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Stress field around axisymmetric partially supported cavities in elastic continuum-analytical solutions

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Abstract. The present paper will be concerned to the investigation of the stress-strain field around the cavity that is loaded or partially loaded at the inner surface by the rotationally symmetric loading. The cavity of the spherical, cylindrical or elliptical shape is situated in a stressed elastic continuum, subjected to the gravitation field. As the contribution to the similar investigations, the paper introduces the new function of loading in the form of the infinite sine series. Besides, in this paper the solution of stresses around an oblong ellipsoid cavity, has been obtained using appropriate curvilinear elliptical coordinates. This analytical approach avoids the solutions of the same problem that lead to expressions that contain rather complex integrations. Thus the presented solutions provide the applicable and explicit expressions for stresses and strains developed in infinite series with easily determinable coefficients by the use of contemporary mathematical packages. The numerical examples are also included to confirm the convergence of the obtained solutions.

Keywords: continuum; cavity; partially supported boundary; loading function.

1. Introduction

An investigation of the influence of heterogeneities (inclusion, cavities, cracks, ...) on the effective properties of the materials is of great interest in applied mathematics and computational mechanics and a vast amount of literature covers this subject. Cavities are good approximations for modeling voids existing in natural materials, such as geo-materials. Particular practical importance of such investigations is related to determination of stress-strain fields around unsupported or supported cavities in a solid rock mass created by excavations for underground structures. These investigations enable a better understanding of interaction between underground structure and rock medium in three-dimensional conditions. The practical consequences that may be derived from the evaluation of disturbances of stresses and strains in vicinity of the cavities formed by underground excavations in solid rocks are related to the essential requirements for the safety of the tunneling works, particularly in cases where three-dimensional geometry of the cavities is playing significant role in the stress-strain changes.

Determination of the stress-strain state around an elliptical cavity (oblong ellipsoid) situated in

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elastic continuum, with unsupported internal boundary, has been recently considered in the work by Lukić *et al*. (2009). Also, in this paper a brief historical review of investigation of the mechanical properties of solids containing spherical, cylindrical and elliptical cavities are given. The researches particularly emphasized are: Neuber (1937), Sternberg and Sadowski (1952), Eshelby (1957, 1959), Lur'e (1964), and the authors of recent studies, Tran-Cong (1997), Chen (2004), Xu *et al*. (1996), Duan *et al*. (2005), Chen *et al*. (2003), Dong *et al*. (2003), Sharma *et al*. (2003), Rahman (2002), Markenscoff (1998a, b), Riccardi and Montheillet (1999), Tsuchida *et al*. (2000), Chen and Lee (2002). Most recently, Ou *et al*. (2008, 2009) obtained the solutions for the elastic fields around a nanosized spheroidal cavity in an elastic medium subjected to arbitrary uniform remote loadings and a uniformly uniaxial tension.

In spite of the fact that stress analysis of an infinite elastic body that contains cavities is a classic topic, the most of the solutions of a general nature lead to rather complex and often unsolvable integrations, that can not provide the usable expressions for stresses and strains. The applicable solutions may be found in the literature for the cavities of spherical shape and for infinite cylinders, while the applicable solutions for ellipsoidal cavities are rather scarce. In this paper the solution of stresses around an oblong ellipsoid cavity that is loaded or partially loaded at the inner surface by the rotationally symmetric loading has been obtained using appropriate curvilinear elliptical coordinates. The derivation of expressions for stresses and strains have been made starting from the solutions of the basic Navier differential equations for the displacements and use of Neuber-Papkovich potentials that are harmonic scalar and vector functions. Applying the Bubnov-Galerkin's method, the formulation of the boundary conditions has been performed using functions satisfying bi-harmonic differential equation, and being developed in infinite series by Legendre's polynomials. This approach made possible to avoid the formulations for stresses and strains that contains cumbersome integrals that determine the displacements and than also to avoid equally or even more laborious procedures to differentiate these expressions in order to obtain the strains. Besides, in order to analyze the support around the opening, the problem of the stress-strain state is extended to the analysis of states in modified conditions, by setting the loading on the inner side of the cavity which simulates the support. The previously elaborated loading functions, developed in infinite series on the basis of Legendre's polynomials Lukić (1998), or defined by Fourier series (Jaeger and Cook 1969), have had the inherent shortcoming at the edges of loaded surface, that is avoided in this paper by application of the loading function based on the infinite serial developed by sine function.

2. Basic equations

We start by the Navier equations for the displacement field, Malvern (1969)

$$(1-2\nu)\nabla^2\mathbf{u}+\nabla(\nabla\cdot\mathbf{u})+\frac{1-2\nu}{G}\mathbf{K} = \mathbf{0} \tag{1}$$

where $\mathbf{K}$ is the constant body force; $\mathbf{u}$ is displacement vector; ν is Poisson's coefficient; G is the shear modulus. If the only body force is the gravitational force, the Laplacian of this body force potential is zero, i.e.

$$\mathbf{K} = \mathbf{0} \tag{2}$$

so the differential Eq. (1) becomes homogeneous

$$(1-2\nu)\nabla^2\mathbf{u}+\nabla(\nabla\cdot\mathbf{u}) = \mathbf{0} \tag{3}$$

Making use of Neuber-Papkovich potentials Φ_0 and $\boldsymbol{\Phi}$ Papkovich (1932), it has been shown that

$$\mathbf{u} = \boldsymbol{\Phi}-\frac{1}{4(1-\nu)}\nabla(\Phi_0+\mathbf{r}\cdot\boldsymbol{\Phi}) \tag{4}$$

where

$$\nabla^2\Phi_0 = 0\,, \quad \nabla^2\boldsymbol{\Phi} = \mathbf{0} \tag{5}$$

Obviously, Φ_0 and $\boldsymbol{\Phi}$ are harmonic scalar and vector functions, respectively. Usually we make use of the potential function ψ

$$\Psi = -\frac{1}{4(1-\nu)}(\Phi_0+\mathbf{r}\cdot\boldsymbol{\Phi}) \tag{6}$$

So defined function is biharmonic, i.e.

$$\nabla^2\nabla^2\Psi = 0 \tag{7}$$

Eq. (7) is basic for the solutions of the problem. Particularly, for the spherical and elliptical cavities they are derived in the form of infinite series of Legendre’s polynomials. The solutions for infinite cylinder have been derived in the form of infinite series of Bessel’s functions.

3. Boundary conditions

Boundary conditions required for determination unknown constants in expressions for stresses are given in terms of stresses acting on the cavity surface. The total stresses around the cavity are determined by superimposing primary stresses acting in the continuum without a cavity and “partial” stresses which are caused by the presence of cavity.

$$\begin{aligned}\sigma_i &= \sigma_i^* + \sigma_i^{pr}\\ \tau_{ij} &= \tau_{ij}^* + \tau_{ij}^{pr}\end{aligned} \tag{8}$$

where σ_i, τ_{ij} = the total stresses, σ_i^*, τ_{ij}^* = partial stresses due to the presence of a cavity and $\sigma_i^{pr}, \tau_{ij}^{pr}$ = primary stresses; (i, j = respective coordinate).

4. Spherical cavity

In the spherical coordinates (r, φ, θ), Fig. 1, for axisymmetric conditions towards z axis are

$$u_\theta = 0 \tag{9}$$

$$\frac{\partial u_r}{\partial\theta} = 0\,, \quad \frac{\partial u_\varphi}{\partial\theta} = 0 \tag{10}$$

Then the Eq. (3) is satisfied by

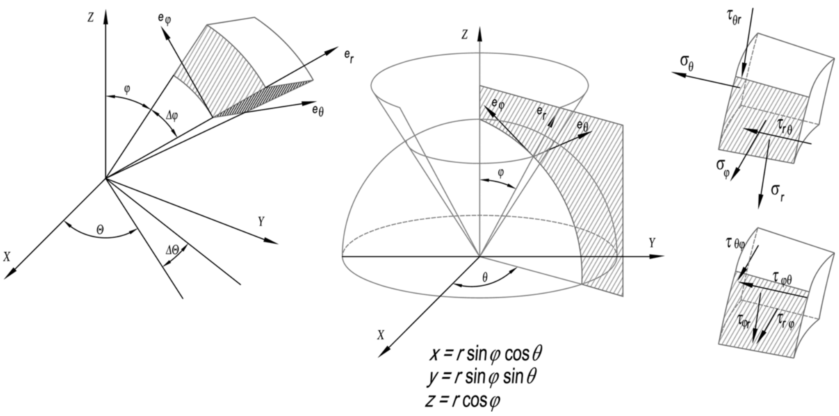

Fig. 1 Spherical coordinate system

$$u_r = -\sum_{n=0}^{\infty}[n(n+3-4\nu)c_n r^{-n} - (n+1)d_n r^{-n-2}]P_n(\cos\varphi) \tag{11}$$

$$u_\varphi = -\sum_{n=0}^{\infty}[(4-n-4\nu)c_n r^{-n} + d_n r^{-n-2}]\frac{dP_n(\cos\varphi)}{d\varphi} \tag{12}$$

where

$$P_n(\mu) = \frac{1}{2^n n!}\frac{d^n}{d\mu^n}(\mu^2-1)^n \tag{13}$$

are Legendre's polynomials ($\mu = \cos\varphi$), and

$$\frac{dP_n}{d\mu} = \frac{n}{\mu^2-1}(\mu P_n - P_{n-1}) \tag{14}$$

Constants c_n and d_n have to be determined from the boundary conditions.

Eqs. (11), (12) and (7) provide the expressions for stresses

$$\sigma_r^* = \frac{E}{1+\nu}\sum_{n=0}^{\infty}\left[n(n^2+3n-2\nu)c_n r^{-n-1} - (n+1)(n+2)d_n r^{-n-3}\right]P_n(\mu)$$

$$\sigma_\varphi^* = \frac{E}{1+\nu}\left\{-\sum_{n=0}^{\infty}\left[n(n^2-2n-1+2\nu)c_n r^{-n-1} - (n+1)^2 d_n r^{-n-3}\right]P_n(\mu) + \sum_{n=0}^{\infty}\left[(4-n-4\nu)c_n r^{-n-1} + d_n r^{-n-3}\right]\operatorname{ctg}\varphi\frac{dP_n(\mu)}{d\varphi}\right\}$$

$$\sigma_\theta^* = \frac{E}{1+\nu}\left\{\sum_{n=0}^{\infty}\left[-n(n+3-4n\nu-2\nu)c_n r^{-n-1} + (n+1)\, d_n r^{-n-3}\right]P_n(\mu) - \sum_{n=0}^{\infty}\left[(4-n-4\nu)c_n r^{-n-1} + d_n r^{-n-3}\right]\operatorname{ctg}\varphi\frac{dP_n(\mu)}{\varphi}\right\}$$

$$\tau^*_{r\varphi} = \frac{E}{1+\nu}\sum_{n=1}^{\infty} -\left[(n^2-2+2\nu)c_n r^{-n-1} + (n+2)d_n r^{-n-3}\right]\frac{dP_n(\mu)}{d\varphi}$$

$$\tau^*_{r\theta} = \tau^*_{\varphi\theta} = 0 \tag{15}$$

Expressions (15) represent the "partial" stresses caused by the presence of a cavity, in the stressed elastic continuum. They are given as infinite series based on Legendre's polynomials $P_n(\cos\varphi)$ and constants c_n and d_n.

Unknown constants c_n and d_n, as mentioned before, are to be determined for partially loaded (supported) cavity surface, from the following boundary conditions

$$\left.\begin{aligned}\sigma_r &= -p(\varphi)\\ \tau_{r\varphi} &= 0\end{aligned}\right\} \quad \text{for} \quad r = R \tag{16}$$

where $p(\varphi)$ is support loading and R is sphere radius. Taking into account Eq. (8) one obtains

$$\left.\begin{aligned}\sigma^*_r &= -\sigma^{pr}_r - p(\varphi)\\ \tau^*_{r\varphi} &= -\tau^{pr}_{r\varphi}\end{aligned}\right\} \quad \text{for} \quad r = R \tag{17}$$

The primary stresses in continuum for "hydrostatic" stress field are defined

$$\sigma^{pr}_r = \sigma^{pr}_\varphi = \sigma^{pr}_\theta = \gamma H$$

$$\tau^{pr}_{r\varphi} = \tau^{pr}_{r\theta} = \tau^{pr}_{\varphi\theta} = 0 \tag{18}$$

with γ is the unit weight of the continuum and H is the height above cavity axis.

The loading function has been defined as infinite sine series

$$p(\varphi) = p\sum_{j=0}^{\infty}\frac{4}{(2j+1)}\sin\frac{(2j+1)\pi}{2\beta}(\beta-\varphi)\,; \quad -\beta \le \varphi \le \beta; \quad (\pi-\beta \le \varphi \le \pi+\beta)$$

$$p(\varphi) = 0\,; \text{ on the unloaded part of the cavity} \tag{19}$$

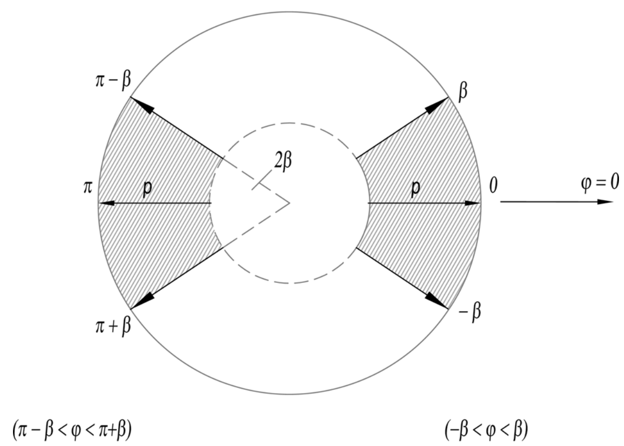

Fig. 2 Partially loaded surface in axial symmetry

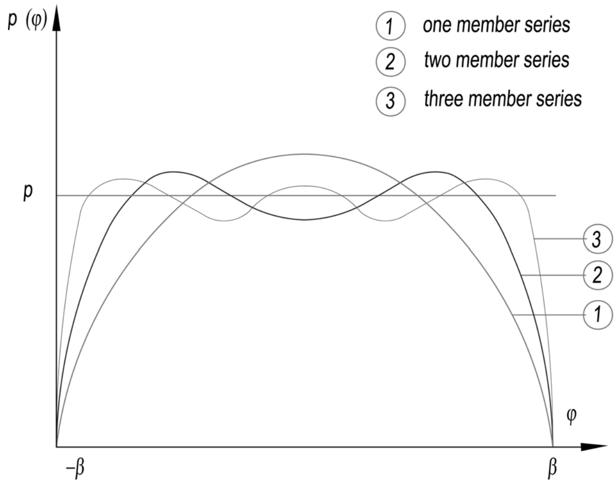

Fig. 3 Graphical presentation of loading function

where β is the given angle that defines loaded surface, Fig. 2 and Fig. 3. The main advantages of this type of loading function are:

- the absence of the discontinuous transition to zero value at the ends of loaded area
- the fast convergence to the central value of the loading within loaded area, and
- the fast convergence to the zero value at the ends of the loaded area trough sine functions.

Expanding of σ_r^{pr}, $\tau_{r\varphi}^{pr}$ and $p(\varphi)$ in infinite series based on Legendre's polynomials, by the use of the general formula

$$f(\mu) = \sum_{n=0}^{\infty} \frac{1}{2}(2n+1)P_n(\mu) \int_{-1}^{1} f(t)P_n(t)\,dt \tag{20}$$

one can obtain

$$\sigma_r^{pr} + p(\varphi) = \sum_{n=0}^{\infty} A_n P_n(\cos\varphi)$$

$$\tau_{r\varphi}^{pr} = \sum_{n=0}^{\infty} B_n \frac{dP_n(\cos\varphi)}{d\varphi} \tag{21}$$

Taking into account Eqs. (18)-(21), from Eq. (15) we find that

$$\frac{E}{1+\nu}\sum_{n=0}^{\infty} [n(n^2+3n-2\nu)c_n R^{-n-1} - (n+1)(n+2)d_n R^{-n-3}]P_n(\cos\varphi) = -\sum_{n=0}^{\infty} A_n P_n(\cos\varphi) \tag{22}$$

$$\frac{E}{1+\nu}\sum_{n=1}^{\infty} -[(n^2-2-2\nu)c_n R^{-n-1} - (n+2)d_n R^{-n-3}]\frac{dP_n(\cos\varphi)}{d\varphi} = -\sum_{n=0}^{\infty} B_n \frac{dP_n(\cos\varphi)}{d\varphi} \tag{23}$$

where

$$A_n = \frac{2n+1}{2}\int_0^{\pi}(\sigma_r^{pr} + p(\varphi))P_n(\cos\varphi)\sin\varphi d\varphi \quad (r = R)$$

$$B_n = \frac{(2n+1)(n-1)!}{2\pi(n-1)!}\int_0^{\pi}\tau_{r\varphi}^{pr}\frac{dP_n(\cos\varphi)}{d\varphi}\sin\varphi d\varphi \tag{24}$$

are determined by development in sine series, Appendix A.

The Eqs. (22) and (23) contains infinite sums on the both sides of the same row, therefore they provide the possibility to form the set of linear equations for determination of unknown constants c_n and d_n.

5. Cylindrical cavity

In deriving the stresses around cylindrical cavity one may refer to potential Ψ defined by Eq. (6) which provides the following relationships for "partial" stresses

$$\sigma_r^* = \frac{\partial}{\partial z}\left(\nu\nabla^2\Psi - \frac{\partial^2\Psi}{\partial r^2}\right)$$

$$\sigma_z^* = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\Psi - \frac{\partial^2\Psi}{\partial z^2}\right]$$

$$\sigma_\varphi^* = \frac{\partial}{\partial z}\left(\nu\nabla^2\Psi - \frac{1}{r}\frac{\partial\Psi}{\partial r}\right)$$

$$\tau_{rz}^* = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\Psi - \frac{\partial^2\Psi}{\partial z^2}\right] \tag{25}$$

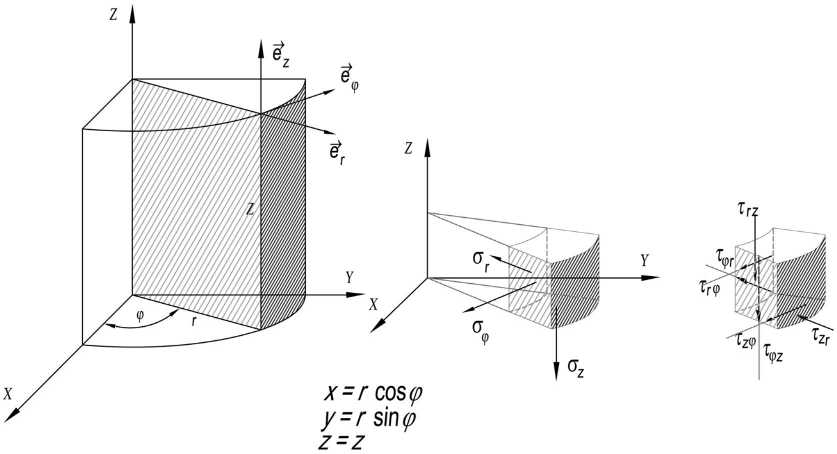

Fig. 4 Cylindrical coordinate system

In cylindrical coordinates (r, φ, z), Fig. 4, operator ∇^2 is defined by

$$\nabla^2 = \frac{\partial}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{\partial z^2} \tag{26}$$

The potential function ψ has been supposed as infinite sine series

$$\Psi = \sum_{k=0}^{\infty} f(kr)\sin kz \tag{27}$$

where $f(kr)$ is unknown function. Substituting this in Eq. (7) we obtain

$$\left(\frac{d^2}{dr^2} + \frac{d}{rdr} - k^2\right)\left(\frac{d^2 f(kr)}{dr^2} + \frac{df(kr)}{rdr} - k^2 f(kr)\right) = 0 \tag{28}$$

Its solution may be written as

$$f(kr) = B_1 K_0(kr) + B_2 r K_1(kr) \tag{29}$$

So, general solution of Eq. (7) has the following form

$$\Psi(r,z) = \sum_{k=0}^{\infty}[B_1 K_0(kr) + B_2 r K_1(kr)]\sin kz \tag{30}$$

where B_1 and B_2 are constants and $K_0(kr)$ and $K_1(kr)$ are the Bessel`s functions of zero and first order.

The expression for $\Psi(r, z)$ given in Eq. (30) introduced in Eq. (25) is providing the solution for the "partial" stress state caused by the presence of cylindrical cavity in elastic continuum subjected to constant stresses in vicinity of the cavity

$$\begin{gathered}
\sigma_r^* = \sum_{k=0}^{\infty} k^2 \cos kz \left\{ K_0(kr)\left[B_2(1-2\nu) - kB_1\right] - K_1(kr)\left(krB_2 + \frac{B_1}{r}\right)\right\} \\
\sigma_\varphi^* = \sum_{k=0}^{\infty} k^2 \cos kz \left[B_2(1-2\nu)K_0(kr) + \frac{B_1}{r}K_1(kr)\right] \\
\sigma_z^* = \sum_{k=0}^{\infty} k^2 \cos kz \{K_0(kr)\left[kB_1 - 2(2-\nu)B_2\right] + B_2 kr K_1(kr)\} \\
\tau_{rz}^* = \sum_{k=0}^{\infty} k^2 \sin kz \{-B_2 kr K_0(kr) - \left[kB_1 - 2(1-\nu)B_2\right] + K_1(kr)\}
\end{gathered} \tag{31}$$

The primary stresses in continuum for "hydrostatic" stress field are defined by

$$\begin{gathered}
\sigma_r^{pr} = \sigma_\varphi^{pr} = \sigma_z^{pr} = \gamma H \\
\tau_{rz}^{pr} = 0
\end{gathered} \tag{32}$$

with γ is the unit weight of the continuum and H is the height above cavity axis.

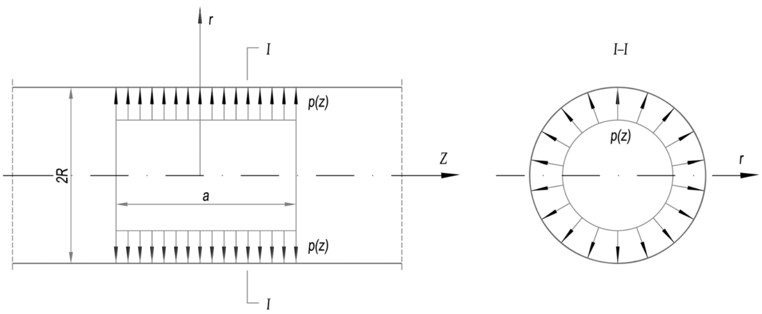

Fig. 5 Axisymmetrical partial loading

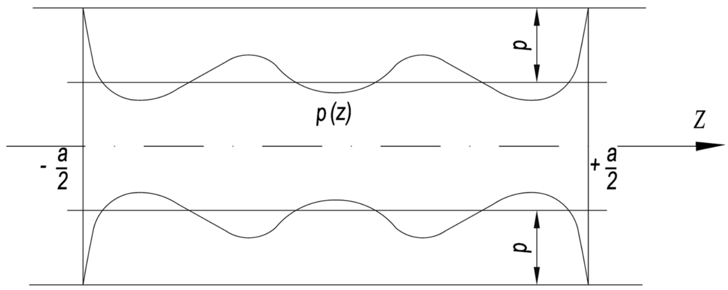

Fig. 6 Graphical view of the loading function

The loading is expressed in the form of infinite sine series

$$p(z) = p\sum_{n=0}^{\infty}\frac{1}{(2n+1)\pi}\sin\frac{(2n+1)\pi}{a}(a-z)\,;\quad 0\le z\le a$$

$$p(z) = 0\,;\quad z\le 0\wedge z\ge a \tag{33}$$

The partially loaded surface of a cylinder by constant loading p within the area defined $-a/2 \le z \le +a/2$ and the shape of loading function are shown on Fig. 5 and Fig. 6.

The unknown constants B_1 and B_2 of Eq. (31) are to be resolved from the boundary conditions

$$\left.\begin{aligned}\sigma_r^* &= -\sigma_r^{pr} - p\left(\frac{z}{R}\right)\\ \tau_{rz}^* &= -\tau_{rz}^{pr}\end{aligned}\right\}\quad \text{for}\quad r = R \tag{34}$$

in the form

$$B_1 = B_2\left[\frac{2(1-\nu)}{k} - r\frac{K_0(kr)}{K_1(kr)}\right]$$

$$B_2 = -\frac{\sigma_r^{pr} + p\sum\limits_{n=0}^{\infty}\frac{4}{(2n+1)\pi}\sin\frac{(2n+1)\pi}{a}(a-z)}{\sum\limits_{k=0}^{\infty}k^2\cos kz\left\{kR\frac{K_0^2(kr)}{K_1(kr)} - K_1(kr)\left[\frac{2}{kR}(1-\nu)+kR\right]\right\}} \tag{35}$$

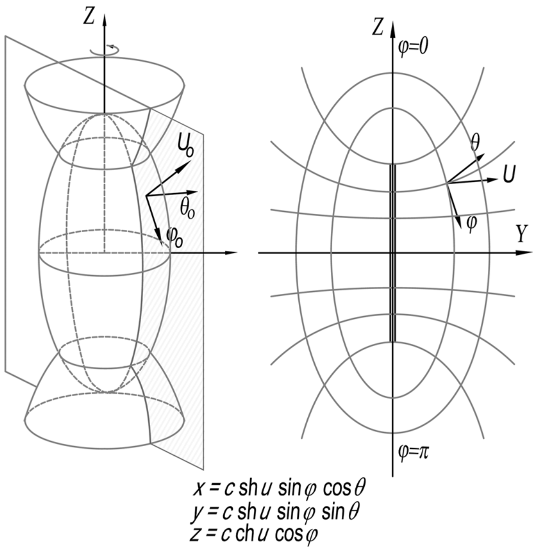

Fig. 7 Oblong ellipsoidal coordinate system

6. Elliptical cavity (oblong ellipsoid)

Introducing the oblong ellipsoidal coordinates, Fig. 7

$$x = c\,\mathrm{sh}u\sin\varphi\cos\varphi\cos\theta;\quad y = c\,\mathrm{sh}u\sin\varphi\sin\theta;\quad z = c\,\mathrm{ch}u\cos\varphi \tag{36}$$

the stress state around elliptic cavity having the shape of rotational oblong ellipsoid can be defined on the basis of solution derived from Eq. (7), where general potential $\Psi = \Phi_0 + x\Phi_1 + y\Phi_2 + z\Phi_3$ (Papkovich-Neuber) in the case of axial symmetry is reduced to

$$\Psi = \Phi_0(u,\varphi) + c\,\mathrm{ch}u\cos\varphi\Phi_3(u,\varphi) \tag{37}$$

where c is focus distance and Φ_0 and Φ_3 are harmonic functions

The stresses for axisymmetrical loading, in oblong elliptical coordinates, expressed by a potential, are expressed in the form

$$\sigma_u^* = -\frac{1}{h^2}\frac{\partial^2\Psi}{\partial u^2} - \frac{c^2}{h^4}\left(\frac{\partial\Psi}{\partial\varphi}\sin\varphi\cos\varphi - \frac{\partial\Psi}{\partial z}\mathrm{ch}\,u\,\mathrm{sh}\,u\right) + \frac{2\alpha}{h^2}\frac{\partial\Phi_3}{\partial u}c\,\mathrm{sh}\,u\cos\varphi + \frac{2c}{\nu h^2}\left(\frac{\partial\Phi_3}{\partial u}\mathrm{ch}\,u\cos\varphi - \frac{\partial\Phi_3}{\partial\varphi}\mathrm{ch}\,u\sin\varphi\right)$$

$$\sigma_\varphi^* = -\frac{1}{h^2}\frac{\partial^2\Psi}{\partial\varphi^2} - \frac{c^2}{h^4}\left(\frac{\partial\Psi}{\partial u}\mathrm{sh}\,u\,\mathrm{ch}\,u - \frac{\partial\Psi}{\partial\varphi}\sin\varphi\cos\varphi\right) - \frac{2\alpha}{h^2}\frac{\partial\Phi_3}{\partial\varphi}c\,\mathrm{ch}\,u\sin\varphi + \frac{2c}{\nu h^2}\left(\frac{\partial\Phi_3}{\partial u}\mathrm{ch}\,u\cos\varphi - \frac{\partial\Phi_3}{\partial\varphi}\mathrm{ch}\,u\sin\varphi\right)$$

$$\sigma_\theta^* = -\frac{1}{h^2}\left(\frac{\partial\Psi}{\partial\varphi}\frac{\cos\varphi}{\sin\varphi} - \frac{\partial\Psi}{\partial u}\frac{\mathrm{ch}\,u}{\mathrm{sh}\,u}\right) + \frac{2c}{\nu h^2}\left(\frac{\partial\Phi_3}{\partial u}\mathrm{ch}\,u\cos\varphi - \frac{\partial\Phi_3}{\partial\varphi}\mathrm{ch}\,u\sin\varphi\right)$$

$$\tau_{u\varphi}^* = -\frac{1}{h^2}\frac{\partial^2\Psi}{\partial u\partial\varphi} + \frac{c^2}{h^4}\left(\frac{\partial\Psi}{\partial\varphi}\mathrm{ch}\,u\,\mathrm{sh}\,u + \frac{\partial\Psi}{\partial u}\sin\varphi\cos\varphi\right) + \frac{\alpha}{h^2}\left(-\frac{\partial\Phi_3}{\partial u}c\,\mathrm{ch}\,u\sin\varphi + \frac{\partial\Phi_3}{\partial\varphi}c\,\mathrm{sh}\,u\cos\varphi\right)$$

$$\tau_{u\vartheta}^* = \tau_{\varphi\vartheta}^* = 0 \tag{38}$$

where

$$h^2 = c^2(\mathrm{ch}^2 u - \cos^2\varphi)$$
$$\alpha = 2\left(1-\frac{1}{\nu}\right) \tag{39}$$

The general solution of Eq. (7) is postulated in the form of infinite series based on associated Legendre's functions of the first and second order

$$\Psi = \sum_{n=0}^{\infty}\sum_{m=0}^{n}[f_{nm}P_n^m(\cos\varphi)Q_n^m(\mathrm{ch}u)] \tag{40}$$

where f_{nm} = unknown coefficients to be defined across the stress boundary conditions.

Using Eqs. (37) and (40) the potential may be expressed as

$$\Psi = \sum_{n=0}^{\infty}\sum_{m=0}^{n} f_{nm}P_n^m(\cos\varphi)Q_n^m(\mathrm{ch}u) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_{nm}P_n^m(\cos\varphi)Q_n^m(\mathrm{ch}u) +$$
$$c\,\mathrm{ch}u\cos\varphi\sum_{n=-1}^{\infty}\sum_{m=0}^{n} C_{nm}P_{n+1}^m(\cos\varphi)Q_{n+1}^m(\mathrm{ch}u) \tag{41}$$

where $\Phi_0(u,\ \varphi)$ and $\Phi_3(u,\ \varphi)$ are harmonics given as series

$$\Phi_{0(u,\,\varphi)} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_{nm}P_n^m(\cos\varphi)Q_n^m(\mathrm{ch}u)$$
$$\Phi_{3(u,\,\varphi)} = \sum_{n=-1}^{\infty}\sum_{m=0}^{n} C_{nm}P_{n+1}^m(\cos\varphi)Q_{n+1}^m(\mathrm{ch}u) \tag{42}$$

For the case of $m = 0$ the solution in Eq. (40) is simpler

$$\Psi = \sum_{n=0}^{\infty}[f_nP_n(\cos\varphi)Q_n(\mathrm{ch}u)] \tag{43}$$

By defined potential in Eq. (37) one may obtain

$$\Psi = \sum_{n=0}^{\infty}A_nP_nQ_n + c\,\mathrm{ch}u\cos\varphi\sum_{n=-1}^{\infty}C_nP_{n+1}Q_{n+1} \tag{44}$$

where the harmonics $\Phi_0(u,\ \varphi)$ and $\Phi_3(u,\ \varphi)$ are given as

$$\Phi_0 = \sum_{n=0}^{\infty}A_nP_nQ_n;\quad \Phi_3 = \sum_{n=-1}^{\infty}C_nP_{n+1}Q_{n+1} \tag{45}$$

with use of $P_n = P_n(\cos\varphi)$ and $Q_n = Q_n(\mathrm{ch}u)$.

The solutions of Eqs. (41) and (44) introduced in Eq. (38) are providing the full tensor of "partial" stresses caused by the presence of the elliptical cavity in the stressed elastic continuum.

Due to the space consuming expressions the stress tensor will be presented here only for the case of $m = 0$, i.e., by the single infinite series

$$\sigma_u^* = \sum_{n=-1}^{\infty} (U_n^1 A_n P_n Q_n + U_n^2 A_n P_n Q_{n+1} + U_n^3 A_n P_{n+1} Q_n + U_n^4 C_n P_{n+1} Q_n + U_n^5 C_n P_n Q_{n+1} + U_n^6 C_n P_{n+1} Q_{n+1})$$

$$\sigma_\varphi^* = \sum_{n=-1}^{\infty} (\varphi_n^1 A_n P_n Q_n + \varphi_n^2 A_n P_{n+1} Q_n + \varphi_n^3 A_n P_n Q_{n+1} + \varphi_n^4 C_n P_n Q_{n+1} + \varphi_n^5 C_n P_{n+1} Q_n + \varphi_n^6 C_n P_{n+1} Q_{n+1})$$

$$\sigma_\theta^* = \sum_{n=-1}^{\infty} (\Theta_n^1 A_n P_n Q_n + \Theta_n^2 A_n P_{n+1} Q_n + \Theta_n^3 A_n P_n Q_{n+1} + \Theta_n^4 C_n P_n Q_{n+1} + \Theta_n^5 C_n P_{n+1} Q_n + \Theta_n^6 C_n P_{n+1} Q_{n+1})$$

$$\tau_{u\varphi}^* = \sum_{n=-1}^{\infty} (T_n^1 A_n P_n Q_n + T_n^2 A_n P_n Q_{n+1} + T_n^3 A_n P_{n+1} Q_n + T_n^4 C_n P_{n+1} Q_{n+1} + T_n^5 C_n P_n Q_n + T_n^6 C_n P_n Q_{n+1} + T_n^7 C_n P_{n+1} Q_n + T_n^8 C_n P_{n+1} Q_{n+1}) \tag{46}$$

Coefficients in Eq. (46) $U_n^i, \varphi_n^i, \Theta_n^i$ (i = 1, 2 ..6) and T_n^i (i = 1, 2 …, 8) are given in the Appendix B. The partial stresses in Eq. (46) vanishes "in infinity". It may be said that for practical problems this is achieved at the distances equal to double size of the cavity.

The boundary conditions are given in terms of stresses

$$\left.\begin{aligned} \sigma_u^* &= -\sigma_u^{pr} - p_k(\varphi) \\ \tau_{u\varphi}^* &= -\tau_{u\varphi}^{pr} \end{aligned}\right\} \quad \text{for} \quad u = u_0 \quad (k = 1, 2) \tag{47}$$

The primary stresses in continuum are defined for "hydrostatic" stress field (in oblong ellipsoidal coordinates)

$$\begin{aligned} \sigma_u^{pr} &= \sigma_\varphi^{pr} = \sigma_\theta^{pr} = \gamma H \\ \tau_{u\varphi}^{pr} &= \tau_{u\theta}^{pr} = \tau_{\varphi\theta}^{pr} = 0 \end{aligned} \tag{48}$$

with γ is the unit weight of the continuum and H is the height above cavity axis.

The same problem that appeared in the cases of spherical and cylindrical cavity, the defining of supporting loading function is also subject of determination for ellipsoidal cavity. The selection of the functiuon is associated with conditions imposed by the solution of differential equation $\nabla^2\nabla^2\Psi = 0$, namely the continuity of the loading function on the boundary. The supporting loading which is the most frequent case of partially loaded cavity surface is shown on the Fig. 8 and Fig. 3

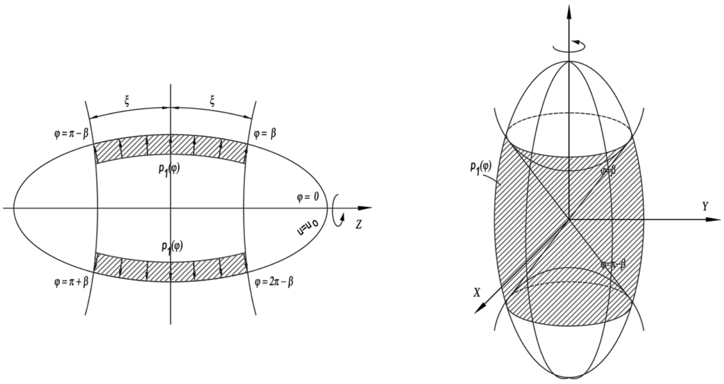

Fig. 8 Support loading on the boundary $p_1(\varphi)$

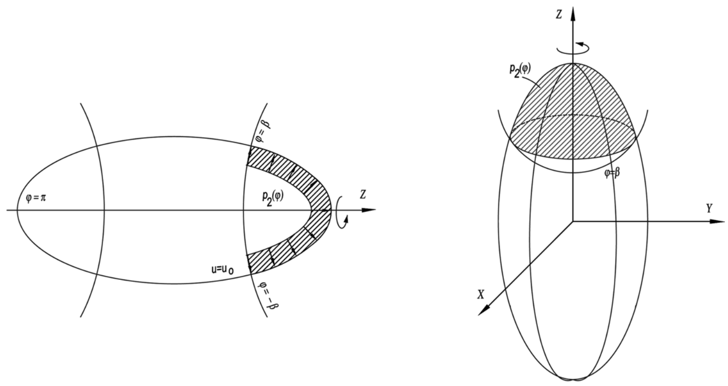

Fig. 9 Support loading on the boundary $p_2(\varphi)$

and its definition is

$$p_1(\varphi) = p\sum_{j=0}^{\infty}\frac{4}{(2j+1)\pi}\sin\frac{(2j+1)\pi}{\pi-2\beta}(\varphi-\beta)\,;\quad \beta\le\varphi\le\pi-\beta;\quad (\pi+\beta\le\varphi\le 2\pi-\beta)$$

$$p_1(\varphi) = 0\,;\ \text{outside the loaded area} \tag{49}$$

and case of partially loaded cavity surface is shown on the Fig. 9 and Fig. 3

$$p_2(\varphi) = p\sum_{j=0}^{\infty}\frac{4}{(2j+1)\pi}\sin\frac{(2j+1)\pi}{2\beta}(\varphi+\beta)\,;\quad -\beta\le\varphi\le\beta$$

$$p_2(\varphi) = 0\,;\ \text{outside the loaded area} \tag{50}$$

Expanding of σ_r^{pr}, $\tau_{r\varphi}^{pr}$ and $p_k(\varphi)$ in infinite series based on Legendre's polynomials, by the use of the general formula Eq. (20) and substituting (48)-(50) into Eq. (46), we obtain the set of equations (with m = 0) for unknown constants

$$\sum_{n=-1}^{\infty}\left[A_n\,(a_{n-2}^1P_{n-2}+a_n^2P_n+a_{n+2}^3P_{n+2})+C_n(b_{n-4}^1P_{n-4}+b_{n-2}^2P_{n-2}+b_n^3P_n+b_{n+2}^4P_{n+2}+b_{n+4}^5P_{n+4})\right]$$

$$=\sum_{n=0}^{\infty}\left[k_n(q_{n-4}^1P_{n-4}+q_{n-2}^2P_{n-2}+q_n^3P_n+q_{n+2}^4P_{n+2}+q_{n+4}^5P_{n+4})\right] \tag{51}$$

and

$$\sum_{n=-1}^{\infty}\left[A_n(a_{n-3}^4P_{n-3}+a_{n-1}^5P_{n-1}+a_{n+1}^6P_{n+1}+a_{n+3}^7P_{n+3})+C_n(b_{n-3}^6P_{n-3}+b_{n-1}^7P_{n-1}+b_{n+1}^8P_{n+1}+b_{n+3}^9P_{n+3}+b_{n+5}^{10}P_{n+5})\right]$$

$$=\sum_{n=-1}^{\infty}\left[l_n\left(s_{n-5}^1P_{n-5}+s_{n-3}^2P_{n-3}+s_{n-1}^3P_{n-1}+s_{n+1}^4P_{n+1}+s_{n+3}^5P_{n+3}+s_{n+5}^6P_{n+5}\right)\right] \tag{52}$$

On the basis of derived expressions (51) and (52) it may be concluded that they are analogous to the expressions (23) and (24) given in the paper (Lukić *et al*. 2009) for unsupported cavity, but with the difference that for the elliptic cavity the coefficients k_n are to be obtained on the basis of development of primary stresses σ_u^{pr} and supporting loading $p(\varphi)$, by Legendre's polynomials (Appendix A). The remaining coefficients are given in Appendix C of the paper.

7. Numerical examples

The numerical implementation of the analytical solutions described previously has been elaborated for the example of elliptical cavity, together with results for spherical and cylindrical cavities of similar size, situated in the same continuum under axisymmetric primary stresses. The stresses have been computed for three cited cavities on the basis of the following data:

- Continuum properties: $\gamma = 28$ kN/m^3 ; $\nu = 0.3$; $E = 20 \times 10^6$ kN/m^2
- Geometry: $H = 100$ m
  1. spherical cavity $r_o = 2$ m; $\Delta r = 0.2$ m
  2. cylindrical cavity $r_o = 2$ m; $\Delta r = 0.2$ m
  3. oblong ellipsoidal cavity $u_o = 1.44$ m; $\Delta u = 0.1$ m; $c = 3$ m

where Δr = the applied increment of radial coordinate for estimation of stress field at consecutive radial distances and Δu = the applied increment of u-coordinate for estimation of stress field at consecutive oblong ellipsoidal surfaces.

- The loading on the inner side of the cavity
  1. spherical cavity $p = 200$ kN/m^2 ; $\beta = \pi/4$
  2. cylindrical cavity $p = 200$ kN/m^2 ; $a = 2$ m
  3. oblong ellipsoidal cavity $p = 200$ kN/m^2 ; $\beta = \pi/4$

For presentation and numerical interpretation of the obtained analytical results, particularly for the elongated oblong rotational ellipsoid, the numerical case that has been selected provides the possibility to derive many conclusions on the basis of the presented diagrams of the obtained results. First of all the obtained results are showing the inpact of the geometry of the elliptic cavity on the values of stresses, then the significant result is obtained by the impact of the internal loading on the state of stresses around the cavity. Morever, possibly the most important result is that the obtained values shown on diagrams clearly confirm that the stresses obtained for an elongated ellipsoidal cavity are bounded by the stress state for spherical cavity on one side, and by the stress state for infinitive cylindrical cavity on the other side. This has been presented within the frames of numerical case by the diagrams on Fig. 10 and Fig. 11. It also shown that the influence of the cavity to the initial stress state is vanishing at the distance of appr. four diameters of the cavity i.e., the stress state become equal to the initial one, that is the well known feature confirmed by other means, particularly within the frame of rock mechanics.

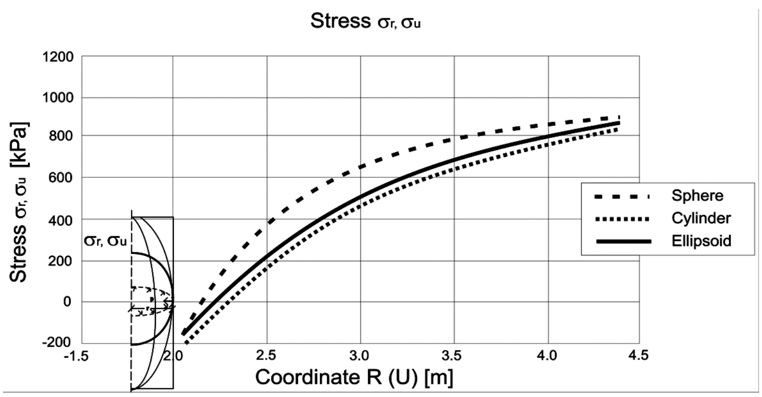

Fig. 10 Radial stresses for considered cavities

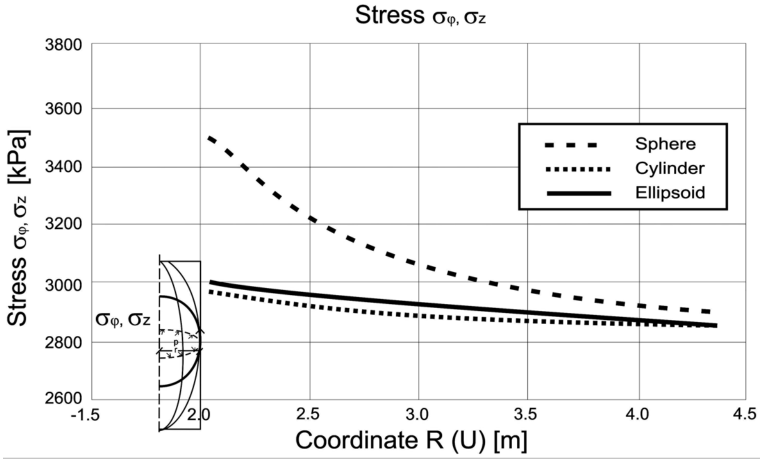

Fig. 11 Tangential stresses for considered cavities

8. Conclusions

This paper represents the study of the stress-strain states around unsupported and partially supported cavities in the stressed continuum. The similarity of the resolution of the problems for the cavities of different shapes has been elaborated, and importance of application of adequate coordinates has been confirmed, as the aid for simplifying the mathematical operations.

The presently available studies of stress concentration in rock mass were mainly concerned with analytical solutions for spherical and infinite cylinder cavity. The extension of these solutions to a more general shape (of an oblong ellipsoid), that is most common shape of underground rock excavations, has been considered in this paper. The solutions for all cavity shapes are also presented for the purpose of comparison of their numerical outcomes.

Besides, it has been demonstrated that the introduced single parametric loading function in the form of infinitive sine series, may be very useful for further studies of stress states around cavities. Some advantages in the use of such a function have been noted.

Acknowledgements

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Notations

A_n, B_n	: coefficients
A_{nm}, C_{nm}	: coefficients ($m = 0 \rightarrow A_n$, C_n)
B_1, B_2	: constants
c	: focus distance
c_n, d_n	: constants
$\mathbf{\Phi}$	: harmonic vector function
Φ_0	: harmonic scalar function
f_{nm}	: unknown coefficients
G	: shear modulus
H	: height above cavity axis (m)
$\mathbf{K}$	: constant force
$K_0(kr)$	: Bessel's function of zero order
$K_1(kr)$	: Bessel's function of first order
p	: constant loading
$p(\varphi)$	: support loading function
$p_k(\varphi)$	: support loading function (k = 1,2)
$P_n(\cos\varphi)$	: Legendre's polynomials of first order
R	: sphere radius (m)
r, φ, θ	: spherical coordinates
r, φ, z	: cylindrical coordinates
$\mathbf{u}$	: displacement vector
u, φ, θ	: oblong ellipsoidal coordinates
$Q_n(chu)$	: Legendre's polynomials of second order
β	: given angle that defines loaded surface
γ	: unit weight of the continuum
Δr	: increment of radial coordinate
Δu	: applied increment of u-coordinate
σ_i, $\tau_{i,j}$	: total stresses
σ_i^*, $\tau_{i,j}^*$	: partial stresses due to the presence of a cavity
σ_i^{pr}, $\tau_{i,j}^{pr}$	: primary stresses
ν	: Poisson's coefficient
Ψ	: potential function

Appendix A

The development of a function in the trigonometric series and Legendre's polynomials is defined by expression

$$f = \sum_{n=0}^{\infty} \sum_{m=0}^{n} H_{nm} P_n^m(\cos\varphi)\cos m\theta$$

where the coefficients H_{nm} are defined by expression

$$H_{nm} = \frac{(n-m)!(2n+1)!}{(n+m)!2\pi\lambda_m} \int_0^{2\pi} d\theta \int_0^{\pi} f \times P_n^m(\cos\varphi)\cos m\theta \sin\varphi d\varphi$$

where $\lambda_0 = 2$ $(m = 0)$ and $\lambda_m = 1$ $(m \neq 0)$

In case of : $m = 0$ one may obtain

$$f = \sum_{n=0}^{\infty} H_n P_n(\cos\varphi)$$

where H_n is to be determined by expression

$$H_n = \frac{(2n+1)}{2} \int_0^{\pi} f \times P_n(\cos\varphi)\sin\varphi d\varphi$$

In the literature known development of the function is

$$F = \sum_{n=0}^{\infty} H_n P_n^{(1)}(\cos\varphi)$$

where H_n is defined by

$$H_n = \frac{(2n+1)(n-1)!}{2\pi(n+1)!} \int_0^{\pi} f \times P_n^{(1)}(\cos\varphi)\sin\varphi d\varphi$$

where

$$P_n^{(1)}(\cos\varphi) = \frac{dP_n(\cos\varphi)}{d\varphi}$$

Appendix B

Coefficients in expressions defining stresses Eq. (46)

$$U_n^1 = -\frac{(n+1)}{h^2 \operatorname{sh}^2 u}\left[1+(n+2)\operatorname{sh}^2 u\right]$$

$$U_n^2 = \frac{\operatorname{ch} u}{h^2}(n+1)\left(\frac{1}{\operatorname{sh}^2 u}+\frac{c^2}{h^2}\right)$$

$$U_n^3 = -\frac{c^2}{h^4}(n+1)\cos\varphi$$

$$U_n^4 = -\frac{c}{h^2}(n+1)\cos\varphi\left[\operatorname{ch}^2 u\left(\frac{1}{\operatorname{sh}^2 u}+\frac{c^2}{h^2}\right)+\alpha\right]$$

$$U_n^5 = \frac{c}{h^2}(n+1)\operatorname{ch} u\left(\frac{c^2}{h^2}\cos^2\varphi+2-\alpha\right)$$

$$U_n^6 = \frac{c}{h^2}\operatorname{ch}u\cos\varphi\left\{-2n-3+(n+1)\left[\frac{2}{\operatorname{sh}^2 u}-\frac{\operatorname{ch}^2 u}{\operatorname{sh}^2 u}-n+2\alpha\right]\right\}+\frac{c^3}{h^4}\cos\varphi\{\operatorname{sh}u\sin^2\varphi+\operatorname{ch}u\left[\operatorname{sh}^2 u-(n+1)(\cos^2\varphi-\operatorname{ch}^2 u)\right]\}$$

$$\varphi_n^1 = \frac{n+1}{h^2\sin^2\varphi}\left[(n+2)\sin^2\varphi-1\right]$$

$$\varphi_n^2 = \frac{n+1}{h^2}\cos\varphi\left(\frac{1}{\sin^2\varphi}+\frac{c^2}{h^2}\right)$$

$$\varphi_n^3 = -\frac{c^2}{h^4}(n+1)\operatorname{ch}u$$

$$\varphi_n^4 = \frac{c(n+1)}{h^2}\operatorname{ch}u\left(\alpha-\frac{\cos^2\varphi}{\sin^2\varphi}-\frac{c^2}{h^2}\cos^2\varphi\right)$$

$$\varphi_n^5 = \frac{c}{h^2}(n+1)\cos\varphi\left(\frac{c^2}{h^2}\operatorname{ch}^2 u+\alpha-2\right)$$

$$\varphi_n^6 = \frac{c}{h^2}\operatorname{ch}u\cos\varphi\left\{(n+1)(1-2\alpha)+\frac{1}{\sin^2\varphi}\left[(n+1)(n+2)-(n+1)^2\cos^2\varphi\right]\right\}$$

$$\Theta_n^1 = \frac{n+1}{h^2}\left(\frac{\cos^2\varphi}{\sin^2\varphi}+\frac{\operatorname{ch}^2 u}{\operatorname{sh}^2 u}\right)$$

$$\Theta_n^2 = -\frac{n+1}{h^2}\frac{\cos\varphi}{\sin^2\varphi}$$

$$\Theta_n^3 = -\frac{n+1}{h^2}\frac{\operatorname{ch}u}{\operatorname{sh}^2 u}$$

$$\Theta_n^4 = \frac{(n+1)c}{h^2}\operatorname{ch}u\left(\frac{\cos^2\varphi}{\sin^2\varphi}+2-\alpha\right)$$

$$\Theta_n^5 = \frac{(n+1)c}{h^2}\cos\varphi\left(\frac{\operatorname{ch}^2 u}{\operatorname{sh}^2 u}-2+\alpha\right)$$

$$\Theta_n^6 = -\frac{(n+1)c}{h^2}\operatorname{ch}u\cos\varphi\left(\frac{\cos^2\varphi}{\sin^2\varphi}+\frac{\operatorname{ch}^2 u}{\operatorname{sh}^2 u}\right)$$

$$T_n^1 = -\frac{(n+1)\,(n+2)}{h^2}\frac{\operatorname{ch}u\cos\varphi}{\sin\varphi\operatorname{sh}u}$$

$$T_n^2 = \frac{n+1}{h^2}\frac{\cos\varphi}{\operatorname{sh}u}\left(\frac{n+1}{\sin\varphi}+\frac{c^2}{h^2}\sin\varphi\right)$$

$$T_n^3 = \frac{n+1}{h^2}\frac{\operatorname{ch}u}{\sin\varphi}\left(\frac{n+1}{\operatorname{sh}u}+\frac{c^2}{h^2}\operatorname{sh}u\right)$$

$$T_n^4 = -\frac{(n+1)^2}{h^2}\frac{1}{\operatorname{sh}u\sin\varphi}$$

$$T_n^5 = -\frac{(n+1)^2}{h^2}\frac{\operatorname{ch} u \cos\varphi}{\operatorname{sh} u \sin\varphi}$$

$$T_n^6 = \frac{(n+1)c}{h^2}\frac{\cos\varphi}{\sin\varphi}\left[(1-\alpha)\operatorname{sh} u + (n+1)\frac{\operatorname{ch}^2 u}{\operatorname{sh} u} - \frac{c^2}{h^2}\operatorname{ch}^2 u \operatorname{sh} u\right]$$

$$T_n^7 = \frac{(n+1)c}{h^2}\frac{\operatorname{ch} u}{\operatorname{sh} u}\left[(\alpha-1)\sin\varphi + (n+1)\frac{\cos^2\varphi}{\sin\varphi} - \frac{c^2}{h^2}\cos^2\varphi\sin\varphi\right]$$

$$T_n^8 = \frac{c}{h^2}\left\{\operatorname{sh} u \sin\varphi + \frac{\operatorname{ch}^2 u}{\operatorname{sh} u}\left[(n+1)(1-\alpha)\sin\varphi\right] + \frac{\cos^2\varphi}{\sin\varphi}\left[(n+1)(\alpha-1)\operatorname{sh} u\right] - \frac{\operatorname{ch}^2 u\cos^2\varphi}{\operatorname{sh} u \sin\varphi}(n+1)^2\right\}$$
$$+\frac{c^3}{h^4}\left\{\operatorname{ch}^2 u \operatorname{sh} u\left[(n+1)\frac{\cos^2\varphi}{\sin\varphi} - \sin\varphi\right] + \cos^2\varphi\sin\varphi\left[\operatorname{sh} u + (n+1)\frac{\operatorname{ch}^2 u}{\operatorname{sh} u}\right]\right\}$$

Appendix C

Coefficients related to unknown constants in expressions Eq. (51) and Eq. (52)

$$a_{n-2}^1 = \left(-\frac{2}{3}A_n^2 - \frac{2}{3}nA_n^4\right)\frac{3}{2}\frac{a_{n-2}}{a_n}\frac{2n-3}{2n+1} + A_n^4 n\frac{a_{n-2}}{a_{n-1}}\frac{2n-3}{2n-1}$$

$$a_n^2 = \left(A_n^0 + \frac{2}{3}A_n^2 - \frac{n}{3}A_n^4\right) + \left(-\frac{2}{3}A_n^2 - \frac{2}{3}A_n^4\right)\frac{a_{n-1}}{a_{n+1}}\frac{2n+1}{2n+3} + A_n^4 n\frac{a_{n-1}}{a_n}$$

$$a_{n+2}^3 = \left(-\frac{2}{3}A_n^2 - \frac{2}{3}nA_n^4\right)\frac{3}{2}\frac{a_n}{a_{n+2}}$$

$$b_{n-4}^1 = 0$$

$$b_{n-2}^2 = \left[-A_n^3 - A_n^9(n+1) + A_n^7(n+1)\right]\frac{a_{n-2}}{a_{n+1}}\frac{2n-3}{2n+3} + \left[A_n^9(n+1) - A_n^7(n+1)\right]\frac{a_{n-2}}{a_n}\frac{2n-3}{2n+1}$$

$$b_n^3 = A_n^9(n+1)\frac{1}{3} + A_n^5(n+1) + A_n^7(n+1)\frac{2}{3} + \left[A_n^1 + \frac{2}{5}A_n^3 - \frac{3}{5}A_n^9(n+1) - A_n^5(n+1) - \frac{2}{5}A_n^7(n+1)\right]\frac{a_n}{a_{n+1}}\frac{2n+1}{2n+3}$$
$$+\left[-\frac{3}{5}A_n^3 - \frac{3}{5}A_n^9(n+1) + \frac{3}{5}A_n^7(n+1)\right]\frac{a_{n-1}}{a_{n+2}}\frac{2n+1}{2n+5} + \left[A_n^9(n+1)\frac{2}{3} - A_n^7(n+1)\frac{2}{3}\right]\frac{a_{n-1}}{a_{n+1}}\frac{2n+1}{2n+3}$$

$$b_{n+2}^4 = \left[A_n^1\frac{2}{5}A_n^3 - A_n^5(n+1) - \frac{2}{5}A_n^7(n+1) - \frac{3}{5}A_n^9(n+1)\right]\frac{a_{n+1}}{a_{n+2}} + \left[-\frac{3}{5}A_n^3 + \frac{3}{5}A_n^7(n+1) - \frac{3}{5}A_n^9(n+1)\right]\frac{a_n}{a_{n+3}}\frac{2n+5}{2n+7}$$
$$+\left[-A_n^7(n+1) + A_n^9(n+1)\right]\frac{a_n}{a_{n+2}}$$

$$b_{n+4}^5 = \left[-A_n^3 + A_n^7(n+1) - A_n^9(n+1)\right]\frac{a_{n+1}}{a_{n+4}}$$

$$a_{n-3}^4 = \left(B_n^2 n - B_n^4\right)\frac{a_{n-3}}{a_n}\frac{2n-5}{2n+1} - B_n^2 n\frac{a_{n-3}}{a_{n-1}}\frac{2n-5}{2n-1}$$

$$a_{n-1}^5 = \left(B_n^0 n - \frac{2}{3}B_n^2 n\right) + \left(-B_n^0 n - B_n^2 n\frac{2}{5} + \frac{2}{5}B_n^4\right)\frac{a_{n-1}}{a_n}\frac{2n-1}{2n+1} + \left(\frac{3}{5}B_n^2 n - \frac{3}{5}B_n^4\right)\frac{a_{n-2}}{a_{n+1}}\frac{2n-1}{2n+3} - \frac{2}{3}B_n^2 n\frac{a_{n-2}}{a_n}\frac{2n-1}{2n+1}$$

$$a_{n+1}^{6}=\left(-B_n^0 n-\frac{2}{5}B_n^2 n+\frac{2}{5}B_n^4\right)\frac{a_n}{a_{n+1}}+\left(\frac{3}{5}B_n^2 n-\frac{3}{5}B_n^4\right)\frac{a_{n-1}}{a_{n+2}}\frac{2n+3}{2n+5}-B_n^2 n\frac{a_{n-1}}{a_{n+1}}$$

$$a_{n+3}^{7}=\left(B_n^2 n-B_n^4\right)\frac{a_n}{a_{n+3}}$$

$$b_{n-3}^{6}=\left[B_n^5+B_n^7\left(n+1\right)-B_n^9\right]\frac{a_{n-3}}{a_{n+1}}\frac{2n-5}{2n+3}-B_n^7\left(n+1\right)\frac{a_{n-3}}{a_n}\frac{2n-5}{2n+1}$$

$$b_{n-1}^{7}=\left[-B_n^1-B_n^3\left(n+1\right)-\frac{8}{7}B_n^5-\frac{1}{7}B_n^7\left(n+1\right)+\frac{1}{7}B_n^9\right]\frac{a_{n-1}}{a_{n+1}}\frac{2n-1}{2n+3}+\left[\frac{4}{7}B_n^5+\frac{4}{7}B_n^7\left(n+1\right)-\frac{4}{7}B_n^9\right]\frac{a_{n-2}}{a_{n+2}}\frac{2n-1}{2n+5}$$

$$+\left[B_n^3\left(n+1\right)+B_n^7\left(n+1\right)\frac{2}{5}\right]\frac{a_{n-1}}{a_n}\frac{2n-1}{2n+1}-\frac{3}{5}B_n^7\left(n+1\right)\frac{a_{n-2}}{a_{n+1}}\frac{2n-1}{2n+3}$$

$$b_{n+1}^{8}=\left[\frac{2}{3}B_n^1-\frac{1}{3}B_n^3\left(n+1\right)+\frac{8}{15}B_n^5-\frac{2}{15}B_n^7\left(n+1\right)+\frac{2}{15}B_n^9\right]$$

$$+\left[-\frac{2}{3}B_n^1-\frac{2}{3}B_n^3\left(n+1\right)-\frac{16}{21}B_n^5-\frac{2}{21}B_n^7\left(n+1\right)+\frac{2}{21}B_n^9\right]\frac{a_n}{a_{n+2}}\frac{2n+3}{2n+5}$$

$$+\left[\frac{18}{35}B_n^5+\frac{18}{35}B_n^7\left(n+1\right)-\frac{18}{35}B_n^9\right]\frac{a_{n-1}}{a_{n+3}}\frac{2n+3}{2n+7}+\left[B_n^3\left(n+1\right)+B_n^7\left(n+1\right)\frac{2}{5}\right]\frac{a_n}{a_{n+1}}-\frac{3}{5}B_n^7\left(n+1\right)\frac{a_{n-1}}{a_{n+2}}\frac{2n+3}{2n+5}$$

$$b_{n+3}^{9}=\left[-B_n^1-B_n^3\left(n+1\right)-\frac{8}{7}B_n^5-\frac{1}{7}B_n^7\left(n+1\right)+\frac{1}{7}B_n^9\right]\frac{a_{n+1}}{a_{n+3}}+\left[\frac{4}{7}B_n^5+\frac{4}{7}B_n^7\left(n+1\right)-\frac{4}{7}B_n^9\right]\frac{a_n}{a_{n+4}}\frac{2n+7}{2n+9}-B_n^7\left(n+1\right)\frac{a_n}{a_{n+3}}$$

$$b_{n+5}^{10}=\left[B_n^5+B_n^7\left(n+1\right)-B_n^9\right]\frac{a_{n+1}}{a_{n+5}}$$

where

$$A_n^0=2c^2(n+1)\operatorname{ch}u_0 Q_{n+1}-c^2(n+1)\left[(n+2)\operatorname{sh}^2 u_0+2\right]Q_n$$

$$A_n^2=c^2\frac{(n+1)}{\operatorname{sh}^2 u_0}\left[\operatorname{ch}u_0 Q_{n+1}-(n\operatorname{sh}^2 u_0+\operatorname{ch}^2 u_0)Q_n\right]$$

$$A_n^4=c^2 Q_n$$

$$A_n^1=(n+1)c^3\left\{\left[\operatorname{sh}^2 u_0\operatorname{ch}u_0(\alpha-n)+2\operatorname{ch}u_0\right]Q_{n+1}-(\alpha\operatorname{sh}^2 u_0+2\operatorname{ch}^2 u_0)Q_n\right\}$$

$$A_n^3=(n+1)c^3\left\{\operatorname{ch}u_0\left[(\alpha-n)+\frac{1}{\operatorname{sh}^2 u_0}(2\text{-}\operatorname{ch}^2 u_0)\right]Q_{n+1}-\left(\alpha+\frac{\operatorname{ch}^2 u_0}{\operatorname{sh}^2 u_0}\right)Q_n\right\}$$

$$A_n^5=(2-\alpha)c^3\operatorname{sh}^2 u_0\operatorname{ch}u_0 Q_{n+1}$$

$$A_n^7=(2-\alpha)c^3\operatorname{ch}u_0 Q_{n+1}$$

$$A_n^9=c^3\operatorname{ch}u_0 Q_{n+1}$$

$$B_n^0=c^2\operatorname{sh}u_0(n+1)Q_{n+1}-c^2 n\operatorname{sh}u_0\operatorname{ch}u_0 Q_n$$

$$B_n^2=c^2\frac{(n+1)}{\operatorname{sh}u_0}(Q_{n+1}-\operatorname{ch}u_0 Q_n)$$

$$B_n^4=c^2\frac{n+1}{\operatorname{sh}u_0}(Q_{n+1}-\operatorname{ch}u_0 Q_n)$$

$$B_n^1 = c^3 \operatorname{sh} u_0 \{\operatorname{sh}^2 u_0 + \operatorname{ch}^2 u_0 [n(1-\alpha)-\alpha]\} Q_{n+1} - c^3 \operatorname{sh} u_0 \operatorname{ch} u_0 (1-\alpha)(n+1) Q_n$$

$$B_n^3 = c^3 \operatorname{sh} u_0 \left[(1-\alpha) \operatorname{sh}^2 u_0 + n \operatorname{ch}^2 u_0 \right] Q_{n+1} - c^3 \operatorname{sh} u_0 \operatorname{ch} u_0 (n+1) Q_n$$

$$B_n^5 = c^3 \left[\operatorname{sh} u_0 + (1-\alpha)(n+1) \frac{\operatorname{ch}^2 u_0}{\operatorname{sh} u_0} \right] Q_{n+1} - c^3 (1-\alpha)(n+1) \frac{\operatorname{ch} u_0}{\operatorname{sh} u_0} Q_n$$

$$B_n^7 = c^3 \left[(1-\alpha) \operatorname{sh} u_0 + (n+1) \frac{\operatorname{ch}^2 u_0}{\operatorname{sh} u_0} \right] Q_{n+1} - c^3 (n+1) \frac{\operatorname{ch} u_0}{\operatorname{sh} u_0} Q_n$$

$$B_n^9 = c^3 \left[\operatorname{sh} u_0 + (n+1) \frac{\operatorname{ch}^2 u_0}{\operatorname{sh} u_0} \right] Q_{n+1} - c^3 (n+1) \frac{\operatorname{ch} u_0}{\operatorname{sh} u_0} Q_n$$

Remaining coefficients are

$$q_{n-4}^1 = c^4 \frac{a_{n-4}}{a_n} \frac{2n-7}{2n+1}$$

$$q_{n-2}^2 = c^4 \left[\left(-2 \operatorname{sh}^2 u_0 - \frac{8}{7} \right) \frac{a_{n-2}}{a_n} \frac{2n-3}{2n+1} + \frac{4}{7} \frac{a_{n-3}}{a_{n+1}} \frac{2n-3}{2n+3} \right]$$

$$q_n^3 = c^4 \left[\left(\operatorname{sh}^4 u_0 + \frac{4}{3} \operatorname{sh}^2 u_0 + \frac{8}{15} \right) + \left(-\frac{4}{3} \operatorname{sh}^2 u_0 - \frac{16}{21} \right) \frac{a_{n-1}}{a_{n+1}} \frac{2n+1}{2n+3} + \frac{18}{35} \frac{a_{n-2}}{a_{n+2}} \frac{2n+1}{2n+5} \right]$$

$$q_{n+2}^4 = c^4 \left[\left(-2 \operatorname{sh}^2 u_0 - \frac{8}{7} \right) \frac{a_n}{a_{n+2}} + \frac{4}{7} \frac{2n+5}{2n+7} \right]$$

$$q_{n+4}^5 = c^4 \frac{a_n}{a_{n+4}}$$

$$s_{n-5}^1 = c^4 \frac{a_{n-5}}{a_n} \frac{2n-9}{2n+1}$$

$$s_{n-3}^2 = c^4 \left[\left(\frac{10}{9} - 2 \operatorname{ch}^2 u_0 \right) \frac{a_{n-3}}{a_n} \frac{2n-5}{2n+1} + \frac{5}{9} \frac{a_{n-4}}{a_{n+1}} \frac{2n-5}{2n+3} - \frac{a_{n-3}}{a_{n+1}} \frac{2n-5}{2n+3} \right]$$

$$s_{n-1}^3 = c^4 \left[\left(\operatorname{ch}^4 u_0 - \frac{6}{5} \operatorname{ch}^2 u_0 + \frac{3}{7} \right) \frac{a_{n-1}}{a_n} \frac{2n-1}{2n+1} + \left(\frac{2}{3} - \frac{6}{5} \operatorname{ch}^2 u_0 \right) \frac{a_{n-2}}{a_{n+1}} \frac{2n-1}{2n+3} + \frac{10}{21} \frac{a_{n-3}}{a_{n+2}} \frac{2n-1}{2n+5} \right.$$
$$\left. + \left(2 \operatorname{ch}^2 u_0 - \frac{6}{7} \right) \frac{a_{n-1}}{a_{n+1}} \frac{2n-1}{2n+3} - \frac{4}{7} \frac{a_{n-2}}{a_{n+2}} \frac{2n-1}{2n+5} \right]$$

$$s_{n+1}^4 = c^4 \left[\left(\operatorname{ch}^4 u_0 - \frac{6}{5} \operatorname{ch}^2 u_0 + \frac{3}{7} \right) \frac{a_n}{a_{n+1}} + \left(\frac{2}{3} - \frac{6}{5} ch^2 u_0 \right) \frac{a_{n-1}}{a_{n+2}} \frac{2n+3}{2n+5} + \frac{10}{21} \frac{a_{n-2}}{a_{n+3}} \frac{2n+3}{2n+7} \right.$$
$$\left. + \left(-\operatorname{ch}^4 u_0 + \frac{2}{3} \operatorname{ch}^2 u_0 - \frac{1}{5} \right) + \left(\frac{4}{3} \operatorname{ch}^2 u_0 - \frac{4}{7} \right) \frac{a_n}{a_{n+2}} \frac{2n+3}{2n+5} - \frac{18}{35} \frac{a_{n-1}}{a_{n+3}} \frac{2n-3}{2n+7} \right]$$

$$s_{n+3}^5 = c^4 \left[\left(\frac{10}{9} - 2 \operatorname{ch}^2 u_0 \right) \frac{a_n}{a_{n+3}} + \frac{5}{9} \frac{a_{n-1}}{a_{n+4}} \frac{2n+7}{2n+9} + \left(2 \operatorname{ch}^2 u_0 - \frac{6}{7} \right) \frac{a_{n+1}}{a_{n+3}} - \frac{4}{7} \frac{a_n}{a_{n+4}} \frac{2n+7}{2n+9} \right]$$

$$s_{n+5}^6 = c^4 \left(\frac{a_n}{a_{n+5}} - \frac{a_{n+1}}{a_{n+5}} \right)$$