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# Effects of anisotropy and curvature on free vibration characteristics of laminated composite cylindrical shallow shells

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**Abstract.** This paper presents effects of anisotropy and curvature on free vibration characteristics of cross-ply laminated composite cylindrical shallow shells. Shallow shells have been considered for different lamination thickness, radius of curvature and elasticity ratio. First, kinematic relations of strains and deformation have been showed. Then, using Hamilton's principle, governing differential equations have been obtained for a general curved shell. In the next step, stress-strain relation for laminated, cross-ply composite shells has been given. By using some simplifications and assuming Fourier series as a displacement field, differential equations are solved by matrix algebra for shallow shells. The results obtained by this solution have been given tables and graphs. The comparisons made with the literature and finite element program (ANSYS).

**Keywords:** structural composites; vibration; anisotropy; shell theory; finite element method (FEM).

## 1. Introduction

A composite is a structural material, which consists of combining two or more constituents on a macroscopic scale to form a useful material. The goal of this three dimensional composition is to obtain a property which none of the constituents possesses. In other words, the target is to produce a material that possesses higher performance properties than its constituent parts for a particular purpose. Some of these properties are mechanical strength, corrosion resistance, high temperature resistance, heat conductibility, stiffness, lightness and appearance. In accordance with this definition, the following conditions must be satisfied by the composite material. It must be man-made and not natural. It must comprise of at least two different materials with different chemical components separated by distinct interfaces. Different materials must be put together in a three dimensional unity. It must possess properties, which none of the constituents possesses alone and that must be the aim of its production. The material must behave as a whole, i.e., the fiber and the matrix material (material surrounding the fibers) must be perfectly bonded. As a structural material, composites offer lower weight and higher strength.

Shells are common structural elements in many engineering structures, including concrete roofs,

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exteriors of rockets, ship hulls, automobile tires, containers of liquids, oil tanks, pipes, aerospace etc. A shell can be defined as a curved, thin-walled structure. It can be made from a single layer or multilayer of isotropic or anisotropic materials. Shells can be classified according to their curvatures. Shallow shells are defined as shells that have rise of not more than one fifth the smallest planform dimension of the shell (Qatu 2004).

Shells are three-dimensional (3D) bodies bounded by two relatively close, curved surfaces. The 3D equations of elasticity are complicated that's why all shell theories (thin, thick, shallow and deep, etc.) reduce the 3D elasticity problem into a 2D one. This is done usually by Classical Lamination Theory-CLT and Kirchhoff hypothesis.

A number of theories exist for layered shells. Many of these theories were developed originally for thin shells and based on the Kirchhoff-Love kinematic hypothesis that straight lines normal to the undeformed mid-surface remain straight and normal to the middle surface after deformation. Among these theories Qatu (2004) uses energy functional to develop equation of motion. Many studies have been performed on characteristics of shallow shells (Qatu 1991, 1992a, 1993a, b). Recently, Latifa and Sinha (2005) have used an improved finite element model for the bending and free vibration analysis of doubly curved, laminated composite shells having spherical and ellipsoidal shapes. Large-amplitude vibrations of circular cylindrical shells subjected to radial harmonic excitation in the spectral neighborhood of the lowest resonance are investigated by Amabili (2003). Gautham and Ganesan (1997) deal with the free vibration characteristics of isotropic and laminated orthotropic spherical caps. Liew *et al.* (2002) has presented the elasticity solutions for free vibration analysis of doubly curved shell panels of rectangular planform. Grigorenko and Yaremchenko (2007) have analyzed the stress-strain state of a shallow shell with rectangular planform and varying thickness. Djoudi and Bahai (2003) have presented a cylindrical strain based shallow shell finite element which is developed for linear and geometrically non-linear analysis of cylindrical shells.

In this paper various parameters affecting free vibration characteristic of symmetric, cross-ply, composite, shallow shells have been examined. The shells have square planform. The Ratio of elasticity modulus ($E_1/E_2$) of anisotropic composites has been considered as a first parameter. For various $E_1/E_2$ values solutions are obtained from computer program written using following theory. Furthermore, for the same ratios, problem is modeled by finite element method also (Reddy 1993). For the solution of problem by finite element method a commercial program, named ANSYS, has been used. $a/R$ (ratio of shell length to radius of shell) is considered as a second parameter for frequencies of the shell structure. Starting from $a/R = 0$ to 0.1 various values are examined by both computer program and ANSYS. Various $a/h$ (ratio of shell length to thickness of shell) values are used as a third parameter. The results obtained from analysis have been compared with literature and ANSYS by using tables and graphs.

## 2. Theories

A lamina is made of isotropic homogeneous reinforcing fibers and an isotropic homogeneous material surrounding the fibers, called matrix material (Fig. 1). Therefore, the stiffness of the lamina varies from point to point depending on whether the point is in the fiber, the matrix or the fiber and matrix interface. Because of these variations, macro-mechanical analysis of a lamina is based on average properties.

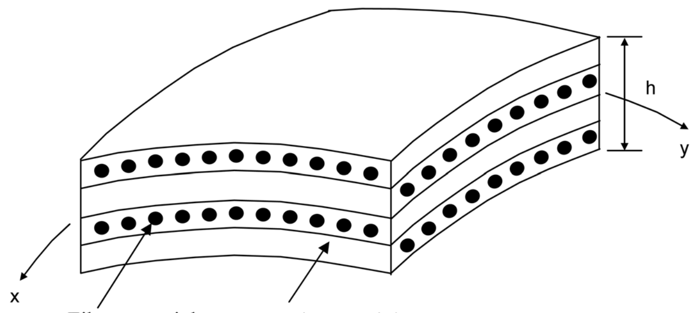


Fig. 1 Fiber and matrix materials in laminated composite shallow shell

There are many theories of shells. Classical shell theory, also known as Kirchhoff-Love kinematic hypothesis, assumes that "The normals to the middle surface remain straight and normal to the mid-surface when the shell undergoes deformation". However, according to first order shear deformation theory "The transverse normals do not remain perpendicular to the mid-surface after deformation" (Reddy 2003). In addition, classical lamination theory says "laminas are perfectly bonded" (Gurdal *et al.* 1998, Hyer 1997, Reddy and Miravete 1995, Jones 1984). The theory of shallow shells can be obtained by making the following additional assumptions to thin (or classical) and thick (or shear deformation) shell theories. It will be assumed that the deformation of the shells is completely determined by the displacement of its middle surface. The derivation of equations of motion is based on two assumptions. The first assumption is that the shallow shell has small deflections. The second assumption is that the shallow shell thickness is small compared to its radii of curvature. Also, the radii of curvature are very large compared to the in-plane displacement. Curvature changes caused by the tangential displacement component $u$ and $v$ are very small in a shallow shell, in comparison with changes caused by the normal component $w$.

### 2.1 Geometric properties

The vectorial equation of the undeformed surface could be written by the $x$ and $y$ cartesian coordinates as

$$\bar{r} = \bar{r}(x, y) \tag{1}$$

a small increment in $\vec{r}$ vector is given as

$$d\bar{r} = \bar{r}_{,x}dx + \bar{r}_{,y}dy \tag{2}$$

where $\bar{r}_{,x}$ is the small increment in $x$ direction and $\bar{r}_{,y}$ is the small increment in $y$ direction (Fig. 2). The differential length of the shell surface could be found by dot product of $d\bar{r}$ by itself

$$ds^2 = d\bar{r}.d\bar{r} = A^2dx^2 + B^2dy^2 \tag{3}$$

where $A$ and $B$ are referred as Lame parameters and defined as

$$A^2 = \bar{r}_{,x}.\bar{r}_{,x}, \quad B^2 = \bar{r}_{,y}.\bar{r}_{,y} \tag{4}$$

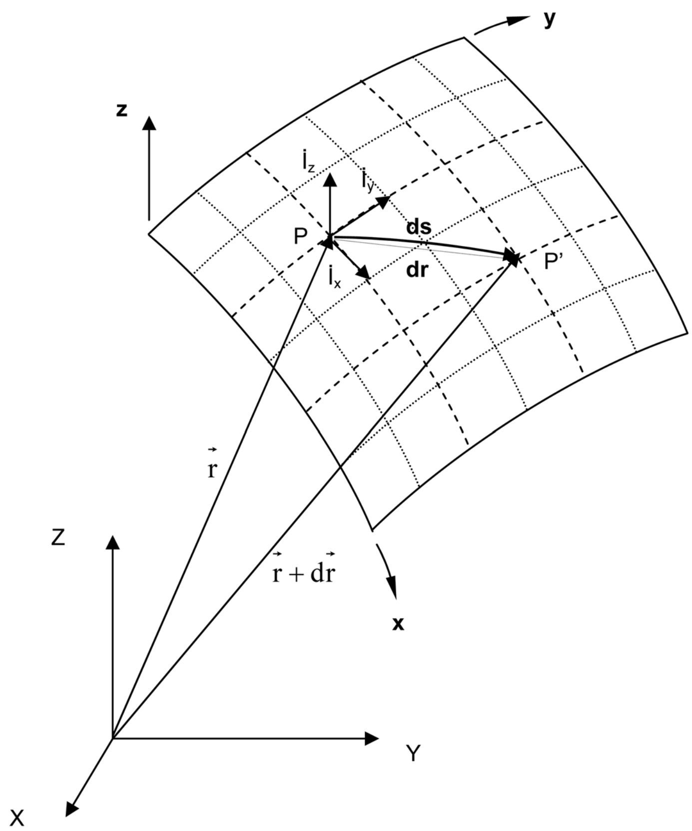


Fig. 2 Coordinates of shell mid-surface

Eq. (3) is called first fundamental form of the surface. Tangent vector to the surface could be obtained by taking derivative of Eq. (1) with respect to surface length. Then, applying Frenet's formula to the derivative of tangent vector and multiplying both sides by unit normal vector gives second quadratic form.

### 2.2 Kinematics of displacement

Let the position of a point, on a middle surface, shown by $\bar{r}(x,y)$. If this point undergoes the displacement by the amount of $\overline{U}$ then, final position of that point could be given as

$$\bar{r}'(x,y) = \bar{r}(x,y) + \overline{U} \tag{5}$$

where $\overline{U}$ is the displacement field of the point and defined as

$$\overline{U} = u\bar{i}_x + v\bar{i}_y + w\bar{i}_z \tag{6}$$

where $\bar{i}_x, \bar{i}_y$ and $\bar{i}_z$ are the unit vectors in the direction of *x*, *y* and *z*. *u*, *v*, and *w* are the displacements in the direction of *x*, *y* and *z* respectively. Using Eqs. (5) and (6) strains are calculated as

$$\varepsilon_x = \frac{1}{\left(1 + z/R_x\right)}\left(\frac{1}{A}\frac{\partial u}{\partial x} + \frac{v}{AB}\frac{\partial A}{\partial y} + \frac{w}{R_x}\right)$$

$$\varepsilon_y = \frac{1}{\left(1 + z/R_y\right)}\left(\frac{1}{B}\frac{\partial v}{\partial y} + \frac{u}{AB}\frac{\partial B}{\partial x} + \frac{w}{R_y}\right)$$

$$\varepsilon_z = \partial w/dz$$

$$\gamma_{xy} = \frac{1}{\left(1 + z/R_x\right)}\left(\frac{1}{A}\frac{\partial v}{\partial x} - \frac{u}{AB}\frac{\partial A}{\partial y} + \frac{w}{R_{xy}}\right) + \frac{1}{\left(1 + z/R_y\right)}\left(\frac{1}{B}\frac{\partial u}{\partial y} - \frac{v}{AB}\frac{\partial B}{\partial x} + \frac{w}{R_{xy}}\right)$$

$$\gamma_{xz} = \frac{1}{A\left(1 + z/R_x\right)}\frac{\partial w}{\partial x} + A\left(1 + z/R_x\right)\frac{\partial}{\partial z}\left(\frac{u}{A\left(1 + z/R_x\right)}\right) - \frac{v}{R_{xy}\left(1 + z/R_x\right)}$$

$$\gamma_{yz} = \frac{1}{B\left(1 + z/R_y\right)}\frac{\partial w}{\partial y} + B\left(1 + z/R_y\right)\frac{\partial}{\partial z}\left(\frac{v}{B\left(1 + z/R_y\right)}\right) - \frac{u}{R_{xy}\left(1 + z/R_y\right)} \quad (7)$$

where $R_x$, $R_y$, and $R_{xy}$ are curvatures in $x$-plane, $y$-plane and $xy$-plane respectively.

### 2.3 Stress strain relation

For an orthotropic media there are 9 stiffness coefficients written in local coordinates.

$$[\sigma] = [Q][\varepsilon] \quad (8)$$

where $[\sigma]$ is the stress matrices, $[Q]$ is the stiffness matrices and $[\varepsilon]$ strain matrices. The stresses in global coordinates are calculated by applying transformation rules. Then, the stresses over the shell thickness are integrated to obtain the force and moment resultants. Due to curvatures of the structure, extra terms must be taken into account during the integration. This difficulty could be overcame by expanding the term $[1/(1 + z/R_n)]$ in a geometric series.

### 2.4 Governing equations

Equation of motion for shell structures could be obtained by Hamilton's principle

$$\delta\int_{t_1}^{t_2}(T + W - U)\,dt = 0 \quad (9)$$

where $T$ is the kinetic energy of the structure

$$T = \frac{\varrho}{2}\int\left\{\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right\}dxdydz \quad (10)$$

$W$ is the work of the external forces

$$W = \int_x\int_y(q_x u + q_y v + q_z w + m_x\psi_x + m_y\psi_y)ABdxdy \quad (11)$$

in which $q_x, q_y, q_z$ are the external forces $u$, $v$, $w$ are displacements in $x$, $y$, $z$ direction respectively. $m_x, m_y$, are the external moments and $\psi_x$, $\psi_y$ are rotations in $x$, $y$ directions respectively. $U$ is the strain energy defined as

$$U = \frac{1}{2}\int(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \sigma_{xy}\varepsilon_{xy} + \sigma_{xz}\varepsilon_{xz} + \sigma_{yz}\varepsilon_{yz})dxdydz \tag{12}$$

Solving Eq. (9) gives set of equations called equations of motion for shell structures.

$$\frac{\partial}{\partial x}(BN_x) + \frac{\partial}{\partial y}(AN_{yx}) + \frac{\partial A}{\partial y}N_{xy} - \frac{\partial B}{\partial x}N_y + \frac{AB}{R_x}Q_x + \frac{AB}{R_{xy}}Q_y + ABq_x = AB(\bar{I}_1\ddot{u}^2 + \bar{I}_2\ddot{\psi}_x^2)$$

$$\frac{\partial}{\partial y}(AN_y) + \frac{\partial}{\partial x}(BN_{xy}) + \frac{\partial B}{\partial x}N_{yx} - \frac{\partial A}{\partial y}N_x + \frac{AB}{R_y}Q_y + \frac{AB}{R_{xy}}Q_x + ABq_y = AB(\bar{I}_1\ddot{v}^2 + \bar{I}_2\ddot{\psi}_y^2)$$

$$-AB\left(\frac{N_x}{R_x} + \frac{N_y}{R_y} + \frac{N_{xy} + N_{yx}}{R_{xy}}\right) + \frac{\partial}{\partial x}(BQ_x) + \frac{\partial}{\partial y}(AQ_y) + ABq_z = AB(\bar{I}_1\ddot{w}^2)$$

$$\frac{\partial}{\partial x}(BM_x) + \frac{\partial}{\partial y}(AM_{yx}) + \frac{\partial A}{\partial y}M_{xy} - \frac{\partial B}{\partial x}M_y - ABQ_x + \frac{AB}{R_x}P_x + ABm_x = AB(\bar{I}_2\ddot{u}^2 + \bar{I}_3\ddot{\psi}_x^2)$$

$$\frac{\partial}{\partial y}(AM_y) + \frac{\partial}{\partial x}(BM_{xy}) + \frac{\partial B}{\partial x}M_{yx} - \frac{\partial A}{\partial y}M_x - ABQ_y + \frac{AB}{R_y}P_y + ABm_y = AB(\bar{I}_2\ddot{v}^2 + \bar{I}_3\ddot{\psi}_y^2) \tag{13}$$

When the shell has small curvature it is referred to as a shallow shell. Shallow shells are defined as shells that have a rise of not more than 1/5th the smallest planform dimension of the shell (Qatu 2004). It has been widely accepted that shallow shell equations should not be used for maximum span to minimum radius ratio of 0.5 or more. For shallow shells, Lame parameters are assumed to equal to one ($A = B = 1$). This gives Eq. (13) in simplified form as

$$\frac{\partial}{\partial x}N_x + \frac{\partial}{\partial y}N_{yx} + q_x = \bar{I}_1\ddot{u}^2 + \bar{I}_2\ddot{\psi}_x^2$$

$$\frac{\partial}{\partial y}N_y + \frac{\partial}{\partial x}N_{xy} + q_y = \bar{I}_1\ddot{v}^2 + \bar{I}_2\ddot{\psi}_y^2$$

$$-\left(\frac{N_x}{R_x} + \frac{N_y}{R_y} + \frac{N_{xy} + N_{yx}}{R_{xy}}\right) + \frac{\partial}{\partial x}Q_x + \frac{\partial}{\partial y}Q_y + q_z = \bar{I}_1\ddot{w}^2$$

$$\frac{\partial}{\partial x}M_x + \frac{\partial}{\partial y}M_{yx} - Q_x + m_x = \bar{I}_2\ddot{u}^2 + \bar{I}_3\ddot{\psi}_x^2$$

$$\frac{\partial}{\partial y}M_y + \frac{\partial}{\partial x}M_{xy} - Q_y + m_y = \bar{I}_2\ddot{v}^2 + \bar{I}_3\ddot{\psi}_y^2 \tag{14}$$

Eq. (14) is defined as equation of motion for thick shallow shell. For thin shallow shells this equation reduces to

$$\frac{\partial}{\partial x}N_x + \frac{\partial}{\partial y}N_{yx} + q_x = \bar{I}_1\ddot{u}^2$$

$$\frac{\partial}{\partial y}N_y + \frac{\partial}{\partial x}N_{xy} + q_y = \bar{I}_1\ddot{v}^2$$

$$-\left(\frac{N_x}{R_x} + \frac{N_y}{R_y} + \frac{2N_{xy}}{R_{xy}}\right) + \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q_z = \bar{I}_1\ddot{w}^2 \tag{15}$$

The Navier type solution can be applied to thick and thin shallow shells. This type solution assumes that the displacement field of the shallow shells could be represented as sine and cosine trigonometric functions.

## 3. Numerical examples

Before proceeding further, the modeling of the shell structure in ANSYS package program has been checked to avoid getting wrong results. A cylindrical shell structure which is solved by Qatu (2004) as an example problem in section 7.3.1.1 has been chosen (Fig. 3).

The studies were made for isotropic steel. The thickness of the shell is $h = 0.02$ in, the length of the shell is $a = 11.74$ in, Radius of the cylindrical shell is $R = 5.836$ in, unit mass is $734 \times 10^{-6}$ lb $s^2/in^4$, modulus of elasticity is $29.5 \times 10^6$ lb/in$^2$ and Poisson's ratio is 0.285. The same cylindrical shell has been solved by Bert *et al.* using Love's shell theory, by Rat and Das who included shear deformation and rotary inertia, by Bray and Eagle using experimental procedure and finally by Qatu using classical shell theory. All the results obtained by researchers have been given in Qatu (2004). The same problem has been solved again by modeling the structure with finite element method and using ANSYS package program. A $160 \times 20$ mesh has been chosen. Each mesh element which is called SHELL99 has 8 degree of freedom. Results of that model prepared in ANSYS have been given in Table 1.

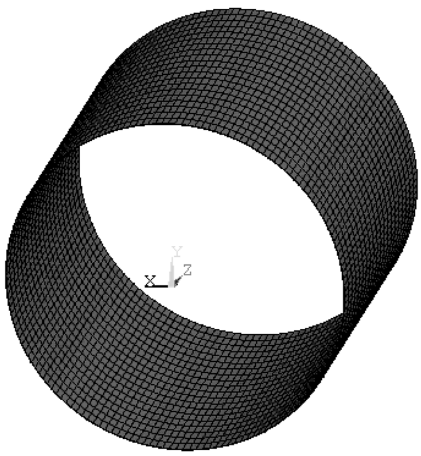


Fig. 3 Cylindrical shells modeled by using ANSYS

Table 1 Natural frequency parameters (Hertz) obtained by using CLSST and ANSYS

| *m* | *n* | CLSST | ANSYS | *m* | *n* | CLSST | ANSYS | *m* | *n* | CLSST | ANSYS |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 5328,25 | 5325,99 | | 0 | 5442,58 | | | 0 | 5458,11 | 5446,97 |
| | 1 | 3270,54 | 3336,81 | | 1 | 4837,71 | 4832,71 | | 1 | 5197,96 | 5205,68 |
| | 2 | 1861,97 | 2144,79 | | 2 | 3725,02 | 3729,93 | | 2 | 4563,85 | 4565,18 |
| | 3 | 1101,78 | 1469,94 | | 3 | 2742,67 | 2799,58 | | 3 | 3813,65 | 3817,14 |
| | 4 | 705,71 | 1061,33 | | 4 | 2018,09 | 2142,28 | | 4 | 3114,51 | 3139,31 |
| | 5 | 497,54 | 803,13 | | 5 | 1515,06 | 1684,79 | | 5 | 2530,39 | 2587,26 |
| | 6 | 400,18 | 642,65 | | 6 | 1174,98 | 1363,45 | | 6 | 2069,45 | 2157,05 |
| | 7 | 380,82 | 556,52 | | 7 | 953,72 | 1139,82 | | 7 | 1719,25 | 1829,20 |
| | 8 | 416,82 | 533,12 | | 8 | 824,39 | 993,07 | | 8 | 1464,11 | 1585,77 |
| | 9 | 488,69 | 561,49 | | 9 | 770,52 | 912,16 | | 9 | 1291,20 | 1414,21 |
| | 10 | 583,96 | 628,94 | | 10 | 778,47 | 889,54 | | 10 | 1190,96 | 1306,51 |
| | 11 | 696,30 | 724,51 | | 11 | 834,33 | 917,13 | | 11 | 1154,97 | 1256,83 |
| | 12 | 822,76 | 840,92 | | 12 | 925,62 | 985,64 | | 12 | 1174,09 | 1259,24 |
| | 13 | 961,95 | 973,94 | | 13 | 1043,11 | 1086,24 | | 13 | 1238,37 | 1306,82 |
| | 14 | 1113,21 | 1121,24 | | 14 | 1180,85 | 1211,93 | | 14 | 1338,48 | 1392,15 |
| 1 | 15 | 1276,17 | 1281,56 | 2 | 15 | 1335,26 | 1357,80 | 3 | 15 | 1466,80 | 1508,31 |
| | 16 | 1450,68 | 1454,19 | | 16 | 1504,23 | 1520,68 | | 16 | 1617,78 | 1649,69 |
| | 17 | 1636,62 | 1638,73 | | 17 | 1686,51 | 1698,52 | | 17 | 1787,61 | 1812,07 |
| | 18 | 1833,93 | 1834,97 | | 18 | 1881,37 | 1890,06 | | 18 | 1973,77 | 1992,46 |
| | 19 | 2042,59 | 2042,76 | | 19 | 2088,34 | 2094,50 | | 19 | 2174,60 | 2188,81 |
| | 20 | 2262,58 | 2262,04 | | 20 | 2307,14 | 2311,34 | | 20 | 2389,01 | 2399,71 |
| | 21 | 2493,87 | 2492,77 | | 21 | 2537,59 | 2540,26 | | 21 | 2616,30 | 2624,22 |
| | 22 | 2736,47 | 2734,93 | | 22 | 2779,58 | 2781,06 | | 22 | 2855,96 | 2861,71 |
| | 23 | 2990,36 | 2988,52 | | 23 | 3033,04 | 3033,61 | | 23 | 3107,69 | 3111,75 |
| | 24 | 3255,54 | 3253,55 | | 24 | 3297,90 | 3297,84 | | 24 | 3371,26 | 3374,05 |
| | 25 | 3532,02 | 3529,36 | | 25 | 3574,13 | 3573,71 | | 25 | 3646,52 | 3648,43 |
| | 26 | 3819,79 | 3818,06 | | 26 | 3861,71 | 3861,21 | | 26 | 3933,35 | 3934,75 |
| | 27 | 4118,84 | 4117,60 | | 27 | 4160,63 | 4160,34 | | 27 | 4231,69 | 4232,96 |
| | 28 | 4429,18 | 4428,72 | | 28 | 4470,86 | 4471,14 | | 28 | 4541,48 | 4543,02 |
| | 29 | 4750,81 | 4751,49 | | 29 | 4792,41 | 4793,63 | | 29 | 4862,68 | 4864,92 |
| | 30 | 5083,73 | 5085,96 | | 30 | 5125,26 | 5127,88 | | 30 | 5195,25 | 5198,69 |

Using results given in Table 1 and results obtained by using ANSYS, the graphs have been drawn. The results have been given for first three ($m$ = 1, 2, 3) longitudinal modes and first thirth ($n$ = 1, 2,…, 30) circumferential modes. The three graphs have been drawn together in Fig. 4.

The correctness of the ANSYS model has been checked in this example problem. The problem has been solved by Qatu, Bert *et al.*, Rath *et al.*, and Bray *et al*. The results obtained by those researchers have been compared by the results obtained by modeling the problem in ANSYS. For three cases graphs have been drawn and a perfect match has been observed with the results.

The governing Eq. (14) (using SDSST theory) and the governing Eq. (15) (using CLSST theory) derived in the theory section are solved by using Mathematica program separately. Furthermore, ANSYS packet program has been used in solution. The geometry of the shell structures has been created using arc-length method in ANSYS. Then, area element has been defined between the arc lines. Finally using SHELL99 finite element, the area has been meshed.

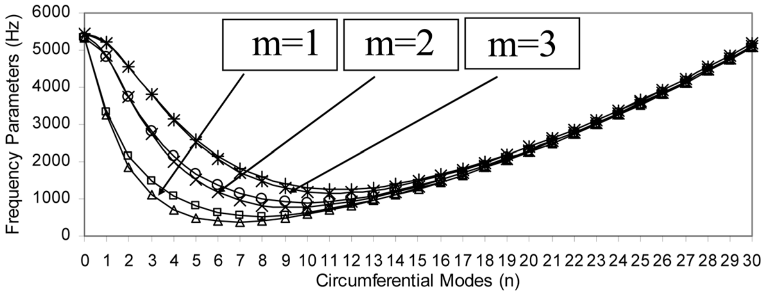


Fig. 4 The results for first three ($m$ = 1, 2, 3) longitudinal modes

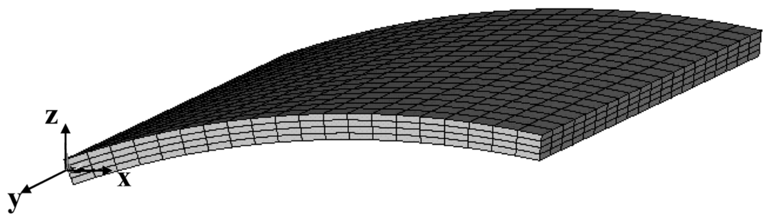


Fig. 5 Cylindrical shallow shell

As an example a simply supported cylindrical shell which has a radius of curvature in one plane and infinite radius of curvature in other plane, is considered (Fig. 5). The shell, in hand, has a square planform where length and width are equal to unity ($a = b = 1$). As a material, a laminated composite has been used with a [0°/90°/90°/0°] symmetrical cross-ply stacking sequence. To determine the effect of anisotropy, shell thickness and radius of curvature on free vibration characteristics of cylindrical shallow shell, problem has been solved for various values. First, ratio of modulus of elasticity ($E_1/E_2$) which is the ratio of modulus of elasticity in fiber direction to matrix direction has been taken as a variable from 1 to 50. Next, effect of shell thickness ratio that ratio of shell width to shell thickness, $a/h$ = 100, 50, 20, 10 and 5, has been examined. Furthermore, radius of curvature has been considered. For different shell width/shell radius ratios which vary from infinity (plate) to 0.1, graphs have been obtained.

For each case, the shell has been solved with three theories. First theory is the classical laminated shallow shell theory (CLSST) which assumes "normals to the middle surface remain straight and normal after deformation". Second theory used in the solution of composite laminated shallow shell is shear deformation shallow shell theory (SDSST). SDSST is similar to CLSST except about transverse normals i.e., the transverse normals do not remain perpendicular to the mid-surface after deformation. The FEM is a powerful numerical method to solve mechanic problems. The FEM is easier and faster than the analytical methods to solve complicated problems. Therefore, simply supported shallow shell of which analytical solution is known, is solved for the verification of the FEM, and chosen to demonstrate the suitability and accuracy of FEM comparing with analytical results. The comparisons indicate that the FEM can be used with good confidence in shallow shell problems of which analytical solution is not known. Entire structure is meshed by finite elements in this theory. Then assuming a suitable displacement fields for each meshing element, the behavior of the structure has been obtained. In this paper, a finite element package program ANSYS has been used. The structure is meshed by 25 × 25 elements. A 8-noded quadratic element is considered as a

Table 2 Natural frequency parameters ($\Omega = \omega a^2\sqrt{\rho/E_2h^2}$) for cross ply laminated [0°/90°/90°/0°] cylindrical shallow shells for shear deformation shallow shell theory (SDSST), classical shallow shell theory (CLSST) and ANSYS ($a/b$ = 1, $E_1/E_2$ = 5, $G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2$ = 0.5, $\upsilon_{12}$ = 0.25 and $K^2$ = 5/6)

| *a/h* | *a/R* | ANSYS | SDSST | CLSST |
|---|---|---|---|---|
| 100 | 0.000 | 8.3182320 | 8.3419561 | 8.3466158 |
| | 0.005 | 8.3554634 | 8.3476910 | 8.3523485 |
| | 0.010 | 8.4660912 | 8.3648721 | 8.3695228 |
| | 0.020 | 8.8945184 | 8.4332445 | 8.4378686 |
| | 0.025 | 9.2028989 | 8.4841601 | 8.4887647 |
| | 0.033 | 9.8354765 | 8.5931268 | 8.5976906 |
| | 0.050 | 11.4494426 | 8.8970644 | 8.9015209 |
| | 0.100 | 17.7495839 | 10.3847602 | 10.3888021 |
| 50 | 0.000 | 8.2824179 | 8.3280318 | 8.3466158 |
| | 0.005 | 8.2918191 | 8.3294614 | 8.3480430 |
| | 0.010 | 8.3198537 | 8.3337484 | 8.3523233 |
| | 0.020 | 8.4309702 | 8.3508743 | 8.3694217 |
| | 0.025 | 8.5132524 | 8.3636951 | 8.3822221 |
| | 0.033 | 8.6884797 | 8.3913259 | 8.4098092 |
| | 0.050 | 9.1705102 | 8.4697626 | 8.4881238 |
| | 0.100 | 11.4231408 | 8.8810714 | 8.8988334 |
| 20 | 0.000 | 8.1342034 | 8.2328074 | 8.3466158 |
| | 0.005 | 8.1357984 | 8.2330311 | 8.3468371 |
| | 0.010 | 8.1403834 | 8.2337022 | 8.3475009 |
| | 0.020 | 8.1586703 | 8.2363860 | 8.3501554 |
| | 0.025 | 8.1722922 | 8.2383982 | 8.3521457 |
| | 0.033 | 8.2017396 | 8.2427437 | 8.3564439 |
| | 0.050 | 8.2852836 | 8.2551448 | 8.3687102 |
| | 0.100 | 8.7224188 | 8.3217438 | 8.4345926 |
| 10 | 0.000 | 7.7777913 | 7.9214445 | 8.3466158 |
| | 0.005 | 7.7783773 | 7.9214963 | 8.3466647 |
| | 0.010 | 7.7795769 | 7.9216516 | 8.3468112 |
| | 0.020 | 7.7843752 | 7.9222728 | 8.3473973 |
| | 0.025 | 7.7879295 | 7.9227386 | 8.3478368 |
| | 0.033 | 7.7956157 | 7.9237449 | 8.3487863 |
| | 0.050 | 7.8176524 | 7.9266189 | 8.3514980 |
| | 0.100 | 7.9353888 | 7.9421093 | 8.3661152 |
| 5 | 0.000 | 6.8597888 | 6.9862196 | 8.3466158 |
| | 0.005 | 6.8603666 | 6.9862292 | 8.3466210 |
| | 0.010 | 6.8607220 | 6.9862580 | 8.3466364 |
| | 0.020 | 6.8620771 | 6.9863732 | 8.3466982 |
| | 0.025 | 6.8630546 | 6.9864596 | 8.3467445 |
| | 0.033 | 6.8652316 | 6.9866463 | 8.3468446 |
| | 0.050 | 6.8714516 | 6.9871794 | 8.3471304 |
| | 0.100 | 6.9048843 | 6.9900556 | 8.3486726 |

Table 3 Natural frequency parameters ($\Omega = \omega a^2 \sqrt{\rho/E_2 h^2}$) for cross ply laminated [0º/90º/90º/0º] cylindrical shallow shells for shear deformation shallow shell theory (SDSST), classical shallow shell theory (CLSST) and ANSYS ($a/b = 1$, $E_1/E_2 = 25$, $G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2 = 0.5$, $\upsilon_{12} = 0.25$ and $K^2 = 5/6$)

| *a/h* | *a/R* | ANSYS | SDSST | CLSST |
|---|---|---|---|---|
| 100 | 0.000 | 15.1848853 | 15.1966195 | 15.2277943 |
| | 0.005 | 15.2714771 | 15.2004253 | 15.2315928 |
| | 0.010 | 15.5291199 | 15.2118367 | 15.2429825 |
| | 0.020 | 16.5179723 | 15.2573956 | 15.2884551 |
| | 0.025 | 17.2247461 | 15.2914739 | 15.3224693 |
| | 0.033 | 18.6565540 | 15.3648371 | 15.3956952 |
| | 0.050 | 22.2396947 | 15.5725053 | 15.6029822 |
| | 0.100 | 35.7649855 | 16.6483393 | 16.6769989 |
| 50 | 0.000 | 15.0831433 | 15.1043949 | 15.2277943 |
| | 0.005 | 15.1052244 | 15.1053385 | 15.2287305 |
| | 0.010 | 15.1708014 | 15.1081688 | 15.2315388 |
| | 0.020 | 15.4301547 | 15.1194844 | 15.2427663 |
| | 0.025 | 15.6217762 | 15.1279651 | 15.2511812 |
| | 0.033 | 16.0279668 | 15.1462701 | 15.2693443 |
| | 0.050 | 17.1349999 | 15.1984395 | 15.3211112 |
| | 0.100 | 22.1675867 | 15.4768842 | 15.5974535 |
| 20 | 0.000 | 14.4766898 | 14.5084431 | 15.2277943 |
| | 0.005 | 14.4808661 | 14.5085857 | 15.2279289 |
| | 0.010 | 14.4918844 | 14.5090135 | 15.2283327 |
| | 0.020 | 14.5356024 | 14.5107244 | 15.2299479 |
| | 0.025 | 14.5682132 | 14.5120074 | 15.2311591 |
| | 0.033 | 14.6385884 | 14.5147787 | 15.2337755 |
| | 0.050 | 14.8377184 | 14.5226923 | 15.2412468 |
| | 0.100 | 15.8694447 | 14.5653161 | 15.2814952 |
| 10 | 0.000 | 12.8759190 | 12.8912312 | 15.2277943 |
| | 0.005 | 12.8778295 | 12.8912607 | 15.2278141 |
| | 0.010 | 12.8808951 | 12.8913493 | 15.2278737 |
| | 0.020 | 12.8932463 | 12.8917036 | 15.2281121 |
| | 0.025 | 12.9023542 | 12.8919693 | 15.2282909 |
| | 0.033 | 12.9222139 | 12.8925433 | 15.2286771 |
| | 0.050 | 12.9787274 | 12.8941828 | 15.2297803 |
| | 0.100 | 13.2795994 | 12.9030251 | 15.2357311 |
| 5 | 0.000 | 9.7176957 | 9.6925563 | 15.2277943 |
| | 0.005 | 9.7194062 | 9.6925597 | 15.2277845 |
| | 0.010 | 9.7204281 | 9.6925699 | 15.2277551 |
| | 0.020 | 9.7244933 | 9.6926106 | 15.2277551 |
| | 0.025 | 9.7274923 | 9.6926412 | 15.2277551 |
| | 0.033 | 9.7340677 | 9.6927071 | 15.2277551 |
| | 0.050 | 9.7527723 | 9.6928956 | 15.2277551 |
| | 0.100 | 9.8531592 | 9.6939123 | 15.2277551 |

Table 4 Natural frequency parameters ($\Omega = \omega a^2 \sqrt{\rho / E_2 h^2}$) for cross ply laminated [0º/90º/90º/0º] cylindrical shallow shells for shear deformation shallow shell theory (SDSST), classical shallow shell theory (CLSST) and ANSYS ($a/b = 1$, $E_1/E_2 = 50$, $G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2 = 0.5$, $\upsilon_{12} = 0.25$ and $K^2 = 5/6$)

| *a/h* | *a/R* | ANSYS | SDSST | CLSST |
|---|---|---|---|---|
| 100 | 0.000 | 20.7615920 | 20.7697041 | 20.8519882 |
| | 0.005 | 20.8861260 | 20.7725750 | 20.8548479 |
| | 0.010 | 21.2546187 | 20.7811850 | 20.8634248 |
| | 0.020 | 22.6662115 | 20.8155881 | 20.8976956 |
| | 0.025 | 23.6739017 | 20.8413518 | 20.9233606 |
| | 0.033 | 25.7098528 | 20.8969019 | 20.9786985 |
| | 0.050 | 30.7856245 | 21.0547801 | 21.1359801 |
| | 0.100 | 49.8238221 | 21.8869447 | 21.9651424 |
| 50 | 0.000 | 20.5170113 | 20.5293619 | 20.8519882 |
| | 0.005 | 20.5487779 | 20.5300691 | 20.8526841 |
| | 0.010 | 20.6430781 | 20.5321904 | 20.8547715 |
| | 0.020 | 21.0158582 | 20.5406734 | 20.8631190 |
| | 0.025 | 21.2911836 | 20.5470330 | 20.8693772 |
| | 0.033 | 21.8740232 | 20.5607652 | 20.8828904 |
| | 0.050 | 23.4588662 | 20.5999425 | 20.9214446 |
| | 0.100 | 30.6143513 | 20.8100439 | 21.1282430 |
| 20 | 0.000 | 19.0690313 | 19.0768068 | 20.8519882 |
| | 0.005 | 19.0751625 | 19.0769093 | 20.8520780 |
| | 0.010 | 19.0914234 | 19.0772169 | 20.8523476 |
| | 0.020 | 19.1564672 | 19.0784473 | 20.8534259 |
| | 0.025 | 19.2048946 | 19.0793699 | 20.8542346 |
| | 0.033 | 19.3093024 | 19.0813631 | 20.8559814 |
| | 0.050 | 19.6045764 | 19.0870555 | 20.8609706 |
| | 0.100 | 21.1261749 | 19.1177388 | 20.8878684 |
| 10 | 0.000 | 15.8118649 | 15.7948517 | 20.8519882 |
| | 0.005 | 15.8149750 | 15.7948703 | 20.8519912 |
| | 0.010 | 15.8198621 | 15.7949262 | 20.8520001 |
| | 0.020 | 15.8394997 | 15.7951499 | 20.8520358 |
| | 0.025 | 15.8540723 | 15.7953176 | 20.8520625 |
| | 0.033 | 15.8857057 | 15.7956800 | 20.8521203 |
| | 0.050 | 15.9756740 | 15.7967152 | 20.8522854 |
| | 0.100 | 16.4526620 | 15.8022988 | 20.8531763 |
| 5 | 0.000 | 10.9302473 | 10.8910270 | 20.8519882 |
| | 0.005 | 10.9327131 | 10.8910274 | 20.8519681 |
| | 0.010 | 10.9344903 | 10.8910288 | 20.8519079 |
| | 0.020 | 10.9415767 | 10.8910341 | 20.8516669 |
| | 0.025 | 10.9468415 | 10.8910382 | 20.8514862 |
| | 0.033 | 10.9582819 | 10.8910469 | 20.8510959 |
| | 0.050 | 10.9908705 | 10.8910719 | 20.8499807 |
| | 0.100 | 11.1651203 | 10.8912067 | 20.8439639 |

Table 5 Natural frequency parameters ($\Omega = \omega a^2\sqrt{\rho/E_2h^2}$) for cross ply laminated [0º/90º/90º/0º] cylindrical shallow shells for ANSYS ($a/b = 1$, $E_1/E_2 = 5$, $G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2 = 0.5$, $\upsilon_{12} = 0.25$ and $K^2 = 5/6$)

| *a/h* | *a/R* | ANSYS (simply supported) | ANSYS (fixed-end supported) |
|---|---|---|---|
| 100 | 0.000 | 8.3182320 | 16.831862 |
| | 0.025 | 9.2028989 | 17.225190 |
| | 0.050 | 11.4494426 | 18.351772 |
| | 0.100 | 17.7495839 | 22.271284 |
| 50 | 0.000 | 8.2824179 | 16.726121 |
| | 0.025 | 8.5132524 | 16.826086 |
| | 0.050 | 9.1705102 | 17.121982 |
| | 0.100 | 11.4231408 | 18.256028 |
| 20 | 0.000 | 8.1342034 | 16.044672 |
| | 0.025 | 8.1722922 | 16.061555 |
| | 0.050 | 8.2852836 | 16.111848 |
| | 0.100 | 8.7224188 | 16.311423 |
| 10 | 0.000 | 7.7777913 | 14.207273 |
| | 0.025 | 7.7879295 | 14.212161 |
| | 0.050 | 7.8176524 | 14.226600 |
| | 0.100 | 7.9353888 | 14.284313 |
| 5 | 0.000 | 6.8597888 | 10.576349 |
| | 0.025 | 6.8630546 | 10.578038 |
| | 0.050 | 6.8714516 | 10.583014 |
| | 0.100 | 6.9048843 | 10.602918 |

Table 6 Natural frequency parameters ($\Omega = \omega a^2\sqrt{\rho/E_2h^2}$) for cross ply laminated [0º/90º/90º/0º] cylindrical shallow shells for ANSYS ($a/b = 1$, $E_1/E_2 = 25$, $G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2 = 0.5$, $\upsilon_{12} = 0.25$ and $K^2 = 5/6$)

| *a/h* | *a/R* | ANSYS (simply supported) | ANSYS (fixed-end supported) |
|---|---|---|---|
| 100 | 0.000 | 15.1848853 | 33.126135 |
| | 0.025 | 17.2247461 | 33.971616 |
| | 0.050 | 22.2396947 | 36.385301 |
| | 0.100 | 35.7649855 | 44.698512 |
| 50 | 0.000 | 15.0831433 | 32.224896 |
| | 0.025 | 15.6217762 | 32.446374 |
| | 0.050 | 17.1349999 | 33.101033 |
| | 0.100 | 22.1675867 | 35.593490 |
| 20 | 0.000 | 14.4766898 | 27.613495 |
| | 0.025 | 14.5682132 | 27.656769 |
| | 0.050 | 14.8377184 | 27.785879 |
| | 0.100 | 15.8694447 | 28.296011 |
| 10 | 0.000 | 12.8759190 | 20.288736 |
| | 0.025 | 12.9023542 | 20.304197 |
| | 0.050 | 12.9787274 | 20.350359 |
| | 0.100 | 13.2795994 | 20.533761 |
| 5 | 0.000 | 9.7176957 | 12.569716 |
| | 0.025 | 9.7274923 | 12.576113 |
| | 0.050 | 9.7527723 | 12.595262 |
| | 0.100 | 9.8531592 | 12.671524 |

Table 7 Natural frequency parameters ($\Omega = \omega a^2\sqrt{\rho/E_2h^2}$) for cross ply laminated [0º/90º/90º/0º] cylindrical shallow shells for ANSYS ($a/b = 1$, $E_1/E_2 = 50$, $G_{12}/E_2 = G_{13}/E_2 = G_{13}/E_2 = 0.5$, $\upsilon_{12} = 0.25$ and $K^2 = 5/6$)

| *a/h* | *a/R* | ANSYS (simply supported) | ANSYS (fixed-end supported) |
|---|---|---|---|
| 100 | 0.000 | 20.7615920 | 45.619966 |
| | 0.025 | 23.6739017 | 46.823099 |
| | 0.050 | 30.7856245 | 50.252560 |
| | 0.100 | 49.8238221 | 62.013315 |
| 50 | 0.000 | 20.5170113 | 43.298782 |
| | 0.025 | 21.2911836 | 43.624668 |
| | 0.050 | 23.4588662 | 44.586552 |
| | 0.100 | 30.6143513 | 48.231493 |
| 20 | 0.000 | 19.0690313 | 33.717305 |
| | 0.025 | 19.2048946 | 33.788658 |
| | 0.050 | 19.6045764 | 34.001205 |
| | 0.100 | 21.1261749 | 34.837356 |
| 10 | 0.000 | 15.8118649 | 22.677674 |
| | 0.025 | 15.8540723 | 22.705309 |
| | 0.050 | 15.9756740 | 22.787902 |
| | 0.100 | 16.4526620 | 23.114987 |
| 5 | 0.000 | 10.9302473 | 13.220198 |
| | 0.025 | 10.9468415 | 13.232260 |
| | 0.050 | 10.9908705 | 13.268337 |
| | 0.100 | 11.1651203 | 13.411531 |

meshing element named as SHELL99. The element has 100 layers to model the composite materials used in the structure. For each layer geometric and material properties is entered to program. Furthermore, thicknesses of each layer, fiber orientations and stacking sequence must be entered carefully. During solution process, subspace and block Lanczos mode extracting methods used separately to calculate first 30 frequencies.

The problem defined (Fig. 5) has been solved by ANSYS and Mathematica program. The results obtained by ANSYS and Mathematica, have been compared in tables and graphs. Non-dimensional natural frequency parameters have been preferred for comparison purposes. Those parameters have been obtained multiplying natural parameters by the term $a^2\sqrt{(\rho/E_2)h^2}$ in which $a$ is the shell planform dimension, $\rho$ is the shell unit mass, $E_2$ is the elasticity modulus of the shell in matrix direction and $h$ is the shell thickness. A simple dimension analysis of the parameter ($\Omega = \omega a^2\sqrt{\rho/E_2h^2}$) gives non-dimensional natural frequency parameters.

Tables 2, 3 and 4 give non-dimensional natural frequency parameters ($\Omega = \omega a^2\sqrt{\rho/E_2h^2}$) varying with shell thickness, shell curvature and shell anisotropy. The planform dimensions of the shell are equal to unity. For each case, three solutions have been carried out. Cylindrical shallow shells have been solved by Mathematica program with the shear deformation shallow shell theory (SDSST) and classical shallow shell theory (CLSST). The results obtained by using both theories are the same given by Qatu (2004) and ANSYS results are in good agreement with the other results.

The non-dimensional frequency parameters are compared for two different boundary conditions, namely simply supported and fixed end of all four edges in Tables 5, 6 and 7. Frequency parameters have been given for various $E_1/E_2$ ratios, $a/R$ ratios and $a/h$ ratios.

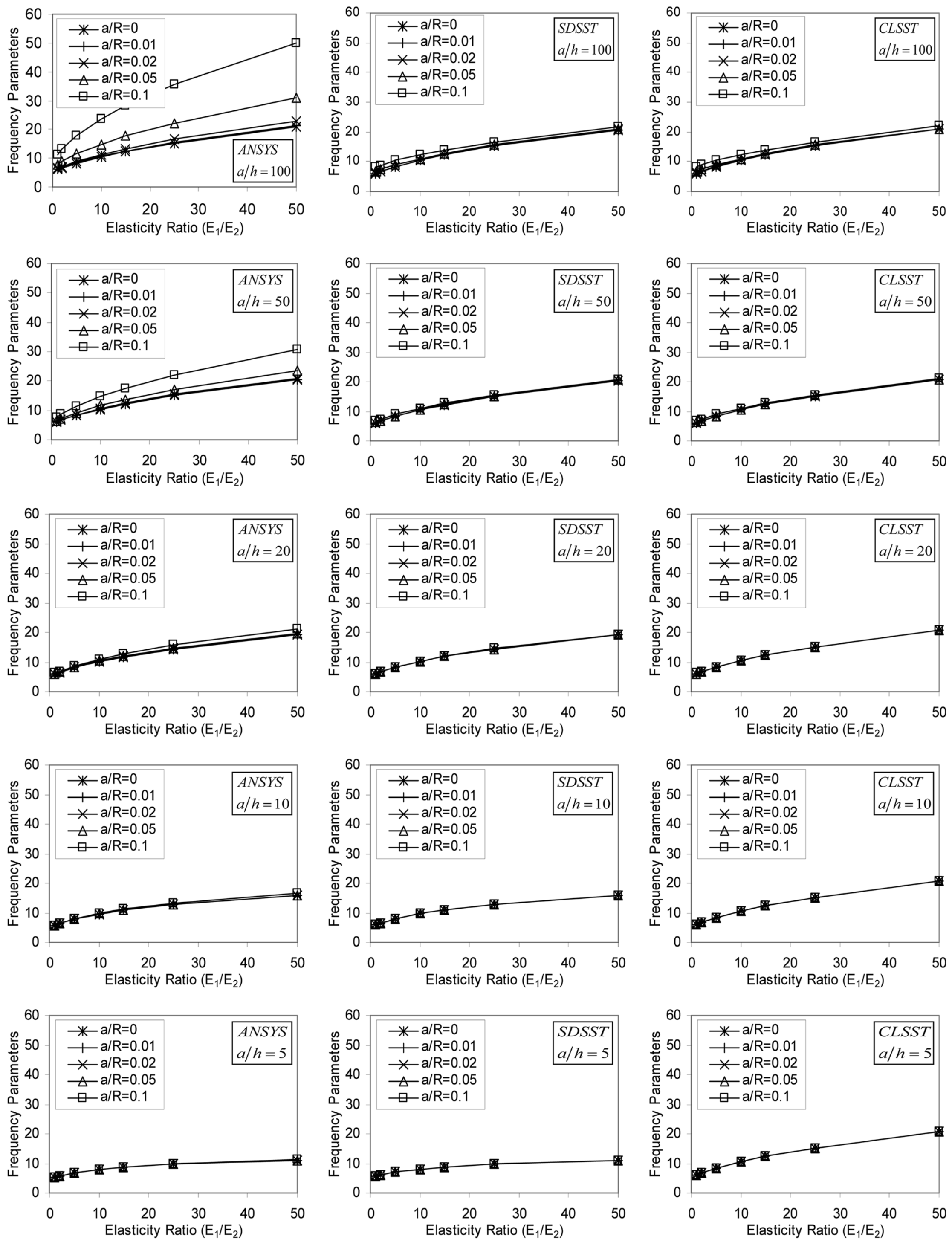


Fig. 6 Natural frequency parameters ($\Omega = \omega a^2\sqrt{\rho/E_2h^2}$) and elasticity ratio effect for cross ply laminated [0º/90º/90º/0º] cylindrical shallow shells for shear deformation shallow shell theory (SDSST), classical shallow shell theory (CLSST) and ANSYS

Fig. 6 shows variation of natural frequency parameters ($\Omega = \omega a^2\sqrt{\rho/E_2h^2}$) versus elasticity ratio effect, shell thickness ratio effect and shell curvature effect for cross ply symmetrically laminated [0º/90º/90º/0º] cylindrical shallow shells. The graphs have been drawn according to results obtained by three theory; shear deformation shallow shell theory (SDSST), classical shallow shell theory (CLSST) and finite element method.

## 4. Conclusions

In this study, free vibration characteristic of symmetrically laminated, cylindrical, composite shallow shells have been investigated by using three different theories. Effects of shell curvatures, shell thicknesses and material anisotropy have been shown with various graphs and tables for shallow shells which have square planform. The tables give non-dimensional natural frequencies versus shell thickness and shell curvatures using three different theories. Each table has been prepared for different material anisotropy value ($E_1/E_2$). Analysis and assumptions used in the SDSST and CLSST is similar to that is used in Qatu (2004). Finite element analysis has been performed by using commercial finite element program named ANSYS.

In the Tables 2, 3 and 4, the following results have been observed. The curvature of shallow shells has the increasing effects on the non-dimensional natural frequencies. As the curvature value increases the non-dimensional natural frequencies also increase. Furthermore, as the curvature value increases the non-dimensional natural frequencies obtained by the solutions of the three theories differ from each other. These differences are mainly caused by the different assumptions between the theories. Next, the thickness effect has been studied. The first important result gained from tables is that, as the thickness increases the results from CLSST (thin shell) differ from other two theories, as expected. The last observation for the thickness is about the range of change of the non-dimensional natural frequencies. The non-dimensional natural frequency for a shallow shell varies in a wide range for the thin shells and closer range for the thick shells. The last observation on tables gives the anisotropy effect. Different material anisotropy values have been considered for each table. A careful examination between tables shows the increase in anisotropy causes increase in the non-dimensional natural frequency values. In addition, this increments also cause the results of ANSYS differ from others.

The non dimensional frequency parameters of fixed-end boundary conditions are higher than simply supported boundary case in Tables 5, 6 and 7. The geometry of the shell also has effects on the fixed-end shallow shells. However, when the shallow shell fixed-end supported, the effects of the shell geometry do not distinct as the simply supported case. Like simply supported case, the effect of the shell curvature on non dimensional frequency parameter has reduced as the shell thickness increase.

When the graphs in Fig. 6 has been examined, It is concluded that assumption of lame parameters equals to unity for shallow shells in the analysis of CLSST and SDSST gets fail as the curvature ratio increase. But luckily, the effect of this assumption decreases as the shell gets thicker. In Fig. 6, frequency parameter versus elasticity ratio graphs have been drawn for different curvature ratios. The solutions obtained by different theories have been given on a separate graph. Graphs in each line of Fig. 6 have been drawn for the same thickness ratio. The differences in frequency parameters of ANSYS solution for the each case of curvature ratios have been represented better than other two theories. Considering shell thickness the results of SDSST for the thick shallow shell case have been coincided with ANSYS solutions but differ with CLSST solutions as expected.

In the literature shallow shells are defined as "shells that have rise of not more than one fifth the smallest planform dimension of the shell". In this study, to verify shallow shell definition, FEM solutions have been compared with CLSST and SDSST solutions, which assumes Lame parameters equal to one ($A = B = 1$), for different situations. As the thickness of the shallow shell increases, the results of FEM and other two theories get closer. Hence, definition of the shallow shell must be done considering the shell thickness. Elasticity ratio i.e., anisotropy, also affects the results. For isotropic case results of three theories agree with each other. However, as the anisotropy increases results of the theories gets differ.

As a conclusion, it could be said that, for shallow shells, no general definition could be done without considering effects of curvature ratio, thickness ratio and anisotropy.

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