Structural Engineering and Mechanics, Vol. 34, No. 4 (2010) 475-490

DOI: http://dx.doi.org/10.12989/sem.2010.34.4.475

New formulation for vibration analysis of Timoshenko beam with double-sided cracks

M.R. Ayatollahi†, R. Hashemi and H. Rokhi

Fatigue and Fracture Laboratory, Department of Mechanical Engineering, Iran University of Science and Technology, Narmak, Tehran 16846, Iran

(Received April 28, 2008, Accepted November 27, 2009)

Abstract. It is the intention of this study to synthesize the effects of double-edge cracks on the dynamic characteristics of a beam. The stiffness matrix is first determined for a Timoshenko beam containing two same-line edge cracks. The presented model is then developed for elements with two parallel double-sided cracks, considering the interaction between the stress fields of adjacent cracks. Finally, a finite element code is implemented, to examine the influence of depth and location of double cracks, on the natural frequencies of the damaged system.

Keywords: crack identification; Timoshenko beam; free vibration; flexibility matrix; double-sided cracks.

1. Introduction

Fatigue or stress/corrosion cracks which are often found in structural members, can lead to mechanical failure due to various loading conditions. Since flaws or cracks present a serious threat to the performance of structures; methods allowing early detection and localization of cracks have been subject of intensive investigation in the recent decades. Several techniques are available to detect cracks in such components. The conventional methods like X-ray, ultrasonic or eddy current-based methods require full scanning of the damaged components. For some particular cases such as turbo-machine fins or pipes, these techniques are costly and sometimes impossible. Hence it has motivated development of alternative methods. Vibration-based detection of crack is one of such techniques. Indeed, cracks alter the dynamic characteristic of structures and can disturb the smooth operation of the machines. Ideally, it should be possible to infer the location and extent of crack from indirect measurements or signals.

To facilitate such methods, a mathematical modeling of the object or structure is necessary. A variety of analytical, numerical and experimental investigations have been done to qualify the crack detection through vibration analysis for different structures with various crack geometries. Among them, several studies have addressed the problem of a cracked beam, because beams are frequently used as design models of some mechanical components. Generally, there are two procedures proposed in technical literatures to introduce the local flexibility generated by the crack into

† Professor, Corresponding author, E-mail: m.ayat@iust.ac.ir

mechanical system. As the first procedure, some researchers such as Chondros and Dimarogones (1979, 1980) and Dimarogonas and Massouros 1980 simulated the effect of a crack as a massless spring (local flexibility), that its equivalent stiffness is computed as a function of crack depth using fracture mechanics methods (Dimarogonas 1976, Boltezar *et al.* 1998). They showed that changes of the natural frequencies and mode shapes of the cracked beam follow definite trends depending on the crack geometry and loading conditions. The trends consequently provide additional information required for accurate crack detection. This model has been successfully used for crack localization in simple structures such as shafts and beams. The extension to the above method also prompted studies of problems related to elastic stability (Anifantis and Dimarogonas 1980), fatigue (Dentsoras and Dimarogonas 1983) and dynamic behavior of cracked shafts (Gounaris *et al.* 1991), etc.

The alternative approach of evaluating the dynamic characteristics of cracked structure is based on the use of finite element method. The second procedure considers the construction of stiffness matrix corresponding to a cracked element, which is later assembled with the other non-cracked elements of system (Gounaris and Dimarogonas 1988, Haisty and Springer 1988). Using this method Qian *et al.* 1990 obtained the dynamic behavior of an Euler-Bernoulli beam with a crack. Christides and Barr 1984 and Shen and Pierre 1990 proposed using the singular elements instead of fine-mesh finite elements, to investigate the free vibration of an Euler Bernoulli beam with double edge cracks.

Also with the intensive use of materials having relatively high transverse shear modulus and the need for beam members with high natural frequencies in aerospace, civil and mechanical engineering applications, a more refined higher-order theory is called for beam structures. Consequently the Timoshenko beam model, which includes shear deformation and rotary inertia effects, is preferable to be used in the investigation of dynamic response of beams. The Timoshenko beam theory has an extended range of applications because it allows treatment of deep beam (depth is large relative to length), short and thin webbed beams where higher modes are excited. However, it introduces some complications not found in the elementary Euler-Bernouli formulation. The dynamic behavior of Timoshenko beam has been studied by various investigators. Among other, Chen and Chen 1988 and Chen and Jeng 1993 used the finite element techniques to compute local flexibility matrix of crack region in a rotating pre-twisted Timoshenko beam. Meanwhile, Kisa and Brandon 2000 adapted the local flexibility procedure for a single edge cracked Timoshenko beam. Loya *et al*. 2006 obtained the natural frequencies for bending vibration of Timoshenko cracked beams with simple boundary conditions. They modeled the beam as two segments connected by two massless springs. In another attempt, Dansheng *et al.* 2007 presented a technique for crack detection in Timoshenko beam based on the first anti-resonance frequency. More recently, an analytical solution for crack identification in the uniform Timoshenko beam with an open edge crack based on bending vibration measurements was developed by Khaji *et al.* 2009. However, to the best of authors' knowledge, the free vibration of Timoshenko beam with double-sided edge cracks has not been addressed in the literature.

The aim of present paper is to investigate the free vibration of Timoshenko beams containing double-sided edge cracks. For this purpose, an improved finite element model is developed analytically for a beam element containing two parallel edge cracks. The following stages are conducted in this study: First, using the concept of compliance matrix that is derived from the local flexibility of the cracks, the stiffness matrix of Timoshenko beam element is obtained for the case of a single edge crack. Next the introduced model is developed for cases that the beam element has two opposite edge cracks with equal and then unequal lengths. Finally, several numerical case studies are carried out to verify the proposed finite element model. In the view of the examples, the effects of depths and locations of the cracks and the interaction in their stress fields are individually investigated on the natural frequencies and mode shapes of the beam. The results show that the proposed finite element model allows an efficient prediction for dynamic behavior of relevant cracked structures and can be used to improve the reliability of the vibration based techniques for diagnostic of double edge cracks.

2. The finite element formulation

In the following section, first the stiffness and mass matrices are extracted for a typical un-cracked Timoshenko beam, taking into account the effect of transverse shear and rotary inertia. Consider a two-noded beam element of length l, where each node has four degrees of freedom (see Fig. 1). Let its degrees of freedom be noted by the non-dimensional transverse deflection ψ, the pure bending slope ϕ and their differentiations with respect to the spatial coordinate ψ' and ϕ'. Thus, the displacement vector of element is expressed as

$$\{\xi^e\}^T = [\psi_i \;\; \phi_i \;\; \psi_i' \;\; \phi_i' \;\; \psi_{i+1} \;\; \phi_{i+1} \;\; \psi_{i+1}' \;\; \phi_{i+i}'] \tag{1}$$

and the strain energy of beam, U can be written as

$$U = \frac{1}{2}EI\int_0^l\left(\frac{d\phi}{dx}\right)^2 dx + \frac{1}{2}kAG\int_0^l\left(\frac{ld\psi}{dx} - \phi\right)^2 dx \tag{2}$$

in which k is the shear correction factor depending on the shape of cross-sections, if the warping effect is not considered. The shear correction factor for rectangular cross-section is 1.2 and for circular section is 1.1.

The shape functions for ψ and ϕ are assumed to be cubic polynomials and are expressed as

$$\psi = \sum_{r=0}^{3} a_r\eta^r, \quad \phi = \sum_{r=0}^{3} b_r\eta^r \tag{3}$$

Substituting Eq. (3) into the equation of strain energy and replacing the coefficients a, b and c in terms of nodal coordinates, Eq. (2) can be rewritten as

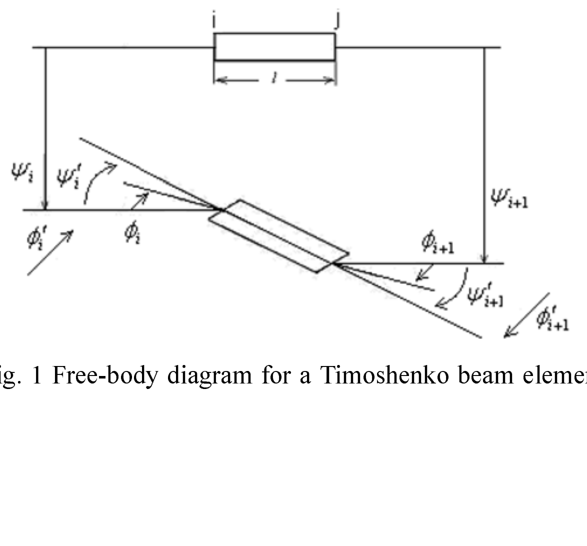

Fig. 1 Free-body diagram for a Timoshenko beam element

$$U = \frac{1}{2}\{\xi^e\}^T[K^e]\{\xi^e\} \tag{4}$$

where $[K^e]$ is the element stiffness matrix

$$[K^e] = \frac{EI}{420l}\begin{bmatrix} 504s & 210s & 42s & 42s & -504s & 210s & 42s & -42s \\ & 156s+504 & -42s & 22s+42 & -210s & 54s-504 & 42s & -13s+42 \\ & & 56s & 0 & -42s & 42s & -14s & -7s \\ & & & 4s+56 & -42s & 13s-42 & 7s & -3s-14 \\ & & & & 504s & -210s & -42s & 42s \\ & & Symm. & & & 156s+504 & -42s & -22s-42 \\ & & & & & & 56s & 0 \\ & & & & & & & 4s+56 \end{bmatrix} \tag{5}$$

and s is the shear deformation parameter of beam

$$s = \frac{kAGl^2}{EI} \tag{6}$$

Considering the defined degrees of freedom for the beam element, the corresponding kinetic energy of Timoshenko beam, T can be written as

$$T = \frac{1}{2}\rho I\int_0^l \dot{\phi}^2 dx + \frac{1}{2}\rho A l^2\int_0^l \dot{\psi}^2 dx \tag{7}$$

where $\dot{\phi}$ and $\dot{\psi}$ are the first derivatives of ϕ and ψ relative to time t.

Substituting Eq. (3) into above equation, the kinetic energy expression can be represented in the form of finite element equation

$$T = \frac{1}{2}\{\dot{\xi}^e\}^T[M^e]\{\dot{\xi}^e\} \tag{8}$$

where $\{\dot{\xi}^e\}$ is the first time derivative of the nodal displacement vector and $[M^e]$ denotes the mass matrix of beam element, that its detail expression is

$$[M^e] = \frac{\rho A l^3}{420}\begin{bmatrix} 156 & 0 & 22 & 0 & 54 & 0 & -13 & 0 \\ & 156R & 0 & 22R & 0 & 54R & 0 & -13R \\ & & 4 & 0 & 13 & 0 & -3 & 0 \\ & & & 4R & 0 & 13R & 0 & -3R \\ & & & & 156 & 0 & -22 & 0 \\ & & Symm. & & & 156R & 0 & -22R \\ & & & & & & 4 & 0 \\ & & & & & & & 4R \end{bmatrix} \tag{9}$$

Here R, the rotary inertia parameter of beam, is defined as

$$R = \frac{I}{Al^2} \tag{10}$$

However on the basis of Hamilton's principle, the variations of the Lagrangian $\int_{t_0}^{t} \delta[T - U] = 0$, provide the dynamic governing equations for the Timoshenko beam element

$$\frac{\partial U}{\partial\{\xi\}} + \frac{d}{dt}\left(\frac{\partial T}{\partial\{\xi\}}\right) = \{0\} \tag{11}$$

Substituting Eqs. (4) and (8) into Eq. (11), the vibration response in the finite element formulation is provided by the well known matrix equation as follows

$$[[K] - \omega_i^2[M]]\{\zeta_i\} = \{0\} \tag{12}$$

where $[K]$ and $[M]$ are obtained from assembling the element matrices $[K^e]$ and $[M^e]$. ω_i and $\{\zeta_i\}$ are the i^{th} natural frequency and mode shape of the faultless beam. The eigenvalue Eq. (12) can be solved to derive the vibration characteristics of Timoshenko beam, ω_i and $\{\zeta_i\}$.

However, a crack on a structural member introduces a local flexibility which is a function of crack properties and loading conditions, hence reduces the total strain energy of structure. In order to study the effect of a crack on the dynamic behavior of an elastic structure, the local flexibility matrix of the cracked member has to be established under general loading. It should be noted that, similar to many pervious investigations the crack is considered in the present work as an open crack model, i.e., the crack in structural element always remains open during vibration. Although several researchers have addressed the problem of beams with breathing cracks (Abraham and Brandon 1995, Douka and Hadjileontaidis 2005) the assumption of open crack is made to avoid the complexity that results from the nonlinear behavior by introducing a breathing crack model.

2.1 Timoshenko beam with single-sided crack

Consider a rectangular cross section beam element with a single edge crack of depth a, along the z-axis (see Fig. 2). The cracked cross-section of beam element is subjected to the following internal forces: an axial force p_1, shear forces p_2, p_3, bending moments p_4, p_5 and a torsion p_6. In addition, the crack produces an additional local displacement u_i between the sections in the right and left sides of the crack. According to the Castigilano's theorem the displacement u_i in the direction i, under the action of force P_i is given, by the following expression

$$u_i = \frac{\partial}{\partial P_i}\int_0^a g da \tag{13}$$

in which, g is the strain energy release rate function. Assuming isotropic linear elastic material behavior, it is possible to derive a closed-form expression for strain energy release rate as

$$g = \frac{1}{E'}\left[\left(\sum_{k=1}^{6} K_{\mathrm{I},k}^2\right) + \left(\sum_{k=1}^{6} K_{\mathrm{II},k}^2\right) + (1+\nu)\left(\sum_{k=1}^{6} K_{\mathrm{III},k}^2\right)\right] \tag{14}$$

where $E' = E/(1 - \nu^2)$ for plane strain, $E' = E$ for plane stress and E is the modulus of elasticity. K_{I}, K_{II} and K_{III} are the mode I, II and III stress intensity factors respectively. The subscript k (k = 1 to 6) is related to the six different loading conditions mentioned earlier. The stress intensity factors

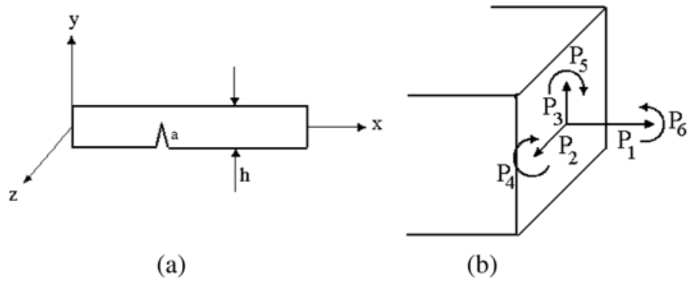

Fig. 2 Schematics of (a) the cracked beam element and (b) the internal forces and their coordinate system

corresponding to these loading types can be evaluated from the literature or from finite element simulation results. It is noteworthy that some of the loading types induce only some specific modes of crack deformation. For example $K_{\mathrm{III},1}$ vanishes because the axial force P_1 cannot normally impose an out of plane sliding. Using a similar argument, only the stress intensity factors $K_{\mathrm{I},1}$, $K_{\mathrm{I},5}$, $K_{\mathrm{I},6}$, $K_{\mathrm{II},2}$, $K_{\mathrm{II},4}$, and $K_{\mathrm{III},3}$ are expected to remain in Eq. (14).

The additional flexibility introduced due to the presence of crack is obtained by the definition of the compliance coefficients as

$$C_{ij} = \frac{\partial u_i}{\partial P_j} = \frac{\partial^2}{\partial P_i \partial P_j}\int_0^a g da \tag{15}$$

Hence, under general loading the flexibility matrix of crack can be derived as

$$[C] = \begin{bmatrix} C_{11} & 0 & 0 & C_{14} & C_{15} & 0 \\ 0 & C_{22} & 0 & 0 & 0 & C_{26} \\ 0 & 0 & C_{33} & 0 & 0 & C_{36} \\ C_{41} & 0 & 0 & C_{44} & C_{45} & 0 \\ C_{51} & 0 & 0 & C_{54} & C_{55} & 0 \\ 0 & C_{62} & C_{63} & 0 & 0 & C_{66} \end{bmatrix} \tag{16}$$

which explains the relation between displacement vector and external applied loads.

Consider a crack that is located in a distance from the left end of beam element. The displacement of both transverse section of the crack are shown by vectors $\{u_L\}$ and $\{u_R\}$ and their corresponding loads by $\{P_L\}$ and $\{P_R\}$, where L and R refer to the left and right sections of crack, respectively. The relations between the loads and the relative displacements between both sections can be represented as

$$-\{P_L\} = \{P_R\} \tag{17}$$

$$\{u_R\} - \{u_L\} = [C](\{P_R\} - \{P_L\}) \tag{18}$$

To determine a cracked finite stiffness matrix, Eqs. (17) and (18) finally can be written in the convenient form at the crack section as

$$\{P\} = [K_{cr}]\{u\} \tag{19}$$

where

$$[K_{cr}] = \begin{bmatrix} [C]^{-1} & -[C]^{-1} \\ -[C]^{-1} & [C]^{-1} \end{bmatrix} \tag{20}$$

is the additional contribution that the crack introduces in the stiffness matrix. In present study the elastic motion in x and z direction is ignored, hence only the diagonal terms of crack compliance matrix $[C]$ i.e., C_{33} and C_{55}, are considered

$$[C] = \begin{bmatrix} C_{33} & 0 \\ 0 & C_{55} \end{bmatrix} \tag{21}$$

Let the crack located between ith and $(i+1)$th elements be considered as a linear spring of translational stiffness K_1 and rotational stiffness K_2. These two constants are the inverse of C_{33} and C_{55} respectively. The relative deflection at the joint point can be expressed in the form of the constraint equations as

$$\phi_i = \phi_{i+1} + \beta_2 \phi_i' \tag{22}$$

$$\psi_i = \psi_{i+1} + \beta_1(\psi_{i+1}' - \phi_{i+1}) \tag{23}$$

$$\psi_i' = \psi_{i+1}' + \beta_2 \phi_i' \tag{24}$$

$$\phi_i' = \phi_{i+1}' \tag{25}$$

where β_1 and β_2, the translational and rotational parameters, are defined as

$$\beta_1 = \frac{kAG}{LK_1} \tag{26}$$

$$\beta_2 = \frac{EI}{LK_2} \tag{27}$$

The Eqs. (22)-(25) lead to construction of a constrained matrix $[C^*]$, which can be used to reduce the stiffness and mass matrices as

$$[K'] = [C^*]^T[K][C^*] \tag{28}$$

$$[M'] = [C^*]^T[M][C^*] \tag{29}$$

Now, if the matrices $[K']$ and $[M']$ are replaced into Eq. (12), our eigenvalue problem takes the effect of a single crack into account.

2.2 Beam element with same-line double-sided cracks

In this section, a beam element with two same-line double-sided cracks is considered (i.e., two cracks on opposite edges and with arbitrary lengths of a and d, as shown in Fig. 3(a)). The stress intensity factors of a single edge cracked beam described in the previous section, can be obtained directly from handbooks of stress intensity factors. However for double-sided cracks of unequal

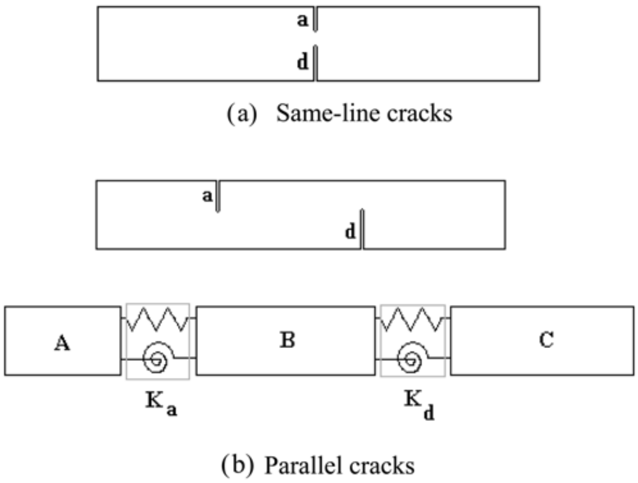

Fig. 3 A beam element with double-sided cracks

length, in order to consider the interaction between stress fields in the vicinity of adjacent cracks, the stress intensity factors and then the compliance matrix should be calculated independently. Indeed, it is known that near the crack tip there are large stress concentrations. Therefore over the cracked zone of beam the stress is not linearly distributed and all components of stress are likely to be non-zero. Due to complexity of the stress field over the bouble-sided cracks, the closed-form stress intensity solution is usually not available. Here a procedure is explained for calculating the compliance matrix of double-sided cracks embedded in a Timoshenko beam element. First, the effective stress intensity factors for the same-line double-sided cracks are defined in terms of stress intensity factors for cracks a and d as

$$K_{\mathrm{I}eff} = \sqrt{K_{\mathrm{I}a}^2 + K_{\mathrm{I}d}^2} \tag{30}$$

$$K_{\mathrm{II}eff} = \sqrt{K_{\mathrm{II}a}^2 + K_{\mathrm{II}d}^2} \tag{31}$$

Substituting $K_{\mathrm{I}eff}$ and $K_{\mathrm{II}eff}$ into Eq. (14), the strain energy release rate g, can be obtained.

This way, the finite element simulation is implemented in the code *Ansys*, to calculate numerically the mode I and II stress intensity factors for each crack. Based on the numerical evaluation of crack tip parameters and using the Gaussian quadrature integration method, the compliance matrix coefficient can be expressed in the form of

$$C_{ij} = \frac{\partial u_i}{\partial P_j} = \frac{\partial^2}{\partial P_i \partial P_j}\left(\frac{a}{2}\right)\left[g\left(\frac{(\sqrt{3}+1)a}{2\sqrt{3}}\right) + g\left(\frac{(\sqrt{3}-1)a}{2\sqrt{3}}\right)\right]; \quad i,j = 3,5;\ P_3 = P, P_5 = M \tag{32}$$

Therefore, using central-difference formula for the second derivative in Eq. (32) and after some transformation and substitution, we arrive at the following expression for the coefficient C_{33}

$$C_{33} = \left(\frac{a}{2}\right)\frac{\{g(a_1, P+\Delta P) + g(a_2, P+\Delta P)\} - 2\{g(a_1, P) + g(a_2, P)\} + \{g(a_1, P-\Delta P) + g(a_2, P-\Delta P)\}}{\Delta P^2} \tag{33}$$

where

$$a_1 = \frac{(\sqrt{3}+1)}{2\sqrt{3}}a \tag{34}$$

$$a_2 = \frac{(\sqrt{3}-1)}{2\sqrt{3}}a \tag{35}$$

$$\Delta P = 0.025P \tag{36}$$

Similarly, Eq. (32) for the coefficient C_{55} can be written as

$$C_{55} = \left(\frac{a}{2}\right)\frac{\{g(a_1, M+\Delta M)+g(a_2, M+\Delta M)\}-2\{g(a_1, M)+g(a_2, M)\}+\{g(a_1, M-\Delta M)+g(a_2, M-\Delta M)\}}{\Delta M^2} \tag{37}$$

where

$$\Delta M = 0.025M \tag{38}$$

Eqs. (33)-(36) imply that, in order to calculate the compliance matrix for the same-line double-sided cracks, the strain energy release rate first should be calculated for typically 12 different combinations of relative crack lengths and loading conditions. Assuming that linear elastic condition is dominant at crack section, the applied shear force P and bending moment M can take any arbitrary values, during the numerical evaluation of g function.

2.3 Beam element with two parallel double-sided cracks

In the following section, the finite element formulation is developed for a Timoshenko beam element containing two parallel double-sided cracks (Fig. 3(b)). Two parallel cracks have arbitrary lengths of a and d. The entire element is considered as combination of three individual segments, i.e., A, B and C, which are connected by two mass less linear springs representing the effect of cracks, as follows

$$\begin{bmatrix} [K_A] & & & & \\ & [K_a] & & & \\ & & [K_B] & & \\ & & & [K_d] & \\ & & & & [K_c] \end{bmatrix}_{24\times 24} \begin{bmatrix} [M_A] & & & & \\ & [0] & & & \\ & & [M_B] & & \\ & & & [0] & \\ & & & & [M_c] \end{bmatrix}_{24\times 24} \tag{39}$$

where $[K_a]$ and $[K_d]$ are the equivalent stiffness matrix due to the presence of cracks a and d. ($[K_A]$, $[K_B]$ and $[K_C]$) and ($[M_A]$, $[M_B]$ and $[M_C]$) are the stiffness and mass matrices corresponding to the segments A, B and C of the beam element and are obtained from Eqs. (5) and (9). The crack flexibility coefficients can be calculated in a similar way as that for two same-line cracks, by adopting the strain energy release rate for each crack in the form of

$$g_a = \frac{1}{E'}[K_{Ia}^2 + K_{IIa}^2] \tag{40}$$

$$g_d = \frac{1}{E'}[K_{Id}^2 + K_{IId}^2] \tag{41}$$

Again, no analytical solution could be found in the literature for parallel edge cracks when there is an interaction between the stress fields. Therefore, one can obtain a relation for the compliance matrix coefficients C_{ij} at the crack location, using the numerical integration and differentiation method described in previous section. For example, the coefficient C_{33} for the second crack of depth d, can be presented as

$$C_{33} = \left(\frac{d}{2}\right)\frac{\{g(d_1, P+\Delta P)+g(d_2, P+\Delta P)\} - 2\{g(d_1, P)+g(d_2, P)\} + \{g(d_1, P-\Delta P)+g(d_2, P-\Delta P)\}}{\Delta P^2} \tag{42}$$

where

$$d_1 = \frac{(\sqrt{3}+1)}{2\sqrt{3}}d \tag{43}$$

$$d_2 = \frac{(\sqrt{3}-1)}{2\sqrt{3}}d \tag{44}$$

Also, the C_{55} can be found from

$$C_{55} = \left(\frac{d}{2}\right)\frac{\{g(d_1, M+\Delta M)+g(d_2, M+\Delta M)\} - 2\{g(d_1, M)+g(d_2, M)\} + \{g(d_1, M-\Delta M)+g(d_2, M-\Delta M)\}}{\Delta M^2} \tag{45}$$

Hence, to determine the compliance matrix for two parallel double-sided cracks, the strain energy release rate should be calculated numerically, using the finite element simulation of the cracked beam for different combinations of crack depth and loading conditions. For example, Eqs. (42) and (45) indicate that, to calculate C_{33} and C_{55} for the second crack, one has to evaluate g function under conditions, summarized in Tables 1 and 2 respectively.

3. Numerical examples

The finite element formulation described in the previous section was implemented in a computer program to calculate the natural frequencies and mode shapes of the Timoshenko beam containing cracks. In order to validate the proposed finite element formulation, a series of numerical examples were studied by the computer program. For some cracked beams of different dimensions, boundary conditions and crack locations. First, the vibration characteristics of a beam with two same-line double-sided cracks are verified by the results presented by previous researchers. The numerical examples then are developed for the case of two parallel double-sided cracked beams. For a more convenient presentation of the results, the frequency ratio (*FR*) and the relative crack depth (*CR*), are defined as

$$FR = \frac{f_{cr}}{f} \tag{46}$$

Table 1 The strain energy release rate functions, for calculation of C_{33}

g function	Cracks depth	Shear force
$g(d_1, P+\Delta P)$	$a, \frac{(\sqrt{3}+1)}{2\sqrt{3}}d$	$P+\Delta P$
$g(d_2, P+\Delta P)$	$a, \frac{(\sqrt{3}-1)}{2\sqrt{3}}d$	$P+\Delta P$
$g(d_1, P)$	$a, \frac{(\sqrt{3}+1)}{2\sqrt{3}}d$	P
$g(d_2, P)$	$a, \frac{(\sqrt{3}-1)}{2\sqrt{3}}d$	P
$g(d_1, P\text{-}\Delta P)$	$a, \frac{(\sqrt{3}+1)}{2\sqrt{3}}d$	$P\text{-}\Delta P$
$g(d_2, P-\Delta P)$	$a, \frac{(\sqrt{3}-1)}{2\sqrt{3}}d$	$P\text{-}\Delta P$

Table 2 The strain energy release rate functions, for calculation of C_{55}

g function	Cracks depth	Bending moment
$g(d_1, M+\Delta M)$	$a, \frac{(\sqrt{3}+1)}{2\sqrt{3}}d$	$M+\Delta M$
$g(d_2, M+\Delta M)$	$a, \frac{(\sqrt{3}-1)}{2\sqrt{3}}d$	$M+\Delta M$
$g(d_1, M)$	$a, \frac{(\sqrt{3}+1)}{2\sqrt{3}}d$	M
$g(d_2, M)$	$a, \frac{(\sqrt{3}-1)}{2\sqrt{3}}d$	M
$g(d_1, M\text{-}\Delta M)$	$a, \frac{(\sqrt{3}+1)}{2\sqrt{3}}d$	$M\text{-}\Delta M$
$g(d_2, M-\Delta M)$	$a, \frac{(\sqrt{3}-1)}{2\sqrt{3}}d$	$M\text{-}\Delta M$

$$CR = \frac{a}{h} \tag{47}$$

where f_{cr} and f are the natural frequencies of cracked and un-cracked beam, respectively. It should be noted that a total number of twelve elements are employed in each finite element analysis of following numerical examples.

3.1 Simply-supported beam with same-line double-sided crack

The first example considers a simply supported beam of length 57.5 cm with a rectangular cross section of 3.175? .9525 cm^2. It contains two cracks of equal depth, which are located exactly at the mid span of beam. The assumed material properties are: E = 200 GPa, ρ = 7850 kg/m^3 and v = 0.3, which are taken from research work of Christides and Barr 1984 and Shen and Pierre 1990 for verification purposes. The boundary conditions for this case at both simply-supported ends are

$$\psi = 0, \quad \varphi' = 0 \tag{48}$$

Figs. 4 and 5 show the variation of frequency ratio for the first and third natural modes, as a function of the relative crack depth CR. It is seen that in both cases there is a reduction in the frequency ratio when the increased crack depth lowers the flexural stiffness in the beam. The obtained results are also compared with similar results presented by Christides and Barr 1984 and Shen and Pierre 1990. The very good consistency existing between our results and those given in above references can be used as verification for the finite element formulation described in this paper.

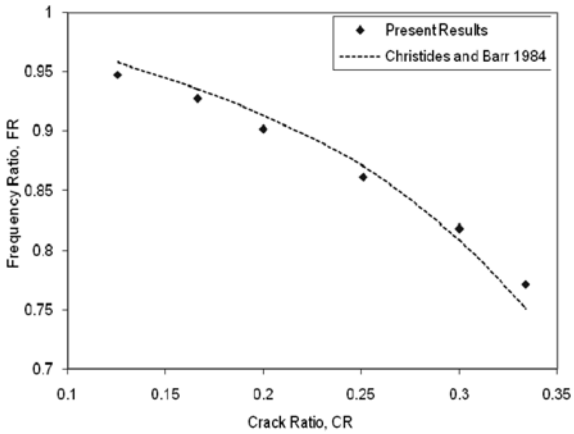

Fig. 4 Variation of the first frequency ratio for the same-line double-cracked beam, versus relative depth of cracks

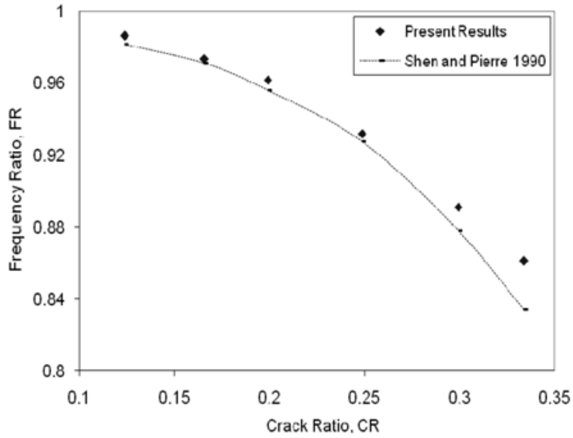

Fig. 5 Variation of the third frequency ratio for the same-line double-cracked beam, versus relative depth of cracks

3.2 Clamped-free beam with two same-line double-sided cracks

Consider a cantilever Timoshenko beam of length 20 cm and uniform cross-section of 2.5 × 0.78 cm^2, with two edge cracks of unequal depth. In this example the first crack has a fixed depth of $h/2$ but the second crack a can take different values. It was assumed that the beam was made of steel with the following material properties: $E = 216$ GPa, $\rho = 7850$ kg/m^3 and $\nu = 0.28$. The boundary condition at the clamped root of this structure can be given by

$$\phi = 0, \quad \psi = 0 \tag{49}$$

Fig. 6 shows the variation of the first relative frequency (FR) as a function of the second crack depth, normalized as $2a/h$. When the depth of second crack vanishes ($a = 0$) the problem turns into a single edge cracked beam for which the numerical result have been reported by Kisa and Bandon 2000 as $f_{cr}/f = 0.98$. Fig. 6 shows the very good agreement existing between the frequency ratio given in above reference and our results for $a = 0$.

3.3 Clamped-free beam with two parallel double-sided cracks

For numerical simulation of parallel double-sided cracks, a cantilever cracked beam of length $L = 95.25$ cm with rectangular cross section of 3.175? .9525 cm^2 is considered. The material properties employed in this example are: $E = 200$ GPa, $\rho = 7850$ Kg/m^3 and $\nu = 0.3$. The first crack with a varying depth is located at $L/3$ from the clamped end. The second crack of fixed depth $d = h/2$ is introduced at a varying distance, Δ from the first one. Based on the described finite element formulation, the variation of first frequency ratio of cracked beam is depicted in Fig. 7, as a function of the distance between cracks and their relative depth (a/d).

It is seen that for any distance Δ, there is a drop in the frequency ratio FR when the relative crack length a/d increases. The reduction in FR is more significant when the distance between the two cracks Δ tends toward zero. This is because the interaction between the stress fields of two cracks

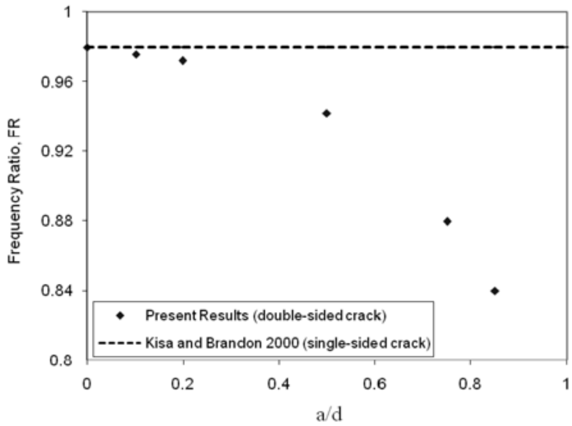

Fig. 6 Variations of the first frequency ratio for the same-line double-cracked beam, versus relative depth of cracks

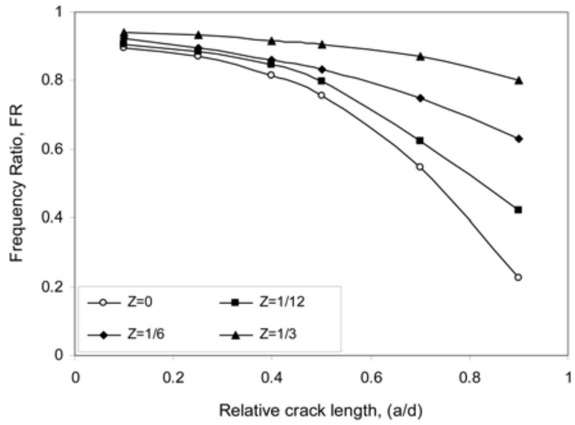

Fig. 7 Variation of *FR* with *a/d* for different values of distance between the two double-sided cracks, $Z = \Delta/L$

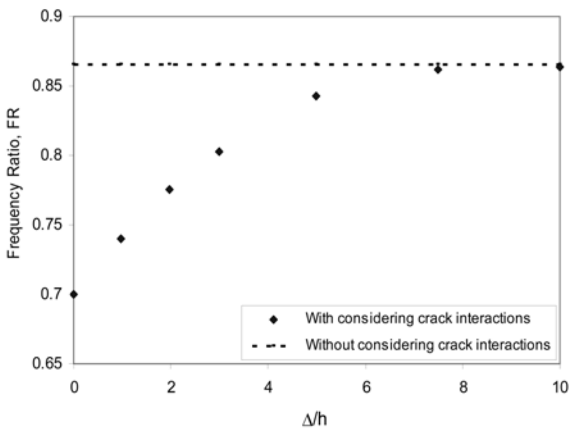

Fig. 8 Effect of distance between the two cracks on *FR* for the first mode of vibration

and hence the reduction in the beam overall stiffness become more considerable when the crack lines approach each other. However, for $\Delta = L/3$ there is only little effect from the cracks interaction. Fig. 8 shows how the relative crack distance Δ/h affects the frequency ratio FR of the first mode of vibration for $d/h = a/d = 0.5$. According to this figure, the distance between the cracks has no influence on the frequency ratio when Δ is greater than 10 times the beam depth *h*. Also shown in Fig. 8 is the frequency ratio for two independent edge cracks. Because in this case no interaction is considered between the stress field of the cracks, the distance between the cracks has no influence on the overall frequency ratio. It is seen that for $\Delta/h > 10$ of two interacting cracks approach the results obtained for two independent edge cracks.

4. Conclusions

A finite element procedure was formulated for a single-edge cracked Timoshenko beam. The formulation was then extended for a beam element containing two double-sided cracks, in which the interaction between the crack tip stresses was taken into account. This way, a new cracked element was addressed in this paper, and the methodology was then validated numerically on the same-line double cracked beams, having cracks of varying depth and different locations. The numerically calculated natural frequencies of Timoshenko beam with various boundary conditions indicated that the obtained results fall close to those presented in previous investigations. The application of present method for the case of parallel double-sided cracks indicated that the relative position of two cracks affects significantly the change in the beam natural frequencies. Indeed, any decrease in the natural frequencies is larger, if the interaction between adjacent cracks is considered by the finite element formulation. Based on the results of this study, when the distance between the cracks increases, the vibration frequencies tend to those values, which are obtained based on the previous finite element simulation with two independent cracks. Consequently, one can conclude that, considering the interaction between the crack tip stresses is important for more accurate prediction of dynamic charactereristic of beams containing double-sided cracks.

References

Abraham, O. and Brandon, J.A. (1995), "The modeling of the opening and closure of a crack", *J. Vib. Acoustic.,* **117**, 370-377.

Anifantis, N. and Dimarogonas, A.D. (1980), "Stability of columns with single crack subjected to follower and vertical loads", *Int. J. Solids Struct.*, **19**, 281-291.

Boltezar, M., Strancar, B. and Kuhelj, A. (1998), "Identification of transverse crack location in flexural vibration of free-free beams", *J. Sound Vib.*, **211**, 729-734.

Chen, L.W. and Chen, C.L. (1988), "Vibration and stability of cracked thick rotating blades", *Comput. Struct.*, **28**, 67-74.

Chen, L.W. and Jeng, C.J. (1993), "Vibration analysis of cracked pre-twisted blades", *Comput. Struct.*, **46**, 133-140.

Chondors, T.G. and Dimarogonas, A.D. (1980), "Identification of cracks in welded joints of complex structures", *J. Sound Vib.*, **69**, 531-538.

Christides, S. and Barr, A.D. (1984), "On-Dimensional theory of cracked Bernoulli-Euler beams", *Int. J. Mech. Sci.*, **26**, 639-648.

Dansheng, W., Hongping, Z., Chuanyao, C. and Yong, X. (2007), "An impedance analysis for crack detection in the Timoshenko beam based on the anti- resonance technique", *Acta Mech. Solida Sin.*, **20**, 228-235.

Dentsoras, A.J. and Dimarogonas, A.D. (1983), "Resonanse controlled fatigue crack propagation", *Eng. Fract. Mech.*, **17**, 381-386.

Dimarogonas, A.D. (1976), *Vibration Engineering*, St. Paul (Minnesota), West publisher.

Dimarogonas, A.D. and Massouros, G. (1980), "Torsional vibration of a shaft with a shaft with a circumferential crack", *Eng. Fract. Mech.*, **15**, 439-444.

Douka, E. and Hadjileontaidis, L.J. (2005), "Time-frequency analysis of the free vibration response of a beam with a breathing crack", *NDT&E Int.,* **38**, 3-10.

Ghondros, T.G. and Dimarogonas, A.D. (1979), "Identification of cracks in circular plates welded at the contour", *Proceedings of ASME Design Eng. Tech. Conf.*, Et. Louis, U.S.A., 79-DET-106.

Gounaris, G. and Dimarogonas, A.D. (1988), "A finite element of a cracked prismatic beam for structural analysis", *Comput. Struct.*, **28**, 309-313.

Gounaris, G., Anifantis, N. and Dimarogonas, A.D. (1991), "Dynamics of cracked hollow beams", *Eng. Fract. Mech.,* **39**, 931-940.

Haisty, B. and Springer, W. (1988), "A general beam element for use in damage assessment of complex structures", *J. Vib. Acoust. Stress Reliab.*, **251**, 13-38.

Khaji, N., Shafiei, M. and Jalalpour, M. (2009), "Closed-form solutions for crack detection problem of Timoshenko beams with various boundary conditions", *Int. J. Mech. Sci.*, **51**, 667-681.

Kisa, M. and Brandon, J. (2000), "The effects of closure of cracked on the dynamics of a cracked cantilever beam", *J. Sound Vib.*, **238**, 1-18.

Loya, J.A., Rubio, L. and Fernández-Sáez, J. (2006), "Natural frequencies for bending vibrations of Timoshenko cracked beams", *J. Sound Vib.*, **290**, 640-653.

Qian, G.L., Gu, S.N. and Jiung, J.S. (1990), "The dynamic behavior and crack detection of a beam with a crack", *J. Sound Vib.*, **138**, 233-243.

Shen, M.H. and Pierre, C. (1990), "Natural modes of Bernoulli-Euler beams with symmetric cracks", *J. Sound Vib.*, **138**, 115-134.

Notations

$[C]$	:	Local flexibility matrix
$[C^*]$	:	Constant constraint matrix
$[K]$	:	Stiffness matrix
$[K']$	:	Reduced stiffness matrix
$[K_{cr}]$	:	Stiffness matrix of crack
$[M]$	:	Mass matrix
$[M']$	:	Reduced mass matrix
$\{P\}$	:	General loading vector
$\{\xi\}$	:	Nodal coordinate vector
a, d	:	Crack depth
A	:	Cross-section area of beam
F	:	Natural frequency
f_{cr}	:	Natural frequency of cracked beam
G	:	Modulus of rigidity
H	:	Beam height
I	:	Second moment of area
G	:	Energy release rate
K	:	Shear correction factor
K_1	:	Translational stiffness
K_2	:	Rotational stiffness
L	:	Element length
M	:	Bending moment
P	:	Shear force
R	:	Rotary inertia parameter
S	:	Shear deformation parameter
T	:	Kinetic energy
U	:	Strain energy
Δ	:	Distance between double cracks
E	:	Modulus of elasticity
K_{I}	:	Mode I stress intensity factor
K_{II}	:	Mode II stress intensity factor
K_{III}	:	Mode III stress intensity factor
β_1	:	Translational flexibility parameter
β_2	:	Rotational flexibility parameter
ϕ	:	Bending slope
η	:	Non-dimensional coordinate
λ	:	Frequency parameter
ν	:	Poisson's ratio
ω	:	Natural frequency
ρ	:	Mass density of material
ψ	:	Non-dimensional deflection