tilde{\bar{t}}_4 \\ \bar{\hat{t}}_1 & \bar{\hat{t}}_2 & \bar{\hat{t}}_3 & \bar{\hat{t}}_4 & \bar{\hat{t}}_1 & \bar{\hat{t}}_2 & \bar{\hat{t}}_3 & \bar{\hat{t}}_4 \\ -\hat{\hat{t}}_1 e^{\kappa_1 L} & -\hat{\hat{t}}_2 e^{\kappa_2 L} & -\hat{\hat{t}}_3 e^{\kappa_3 L} & -\hat{\hat{t}}_4 e^{\kappa_4 L} & \hat{\hat{t}}_1 e^{-\kappa_1 L} & \hat{\hat{t}}_2 e^{-\kappa_2 L} & \hat{\hat{t}}_3 e^{-\kappa_3 L} & \hat{\hat{t}}_4 e^{-\kappa_4 L} \\ -\tilde{\tilde{t}}_1 e^{\kappa_1 L} & -\tilde{\tilde{t}}_2 e^{\kappa_2 L} & -\tilde{\tilde{t}}_3 e^{\kappa_3 L} & -\tilde{\tilde{t}}_4 e^{\kappa_4 L} & \tilde{\tilde{t}}_1 e^{-\kappa_1 L} & \tilde{\tilde{t}}_2 e^{-\kappa_2 L} & \tilde{\tilde{t}}_3 e^{-\kappa_3 L} & \tilde{\tilde{t}}_4 e^{-\kappa_4 L} \\ -\tilde{\bar{t}}_1 e^{\kappa_1 L} & -\tilde{\bar{t}}_2 e^{\kappa_2 L} & -\tilde{\bar{t}}_3 e^{\kappa_3 L} & -\tilde{\bar{t}}_4 e^{\kappa_4 L} & -\tilde{\bar{t}}_1 e^{-\kappa_1 L} & -\tilde{\bar{t}}_2 e^{-\kappa_2 L} & -\tilde{\bar{t}}_3 e^{-\kappa_3 L} & -\tilde{\bar{t}}_4 e^{-\kappa_4 L} \\ -\bar{\hat{t}}_1 e^{\kappa_1 L} & -\bar{\hat{t}}_2 e^{\kappa_2 L} & -\bar{\hat{t}}_3 e^{\kappa_3 L} & -\bar{\hat{t}}_4 e^{\kappa_4 L} & -\bar{\hat{t}}_1 e^{-\kappa_1 L} & -\bar{\hat{t}}_2 e^{-\kappa_2 L} & -\bar{\hat{t}}_3 e^{-\kappa_3 L} & -\bar{\hat{t}}_4 e^{-\kappa_4 L} \end{bmatrix}$$

in which

$$\hat{\hat{t}}_j = E_1 I_1 \kappa_j^3 + P_1 \kappa_j \qquad \tilde{\tilde{t}}_j = E_2 I_2 t_j \kappa_j^3 + P_2 t_j \kappa_j$$
$$\tilde{\bar{t}}_j = -E_1 I_1 \kappa_j^2 \qquad \bar{\hat{t}}_j = -E_2 I_2 t_j \kappa_j^2 \qquad (j = 1-4)$$

Eliminating the coefficients A_i from Eqs. (15) and (16) gives the following relationship between the nodal forces and nodal displacements

$$\{F\} = [H][R]^{-1}\{D\} = [K]\{D\} \tag{17}$$

where $[K]$ is the exact dynamic stiffness matrix of the axially loaded double-beam element. It should be mentioned that the explicit analytical expressions for the terms of the dynamic stiffness matrix could be derived using the symbolic manipulator software such as Maple (1990), although the expressions are too lengthy to list in the paper.

4. Automated Muller root search method

If the dynamic stiffness matrix for each element of the double-beam system is known, the global dynamic stiffness matrix for the whole double-beam system can be assembled in a completely analogous way to that used by the conventional finite element method. Once the overall dynamic stiffness matrix is obtained, the appropriate boundary conditions for the particular double-beam system under consideration are applied to obtain the frequency characteristic determinant, which equals zero when an eigenfrequency of the double-beam system is found. Obviously the resulting frequency characteristic equation is a transcendental function of the frequency, which allows an infinite number of natural frequencies of the double-beam system to be accounted for. Although more advanced algorithms and more complex procedures may be used to determine the frequency values of the characteristic equation, a simple automated Muller root search method (Burden and Faires 1989) is adopted in the present study to determine all the natural frequencies in a given frequency band. In order to prevent missing roots, a visual method involving plotting the characteristic determinant curve for the double-beam system to visually approximate where the value of the determinant equals zero is also implemented. The mode shapes corresponding to the natural frequencies can be found in the usual way by making an arbitrary assumption about one unknown variable of the double-beam system and then calculating the remaining variables in terms of the arbitrarily chosen one.

Let us denote the frequency characteristic determinant by $f(\omega)$ and consider the solution of $f(\omega) = 0$ by Muller's method. Muller's method uses a quadratic approximation to the function f by interpolating a quadratic polynomial through the last three computed points and then determining where this curve crosses the ω axis. The following algorithm is an implementation of Muller's method. This algorithm terminates when a prescribed error tolerance on $|f(\omega)|$ has been achieved or a maximum number of iterations has been performed.

(1) Pick three distinct values of ω, i.e., $\omega_0, \omega_1, \omega_2$. Set $i = 1$ and compute

$$h_i = \omega_i - \omega_{i-1}; \quad h_{i+1} = \omega_{i+1} - \omega_i; \quad \delta_i = (f(\omega_i) - f(\omega_{i-1}))/h_i$$

$$\delta_{i+1} = (f(\omega_{i+1}) - f(\omega_{i-1}))/h_{i+1}; \quad d_i = (\delta_{i+1} - \delta_i)/(h_{i+1} + h_i)$$

(2) Compute

$$b_i = \delta_{i+1} + h_{i+1} d_i; \qquad D_i = (b_i^2 - 4f(\omega_{i+1})d_i)^{1/2}$$

(3) If $|b_i - D_i| < |b_i + D_i|$ then set $E_i = b_i + D_i$; else set $E_i = b_i - D_i$

(4) Set $h_{i+2} = -2f(\omega_{i+1})/E_i$; $\omega_{i+2} = \omega_{i+1} + h_{i+2}$.

where ω_{i+2} is the new approximation of the root ω_{i+1}. The process is continued until a desired accuracy is obtained. The rate of convergence is very high and may be increased if an estimate of the eigenvalue is available. Previously roots may be removed from $f(\omega)$ by dividing it by $\prod_{k=1}^{p} (\omega - \omega_k)$, where ω_k $(k = 1, 2, \ldots, p)$ are the first p roots.

5. Numerical results

The free transverse vibration of the axially loaded double-beam system is investigated by using the dynamic stiffness formulation developed in preceding sections. The natural frequencies and normal mode shapes of a particular double-beam system subjected to the constant axial forces are determined numerically.

The following values of the parameters of the double-beam system are used in the numerical calculations:

$b_1 = 0.01$ m, $h_1 = 0.005$ m, $b_2 = 0.01$ m, $h_2 = 0.01$ m, $L = 1$ m, $E_1 = 2.0 \times 10^{11}$ N/m^2
$E_2 = 2.0 \times 10^{11}$ N/m^2, $\rho_1 = 7600$ kg/m^3, $\rho_2 = 7600$ kg/m^3, $k = 8.0 \times 10^3$ N/m^2, $m_3 = 0.76$ kg/m

where b_i and h_i are the width and height of the beam, respectively. ρ_i is the mass density per unit volume of the beam. The mass per unit length of the beam is $m_i = \rho_i A_i$. The cross-sectional area of the beam is $A_i = b_i h_i$. The second moment of area of the beam section is $I_i = b_i h_i^3/12$. The other symbols have been defined in the section 2.

A detailed free vibration analysis of the above double-beam system is now performed for ten interesting cases of the different boundary conditions. These are

Case I: upper beam clamped-clamped, lower beam clamped-clamped;

Case II: upper beam clamped-simply supported, lower beam clamped-simply supported;

Case III: upper beam clamped-free, lower beam clamped-free beam;

Case IV: upper beam simply supported-simply supported, lower beam simply supported-simply supported;

Case V: upper beam free-free, lower beam simply supported-simply supported;

Case VI: upper beam free-free, lower beam clamped-simply supported;

Case VII: upper beam free-free, lower beam clamped-clamped;

Case VIII: upper beam simply supported-simply supported, lower beam clamped-free;

Case IX: upper beam simply supported-simply supported, lower beam clamped-simply supported;

Case X: upper beam simply supported-simply supported, lower beam clamped-clamped.

The free vibration behaviors of the above-mentioned individual uniform double-beam system with any desired boundary conditions can be studied by using the appropriate stiffness terms of the dynamic stiffness matrix; e.g., by deleting those rows and columns of the matrix corresponding to zero boundary displacements. It should be noted that for the Case I this double-beam system is considered as two equal-length double-beam elements and for the other cases this double-beam system is idealized with one double-beam element.

For above ten cases of the different boundary conditions, the first six natural frequencies of the double-beam system with and without inclusion of the axial forces are calculated and the numerical results are shown in Tables 1-5. Both compressive and tensional axial forces are considered. It should be mentioned that putting $P_1 = 0$ and $P_2 = 0$ in Eq. (17) yields the solutions for the unloaded double-beam system. It is known that the dynamic characteristics such as the natural frequencies of the double-beam system are dependent on the beam properties, spring properties, axial forces and boundary conditions at the beam ends.

When the beam properties, spring properties, axial forces are kept invariant, consider the effect of the boundary conditions on the natural frequencies of the double-beam system. Tables 1-5 show the influence of the boundary conditions on the first six natural frequencies of the double-beam system. Of the ten cases considered, the Case I exhibits the highest natural frequencies. For the first four boundary conditions, i.e., Cases I to IV, in each case the upper and lower beams have the same boundary conditions. As expected, the Case I yields the highest natural frequencies followed by Case II, Case IV and Case III. For the Cases V to VII, in each case the ends of upper beam are free

Table 1 Natural frequency (Hz) of the axially loaded double-beam system for Cases I and II

Mode No.	Case I			Case II		
	$P_1 = 0$ $P_2 = 0$	$P_1 = 700$ N $P_2 = 1000$ N	$P_1 = -700$ N $P_2 = -1000$ N	$P_1 = 0$ $P_2 = 0$	$P_1 = 700$ N $P_2 = 1000$ N	$P_1 = -700$ N $P_2 = -1000$ N
1	26.44	18.72	32.04	20.18	9.84	26.31
2	54.24	48.72	58.03	41.49	35.13	46.24
3	61.32	50.17	71.46	50.65	36.85	61.76
4	116.66	103.12	128.67	100.97	85.83	113.96
5	137.41	131.92	142.74	112.06	105.73	118.18
6	191.47	177.47	204.44	170.97	155.78	184.83

Table 2 Natural frequency (Hz) of the axially loaded double-beam system for Cases III and IV

Mode No.	Case III			Case IV		
	$P_1 = 0$ $P_2 = 0$	$P_1 = 120$ N $P_2 = 200$ N	$P_1 = -120$ N $P_2 = -200$ N	$P_1 = 0$ $P_2 = 0$	$P_1 = 400$ N $P_2 = 600$ N	$P_1 = -400$ N $P_2 = -600$ N
1	5.49	2.95	6.98	14.25	8.18	18.39
2	26.15	22.88	28.49	33.45	29.96	36.61
3	28.33	27.75	29.25	41.28	32.28	48.53
4	53.78	51.48	55.55	86.51	77.22	93.81
5	61.34	58.06	64.67	89.57	85.19	94.84
6	116.66	113.46	119.76	151.66	142.44	160.32

Table 3 Natural frequency (Hz) of the axially loaded double-beam system for Cases V and VI

Mode No.	Case V			Case VI		
	$P_1 = 0$ $P_2 = 0$	$P_1 = 300$ N $P_2 = 500$ N	$P_1 = -300$ N $P_2 = -500$ N	$P_1 = 0$ $P_2 = 0$	$P_1 = 300$ N $P_2 = 500$ N	$P_1 = -300$ N $P_2 = -500$ N
1	13.04	7.79	14.16	16.16	10.26	16.58
2	18.53	13.00	22.19	18.68	13.06	22.29
3	26.75	14.33	31.78	27.63	16.48	35.74
4	34.07	30.78	40.30	41.04	38.32	44.42
5	60.92	48.27	70.81	61.63	48.81	71.92
6	89.97	86.11	93.94	110.76	104.49	114.41

Table 4 Natural frequency (Hz) of the axially loaded double-beam system for Cases VII and VIII

Mode No.	Case VII			Case VIII		
	$P_1 = 0$ $P_2 = 0$	$P_1 = 300$ N $P_2 = 500$ N	$P_1 = -300$ N $P_2 = -500$ N	$P_1 = 0$ $P_2 = 0$	$P_1 = 400$ N $P_2 = 600$ N	$P_1 = -400$ N $P_2 = -600$ N
1	17.67	10.89	17.77	10.90	5.29	14.06
2	18.75	13.24	22.42	24.50	19.93	28.09
3	27.99	17.74	37.11	42.89	34.31	49.74
4	53.60	48.77	55.74	51.46	45.65	56.95
5	61.72	51.55	72.13	88.36	78.83	96.92
6	117.08	106.17	126.97	132.78	127.46	137.82

Table 5 Natural frequency (Hz) of the axially loaded double-beam system for Cases IX and X

Mode No.	Case IX			Case X		
	$P_1 = 0$ $P_2 = 0$	$P_1 = 700$ N $P_2 = 1000$ N	$P_1 = -700$ N $P_2 = -1000$ N	$P_1 = 0$ $P_2 = 0$	$P_1 = 700$ N $P_2 = 1000$ N	$P_1 = -700$ N $P_2 = -1000$ N
1	17.80	7.13	24.05	19.48	9.75	25.70
2	39.45	23.30	45.00	42.01	23.65	54.39
3	43.21	35.92	54.51	53.34	49.45	56.95
4	86.55	69.45	100.49	87.23	69.91	101.57
5	111.86	105.47	118.13	135.42	128.85	140.87
6	151.92	135.31	166.79	153.56	138.03	168.32

and free, and the Case VII shows the higher frequencies followed by Case VI and Case V. Similarly, for the Cases VIII to X, in each case the ends of upper beam are simply supported and simply supported, and the Case X has the higher frequencies followed by Case IX and Case VIII. The lowest fundamental frequency is found for the Case III. It can also be observed from Tables 1-5

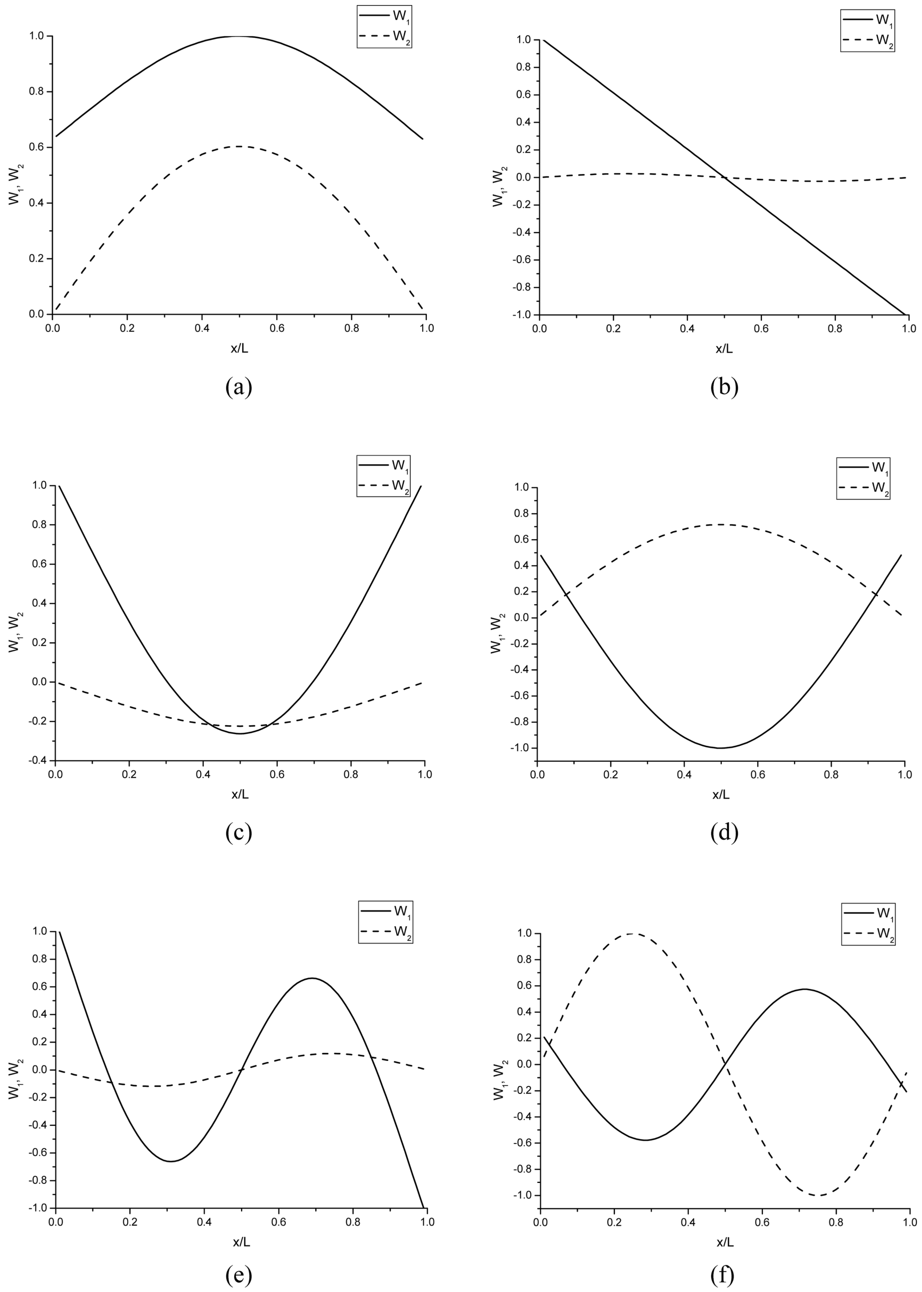

Fig. 4 The first six normal mode shapes of the double-beam system for Case V with $P_1 = 0$ and $P_2 = 0$ (a) mode 1, (b) mode 2, (c) mode 3, (d) mode 4, (e) mode 5, (f) mode 6

that for Cases II, VI and IX, in each case the ends of lower beam are clamped and simply supported, and the Case II produces the higher frequencies followed by Case IX and Case VI.

When the geometric and material properties as well as the boundary conditions of the double-beam system are kept invariant, the effect of the axial forces P_1 and P_2 on the natural frequencies of

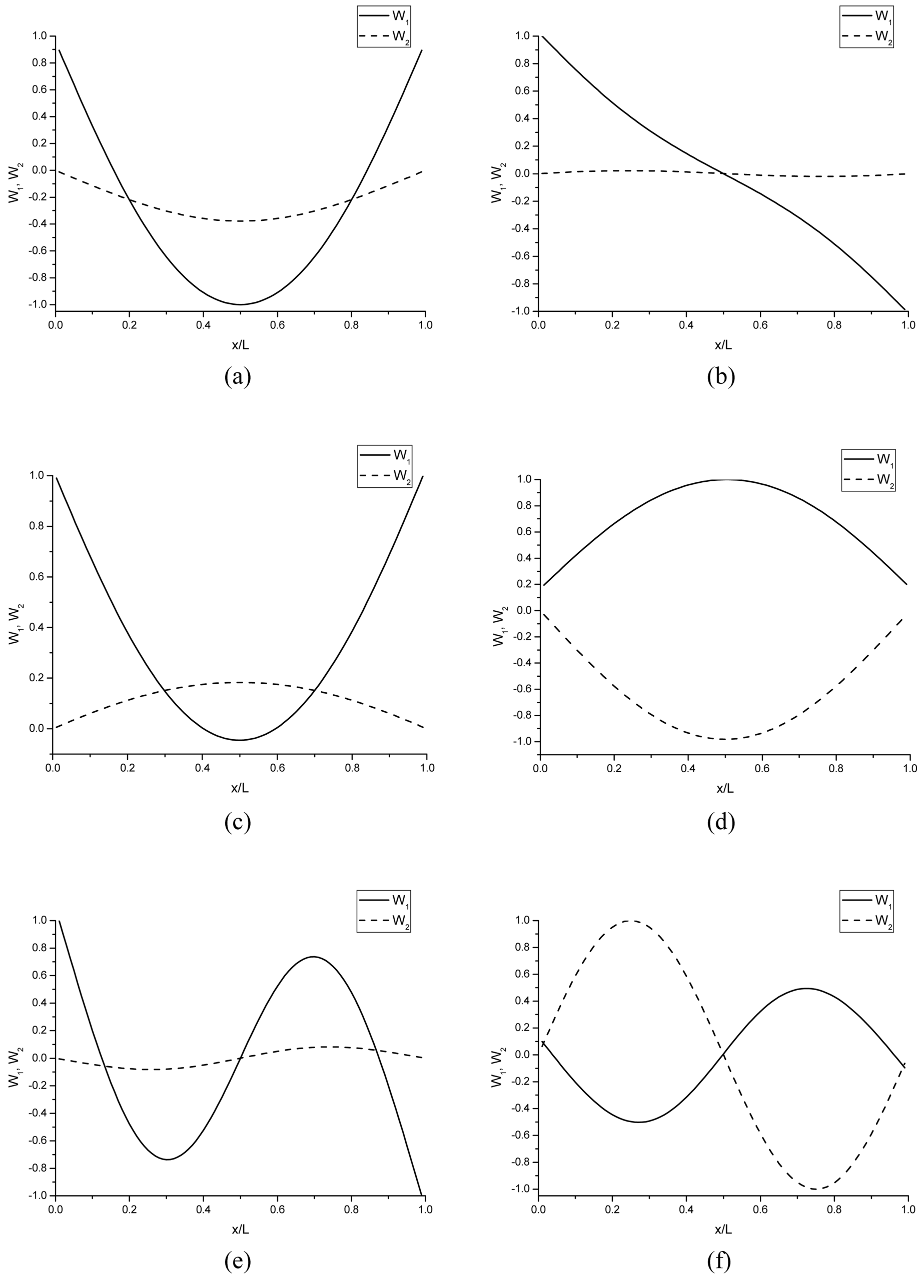

Fig. 5 The first six normal mode shapes of the double-beam system for Case V with $P_1 = 300$ N and $P_2 = 500$ N (a) mode 1, (b) mode 2, (c) mode 3, (d) mode 4, (e) mode 5, (f) mode 6

the double-beam system can also be studied from Tables 1-5. It is evident that the natural frequencies of the double-beam system are sensitive to the variations of the axial forces P_1 and P_2 under consideration. As expected, the compressive axial forces will tend to decrease all the natural frequencies of the double-beam system, whereas the tensile axial forces will increase them. The

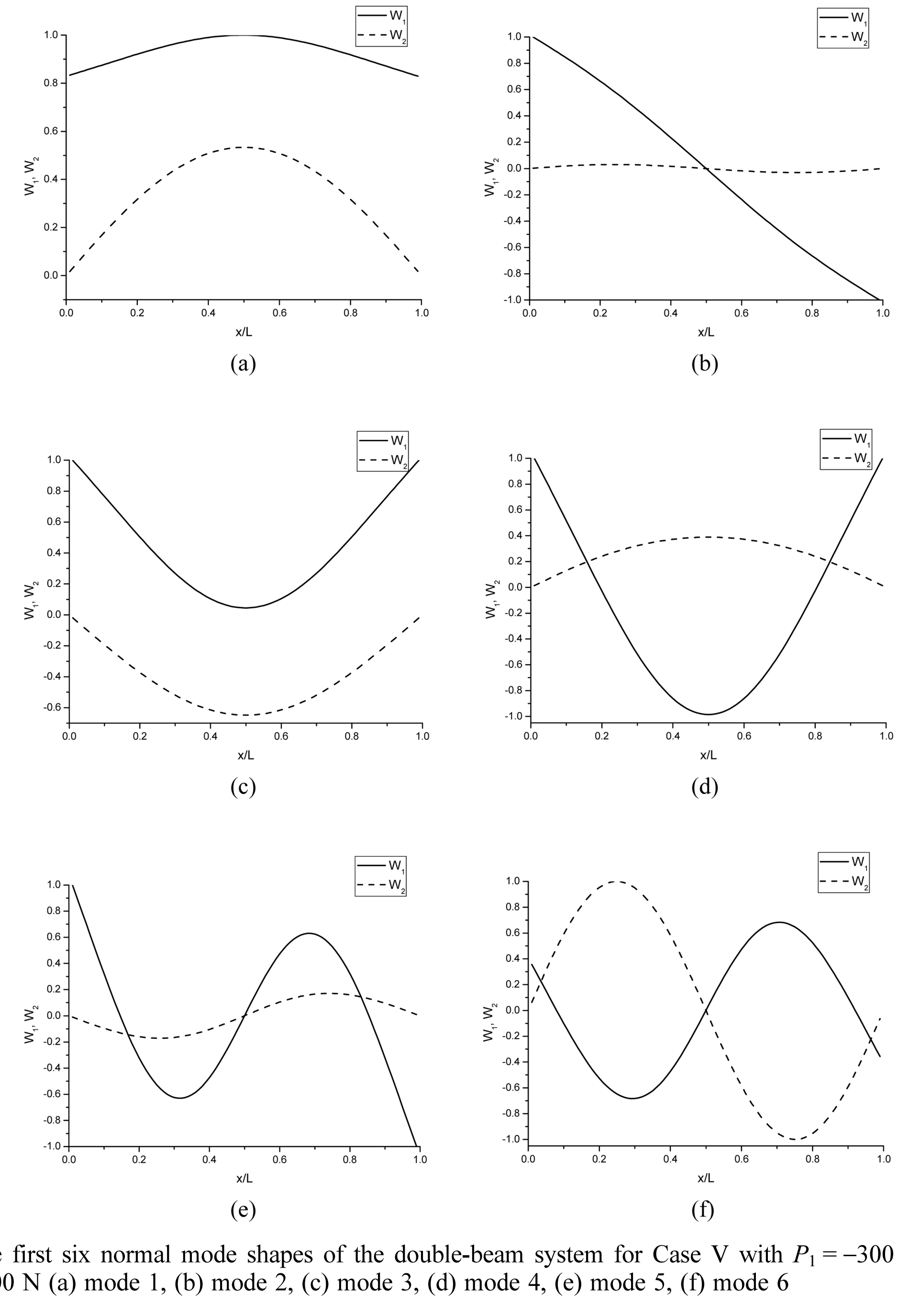

Fig. 6 The first six normal mode shapes of the double-beam system for Case V with $P_1 = -300$ N and $P_2 = -500$ N (a) mode 1, (b) mode 2, (c) mode 3, (d) mode 4, (e) mode 5, (f) mode 6

axial force has a much significant effect on the lower natural frequencies of the double-beam system, but it has no pronounced effect on the higher natural frequencies. From Tables 1-5, it can be seen that the compressive axial force has a more significant effect on the natural frequencies of the double-beam system than the equal and opposite tensile force.

The exact mode shapes of the double-beam system for Case V with and without the axial forces included are calculated and shown in Figs. 4-6. The first six normal mode shapes for the unloaded double-beam system are shown in Figs. 4(a)-(f). The first six normal mode shapes for the axially loaded double-beam system with P_1 = 300 N and P_2 = 500 N are plotted in Figs. 5(a)-(f). The first six normal mode shapes for the axially loaded double-beam system with P_1 = −300 N and P_2 = −500 N are illustrated in Figs. 6(a)-(f). The influence of the axial forces on the normal mode shapes of the double-beam system can be observed from Figs. 4-6.

It is obvious from Figs. 4-6 that the normal mode shapes of the double-beam system are quite sensitive to the variations of the axial forces P_1 and P_2 under consideration.

6. Conclusions

An analytical method for determining the natural frequencies and mode shapes of the axially loaded slender double-beam system with the mass of springs included is developed. The dynamic stiffness matrix is formulated for an axially loaded double-beam element by directly solving the governing differential equations of motion in free vibration. The effects of the axial force and the mass of springs are included, but the effects of the shear deformation and rotary inertia are considered to be negligible and not included in the present development. The dynamic stiffness method is illustrated by a particular axially loaded double-beam system, in which the effects of the axial force and the boundary condition on the natural frequencies and/or mode shapes are extensively investigated.

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