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New accuracy indicator to quantify the true and false modes for eigensystem realization algorithm

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Abstract. The objective of this paper is to apply a new proposed accuracy indicator to quantify the true and false modes for Eigensystem Realization Algorithm using output-based responses. First, a discrete mass-spring system and a simply supported continuous beam were modelled using finite element method. Then responses are simulated under random excitation. Natural Excitation Technique using only response measurements is applied to compute the impulse responses. Eigensystem Realization Algorithm is employed to identify the modal parameters on the simulated responses. A new accuracy indicator, Normalized Occurrence Number-NON, is developed to quantitatively partition the realized modes into true and false modes so that the false portions can be disregarded. Numerical simulation demonstrates that the new accuracy indicator can determine the true system modes accurately.

Keywords: modal identification; eigensystem realization algorithm; accuracy indicator; normalized occurrence number.

1. Introduction

Modal identification is the process of estimating modal parameters from vibration measurements obtained from different locations of a structure. The modal parameters of a structure include the natural frequencies, mode shapes and the damping ratios that influence the response of the structure in a frequency range of interest. Classically, one applies an artificial, measurable input to the system and one measures the output. From these input-output measurements, the experimental modal analysis can be applied to obtain the modal parameters by using different estimation methods. However, cases exist where it is rather difficult to apply an artificial force and where one has to rely upon available ambient excitation sources. It is practically impossible to measure this ambient excitation and the outputs are the only information that can be used by the system identification algorithms. Output-only modal identification is just using structure responses for extracting modal parameters and thereby more complicated due to unknown input and the noisy measurements (Desforges *et al*. 1995).

There are a variety of available approaches to estimate structural modal parameters using output responses. One category is based on time series theory (Pandit and Wu 1983, Andersen 1997) or stochastic subspace technique (Van Oerschee and De Moor 1996, Bodeux and Golinval 2003) which

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directly utilizes the output response signals. Another category uses the free decaying responses where techniques of Ibrahim Time Domain (Ibrahim and Mickulcik 1977) or Eigensystem Realization Algorithm (Juang and Pappa 1985, Mohanty and Rixen 2006) can be adopted to identify the modal parameters of the structure.

Due to measurement noises, excitation type and level, the realized model will be redundant. The interference of spurious numerical modes that result from noises and computation error occurs inevitably in estimated results. Therefore, it is necessary to completely identify and remove spurious modes (Zhang *et al.* 2005). And it is generally much more difficult to establish reliable confidence values for these results. The objective of this paper is to apply a new proposed accuracy indicator to quantify the system and noise modes for Eigensystem Realization Algorithm using output-based responses. First, a discrete mass-spring system and a simply supported continuous beam were modelled using finite element method. Then responses are simulated under random excitation. Natural Excitation Technique (James *et al.* 1993) using only response measurements is applied to compute the impulse responses. Eigensystem Realization Algorithm (ERA) is employed to identify the modal parameters on the simulated responses. A new accuracy indicator, *Normalized Occurrence Number*-NON, is developed to quantitatively partition the realized modes into true system and spurious modes so that the false portions can be eliminated.

2. Basic formulation

2.1 Modal identification using eigensystem realization algorithm

In the Eigensystem Realization Algorithm (Juang and Pappa 1985), the block hankel matrix $H(k)$ is formed as

$$H(k) = \begin{bmatrix} h(k+1) & h(k+2) & \dots & h(k+\alpha) \\ h(k+2) & h(k+3) & & h(k+\alpha+1) \\ \vdots & \vdots & \ddots & \vdots \\ h(k+\beta) & h(k+\beta+1) & \dots & h(k+\alpha+\beta-1) \end{bmatrix} \tag{1}$$

in which $h(k)$ is the $l \times r$ dimensional impulse response matrix of the kth time step. The parameters α and β, correspond to the number of columns and rows of the Hankel matrix, respectively. Theoretically, the rank of $H(k)$ is constant, equivalent to the dimension of the system. For the real systems contaminated by noise, however, there exists rank deficiency. The rank of $H(k)$ will be constant only when the parameters α and β are increased to an extent.

The ERA solution to the system realization problem uses singular value decomposition (Klema and Laub 1980) at $k = 0$, i.e.

$$H(0) = \overline{U}\overline{S}\overline{V}^T \tag{2}$$

in which $\overline{S}$ is a diagonal matrix whose diagonal elements are the singular values in decreasing order. Retaining the N largest singular values of $\overline{S}$ and corresponding vectors of $\overline{U}$ and $\overline{V}$, Eq. (2) may be written as

$$H(0) = [U \;\; *]\begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}[V \;\; *]^T = USV^T \tag{3}$$

Therefore an N dimensional realization is computed as follows

$$\hat{A} = S^{-1/2}U^T H(1) V S^{-1/2}, \quad \hat{B} = S^{1/2}V^T E_r, \quad \hat{C} = E_m^T U S^{1/2} \tag{4}$$

where $H(1)$ is a matrix of the same form as $H(0)$ but whose data are shifted in time by one additional sample. U and V are matrices formed by the first N columns of $\overline{U}$ and $\overline{V}$, respectively. $E_m^T = [I_m \;\; 0_m \;\; \ldots \;\; 0_m]$, $E_r^T = [I_r \;\; 0_r \;\; \ldots \;\; 0_r]$, where $I_m, 0_m$ are unity and zeros matrices of m dimension, respectively, and $I_r, 0_r$ with r dimension.

Transform the realization into modal coordinate using the eigenvlues Z and eigenvector matrix Ψ of $\hat{A}$

$$A' = \Psi^{-1}\hat{A}\,\Psi = Z, \quad B' = \Psi^{-1}\hat{B}, \quad C' = \hat{C}\,\Psi \tag{5}$$

The modal damping rates σ_i and damped modal frequencies ω_i are the real and imaginary parts of the eigenvalues after transformation back to the continuous domain.

$$s_i = \sigma_i \pm j\,\omega_i = \ln(z_i)/\Delta t \tag{6}$$

where Δt is the sampling interval. Modal participation factors and mode shapes are the corresponding rows of B' and columns of C', respectively.

2.2 Acquiring the impulse responses using NExT

As stated in the above section, the ERA uses the impulse responses of the system to construct the hankel matrix. Conventional modal analysis utilizes inverse Fourier transform of Frequency Response Function (FRF) to compute the impulse responses, which requires measurements of both the excitation and the corresponding responses. However, ambient excitation doesn't lend itself to FRF calculation because the ambient loads can't be measured. James *et al.* (1993) showed that the cross correlation functions between the responses of the system are a solution to the homogenous equation of motion.

For an n-degree of freedom (DOF) linear, time invariant dynamic system, the vibration equation can be represented by the motion equation

$$M\ddot{z}(t) + C\dot{z}(t) + Kz(t) = f(t) \tag{7}$$

Where M, C, K are the mass, damping and stiffness matrices, respectively, $z(t)$ is the displacement response vector, $f(t)$ is the excitation force.

For stationary uncorrelated input force, the correlation function between two measured displacement responses $z_i(t)$ and $z_j(t)$ can be showed to be (James *et al.* 1993)

$$M\ddot{R}_{zz}(\tau) + C\dot{R}_{zz}(\tau) + KR_{zz}(\tau) = 0 \tag{8}$$

In Eq. (8), $R_{zz}(\tau)$ is the cross correlation function between $z_i(t)$ and a chosen reference response $z_j(t)$, and is defined as $R_{zz}(\tau) = E[z_i(t+\tau)z_j(t)]$ where $E[\cdot]$ denotes for expected value. Thus the cross correlation function has the same form as the free response solution of Eq. (7) for some initial conditions. Note that the mode shapes of the acceleration responses are identical to those of the displacements, so the acceleration responses can be used as well.

2.3 Indicators for distinguishing the system and noise modes

Because of noise contamination of the measurements, non-linearity, and computer round-off, the block matrix $H(k)$ will usually be of full rank which doesn't, in general, equal to the true order of the system under test. A realization which closely represents the underlying linear dynamics of the system is more desirable. In modal parameter identification, several indicators (Richard *et al.* 1993) such as extended modal amplitude coherence (EMAC), modal amplitude coherence (MPC) and consistent mode indicator (CMI) have been investigated to quantitatively partition the realized mode into system and noise portions so that the noise modes can be disregarded.

3. New accuracy indicator

Wang *et al.* (2005) introduced statistical method to deal with the identified modal frequencies. In the paper, Statistically Averaging Modal Frequency Method (**SAMFM**) is developed to distinguish the system and noise modes. The basic idea of **SAMFM** is as follows. The dynamic characteristics of the system are always reflected in the dynamic responses and are most expressive of themselves. For noise-contaminated data, there maybe exist noise modes in the realized modes for the Hankel matrix of a certain order used for identification and it is difficult to be distinguished. However, when the order of the hankel matrix is varied, the noise modes may not occur though some different noise modes may appear. In any case, the system modes would be exhibited as readily as possible. Hence one can change the order of the hankel matrix and statistically accumulate the identified modes. It is reasonable to consider the modes with the most accumulated number as the system modes. And the modal frequency is the accumulated value divided by the corresponding accumulated number. Since modal frequency was statistically averaged, the effect of noise contamination can be effectively restrained. The computation process is summarized as follows.

Step 1: Determine the highest frequency to be identified f_h and frequency interval Δf, and divide the frequency axis into m segments: $0-\Delta f$, $\Delta f-2\Delta f$, …, $(m-1)\Delta f-f_h$. And set two vectors ff_j and n_j ($j = 1, \ldots, m$) to zeros initially.

Step 2: For a certain value of α, β, construct the Hankel matrix and the modal frequencies f_i ($i = 1, \ldots, nf$) are identified with ERA. Assume $k = \mathrm{int}(f_i/\Delta f)$ where $\mathrm{int}(\bullet)$ means integer. Determine which segment f_i is located in and add 1 to the corresponding counter, i.e., $ff_k = ff_k + f_i$, $n_k = n_k + 1$ ($i = 1, \ldots, nf$).

Step 3: Increase the numbers of column and row of hankel matrix and repeat step 2 until end.

Step 4: Finally the corresponding occurrence number of each segment is n_j ($j = 1, \ldots, m$). And the identified modal frequency of each segment is $f_j = ff_j/n_j$ ($j = 1, \ldots, m$) if n_j is not equal to zero.

In Wang's (Wang *et al.* 2005) paper, the modes corresponding to higher occurrence are considered to be the true system modes of the structure. However, it is still artificially random to determine the exact occurrence as the critical value.

In the present paper, we establish more robust classification criteria for true modal identification. The true system modes of the structure are selected on the basis of a rejection of hypotheses in the statistical sense. Firstly, the occurrence number n_j associated with each segment is treated as a realization of a random variable n. In other words, the collection of the occurrence value, n_j, represents a sample population. For purposes of making a consistent comparison, we wish to classify a mode into one of two groups. We first normalize the values of occurrence n_j, according to the rule

$$NON_j = \frac{n_j - \bar{n}}{\sigma_n} \tag{9}$$

in which the terms $\bar{n}$ and σ_n represent, respectively, the mean and the standard deviation of the collection of n_j values and NON_j denotes normalized occurrence number corresponding to the j-th segment of frequency.

Our next problem is to develop an algorithm that would classify NON_j into system and noise modes using hypothesis testing (Gibson and Melsa 1975). The null hypothesis (i.e., H_0) is: The mode is a noise mode. The alternate hypothesis (i.e., H_1) is: The mode is system mode.

choose H_1: when $NON_j > K$
choose H_0: when $NON_j < K$

in which K is a number that reflects the level of significance of the test. For $K = 2$, the level of significance is 0.023. Hereby, a new accuracy indicator, NON_j, normalized occurrence number, is defined to distinguish the system and noise modes. In the following numerical study, $K = 2$ is chosen for the critical threshold.

4. Numerical examples

Numerical examples are given below to illustrate the procedure of applying the new accuracy indicator to determine the system modes. Two structural models-a simple discrete mass-spring system and a simply supported beam model-are chosen for numerical demonstration. Two cases, noise-free and random noise with noise-to-signal ratio (NSR) equal to 10%, are studied to investigate the sensitivity of noise level to the identification accuracy.

4.1 Discrete mass-spring system

We will begin by illustrating the application of the new accuracy indictor to a simple mass-spring system. The system consists of 5 inertias joined together by 5 springs to form a chain, as shown in Fig. 1. Each spring has a stiffness of 2900 kN/m and each inertia has a mass of 260 kg except the end one with 220 kg. When simulating the responses, 1% damping ratios for all five modes were added to system to form the damping matrix.

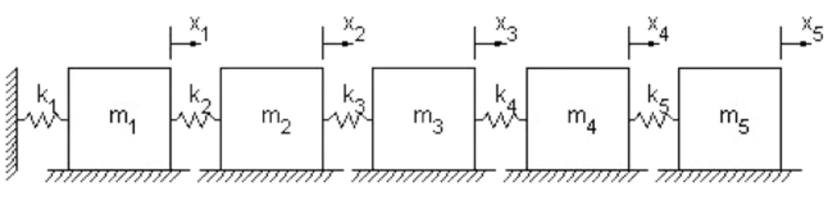

Fig. 1 The discrete mass-spring example

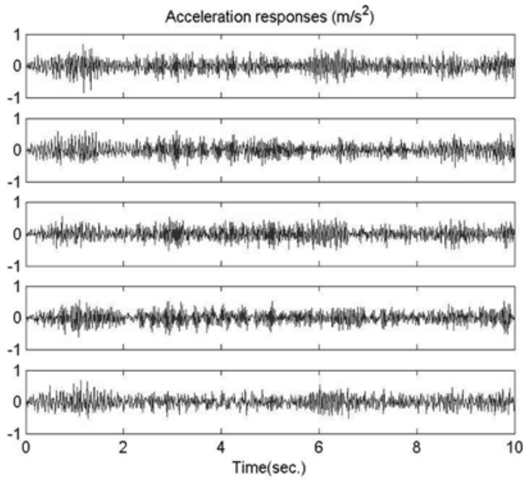

Fig. 2 Acceleration responses at the five masses

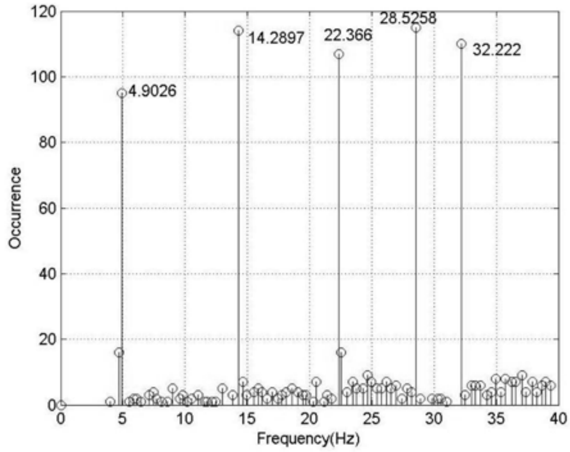

Fig. 3 The occurrence number for noise-free measurements (before normalizing)

Modal analysis was conducted to get the modal frequencies of the system. And gaussian random excitations acting on the five masses are assumed to compute the responses. The sampling interval is 0.01s and totally 2048 data points are acquired for each mass. Shown in Fig. 2 are the acceleration responses at the five masses.

Using the impulse responses from NExT as the input of ERA and varying the rows and columns of the Hankel matrix, the modal frequencies based on the new accuracy indicator- **NON**, are identified. In applying the new indicator, the maximum numbers of columns and rows of the block hankel matrix are 40 and 8, respectively. Fig. 3 shows the occurrence number for noise-free measurements *before normalizing*. From Fig. 3 one can see that the modal frequencies whose occurrence numbers are greater than 80 are supposed to be the true modes. However, when the maximum numbers of columns and rows of the block hankel matrix are changed, 80 may be changed either, i.e., it is not a relatively stable value. One can also see that there are spurious frequencies which occurrence numbers are close to 20. The normalized occurrence number (**NON**) of the identified modes is shown in Fig. 4 for noise free measurements. The theoretical, the identified frequencies, damping ratios and the corresponding errors are listed in Table 1 for comparison. It is clearly seen that the errors for modal frequencies are negligible. The identification of damping ratios is not as easily as eigenfrequencies. The relative errors of the identified damping ratios are more larger than those of the identified modal frequencies. However, the results may be considered reasonable since it is more difficult for damping identification than frequency identification.

An important aspect in the development of modal identification is its sensitivity to uncertainties in the measurements. To study the effect of measurement noise, it is assumed that the simulated acceleration responses are contaminated with Gaussian random noise with noise-to-signal ratio (NSR) equal to 10%. Similarly applying the procedure, the **NON** of the realized modes with noise is shown in Fig. 5 and the modal frequencies are listed in Table 2. For comparison, the theoretical values and the relative errors are also listed in Table 2. It can be seen that the effect of noise on the NON for modal frequencies is negligible. However, the effect of noise on the identification of damping coefficients can not be neglected when comparing the results of Tables 1 and 2. From

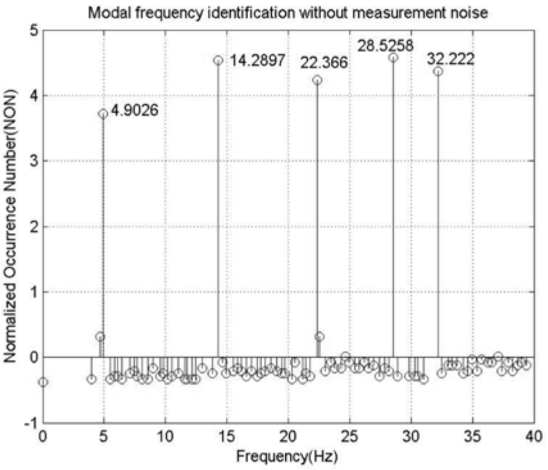

Fig. 4 **NON** of the system for noise-free measurements

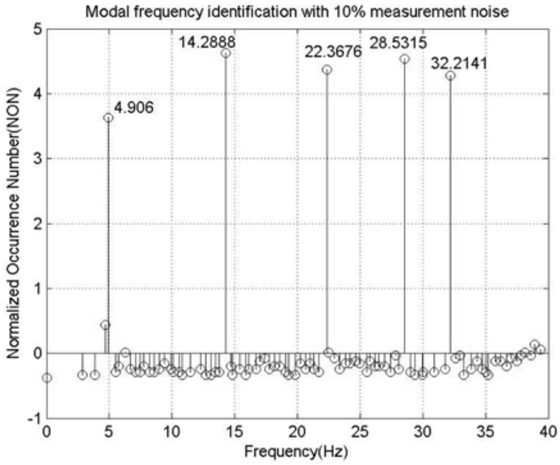

Fig. 5 **NON** of the system with 10% measurement noise

Table 1 Comparison of modal frequencies and damping ratios without measurement noise

Mode	Modal frequencies			Damping ratios		
	Theoretical (Hz)	NON (Hz)	Relative error (%)	Theoretical (Hz)	NON (Hz)	Relative error (%)
1	4.9197	4.9026	-0.349	0.01	0.0138	38.1
2	14.311	14.290	-0.151	0.01	0.0138	37.8
3	22.417	22.366	-0.226	0.01	0.0097	-2.64
4	28.567	28.526	-0.144	0.01	0.0130	30.9
5	32.350	32.222	-0.397	0.01	0.0096	-4.22

Table 2 Comparison of modal frequencies and damping ratios with 10% measurement noise

Mode	Modal frequencies			Damping ratios		
	Theoretical (Hz)	NON (Hz)	Relative error (%)	Theoretical (Hz)	NON (Hz)	Relative error (%)
1	4.9197	4.9060	-0.280	0.01	0.0206	106.6
2	14.311	14.289	-0.157	0.01	0.0175	74.98
3	22.417	22.368	-0.219	0.01	0.0199	99.3
4	28.567	28.532	-0.124	0.01	0.0122	21.6
5	32.350	32.214	-0.421	0.01	0.0095	-4.86

Figs. 4 and 5, one can set the threshold (K) to a higher value (three for example) if one needs a higher confidence level.

4.2 Simply supported beam model

We next illustrate the application of the new accuracy indicator to a discrete model of a continuous structure. The structure considered is a simply supported beam, as shown in Fig. 6. The

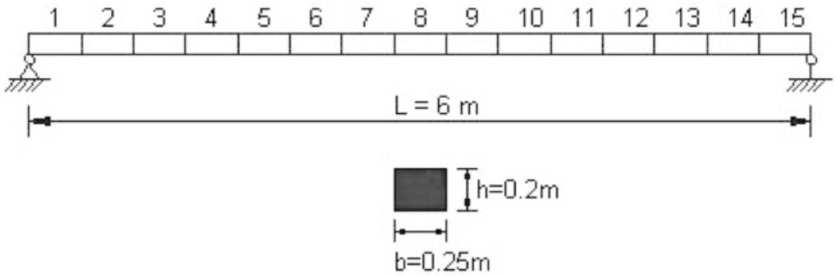

Fig. 6 The simulated simply supported beam

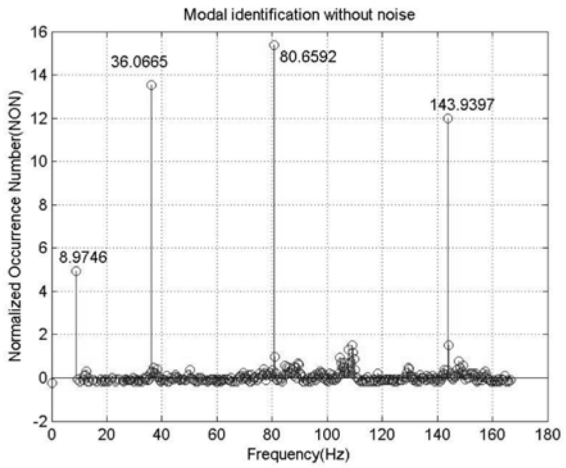

Fig. 7 **NON** of the beam model for noise-free measurements

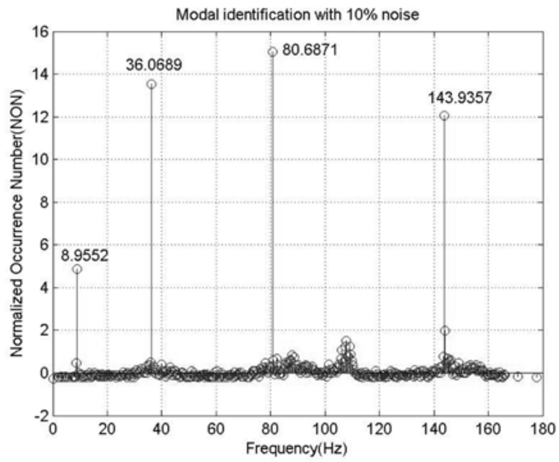

Fig. 8 **NON** of the beam model with 10% measurement noise

simulated beam of 6 m length is equally divided into 15 two-dimensional beam elements as shown in Fig. 6. The density and elastic modulus of the material of the beam are 2500 kg/m^3 and 3.2×10^{10} N/m^2, respectively. Similarly area of cross section and moment of inertia of simulated beam are 0.05 m^2 and 1.66×10^{-4} m^4, respectively.

Modal analysis is carried out to get the FE frequencies and acceleration responses in the transverse direction at the 14 nodes are calculated assuming that the beam is excited by random loading in transverse direction at node 3, 6 and 12. The sampling time interval is 0.003s and 2048 data points are acquired for each DOF.

The similar identification procedure as explained above is carried out. Cases with no noise and with 10% noise are studied. Shown in Fig. 7 is the **NON** of the system modes in case of noise free acceleration responses. And the first four theoretical frequencies from FE model, the identified frequencies and the corresponding errors are listed in Table 3. The maximum error that appeared in frequencies is 0.37% in the 3rd mode.

Fig. 8 shows the NON of the modal frequencies in case of 10% measurement noise for acceleration responses. For comparison, the corresponding theoretical frequencies, identified values and relative errors are listed in Table 4. It can be seen that the maximum error in frequencies is 0.44% in the 1st mode.

Table 3 Comparison of modal frequencies of the beam model without measurement noise

Mode	Theoretical (Hz)	NON (Hz)	Relative error (%)
1	8.9948	8.9746	-0.224
2	35.980	36.066	0.240
3	80.962	80.659	-0.374
4	143.97	143.94	-0.018

Table 4 Comparison of modal frequencies of the beam model with 10% measurement noise

Mode	Theoretical (Hz)	NON (Hz)	Relative error (%)
1	8.9948	8.9552	-0.441
2	35.980	36.0689	0.247
3	80.962	80.6871	-0.339
4	143.97	143.9357	-0.020

5. Conclusions

Modal parameters identification using output-only measurements has received much attention over the past decades. And there are a variety of available approaches to estimate structural modal parameters. Due to measurement noises, the realized modes will unavoidably include fictitious computational modes. And it is generally much more difficult to establish reliable confidence values for these results. In the present paper, a new accuracy indicator, *Normalized Occurrence Number* (**NON**), is proposed to distinguish the system and noise modes for Eigensystem Realization Algorithm. This new accuracy indicator reliably indicates the relative confidence of each identified mode on the basis of hypothesis testing. A critical value is predetermined which reflects the level of significance of the test. Modes with NON greater than this critical value can generally be accepted as system modes.

For verifying the effectiveness and robustness of the new accuracy indicator, two examples are illustrated. Two cases, noise-free measurements and random noise contaminated responses with noise-to-signal ratio (NSR) equal to 10%, are studied to investigate the sensitivity of noise level to the identification accuracy. Numerical simulation demonstrates that the new indicator can determine the system modes properly. The precision of the identified eigenfrequencies is very high and the effect of noise on the NON for modal frequencies is negligible. However, the identification of damping ratios is not as easily as eigenfrequencies. The relative errors of the identified damping ratios are larger than those of the identified modal frequencies. And the effect of noise on the identification of damping coefficients can not be neglected. Further investigation is still needed to improve the damping identification using NON.

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