*Structural Engineering and Mechanics, Vol. 33, No. 2 (2009) 193-213*
DOI: http://dx.doi.org/10.12989/sem.2009.33.2.193 

# Bending, buckling and vibration analyses of nonhomogeneous nanotubes using GDQ and nonlocal elasticity theory

S.C. Pradhan† and J.K. Phadikar

*Department of Aerospace Engineering, Indian Institute of Technology, Kharagpur, West Bengal 721 302, India*

*(Received September 20, 2008, Accepted August 2, 2009)*

**Abstract.** In this paper structural analysis of nonhomogeneous nanotubes has been carried out using nonlocal elasticity theory. Governing differential equations of nonhomogeneous nanotubes are derived. Nanotubes include both single wall nanotube (SWNT) and double wall nanotube (DWNT). Nonlocal theory of elasticity has been employed to include the scale effect of the nanotubes. Nonlocal parameter, elastic modulus, density and diameter of the cross section are assumed to be functions of spatial coordinates. General Differential Quadrature (GDQ) method has been employed to solve the governing differential equations of the nanotubes. Various boundary conditions have been applied to the nanotubes. Present results considering nonlocal theory are in good agreement with the results available in the literature. Effect of variation of various geometrical and material parameters on the structural response of the nonhomogeneous nanotubes has been investigated. Present results of the nonhomogeneous nanotubes are useful in the design of the nanotubes.

**Keywords:** nanotubes; differential quadrature method; nonhomogeneous; bending; vibration; buckling.

## 1. Introduction

Nano sized tubes hold an important area of research for the future structural developments and design in modern structural engineering due to their novel mechanical properties. These nano-tubes have got highly promising applications in nanotube-reinforced ultra-strong composites, MEMS/NEMS devices and smart structures. Since the discovery of carbon nanotubes (CNT) (Iijima 1991) good amount of research work are being reported in the literature (Thostenson *et al.* 2001, Ronald *et al.* 2007). Conducting experiments with nanoscale size specimens is found to be difficult and costly. Therefore, development of appropriate mathematical models for CNTs became an important issue. Generally, three approaches have been developed to model CNTs. These approaches are (a) atomistic (Ball 2001 and Baughman *et al.* 2002) (b) hybrid atomistic-continuum mechanics (Bodily and Sun 2003, Li and Chou 2003) and (c) continuum mechanics. Atomistic approach uses (i) classical molecular dynamics simulation, (ii) tight binding molecular dynamics and (iii) density functional models. Atomistic approach is highly computationally intensive and very expensive. In

† Assistant Professor, Corresponding author, E-mail: scp@aero.iitkgp.ernet.in

hybrid approach CNTs are represented by structural elements. The strain energy is considered to be equivalent of the steric energy. This hybrid approach is computationally less expensive than the atomistic approach. Some researchers employed continuum mechanics approach for the analysis of CNTs. Here single wall carbon nanotubes are modeled by a continuum beam or cylindrical shell elements. This continuum mechanics approach is ideal in analyzing large scale systems containing CNTs. For multi walled CNTs a multi beam model has been developed (Yoon *et al.* 2002, 2003). For more accurate analysis shear deformation theories of beam have been proposed (Wang *et al.* 2006, Wang and Vardan 2005, Aydogdu 2008). Eringen (1983, 2002) developed nonlocal elasticity theory. In this nonlocal elasticity theory scale effect of structures is taken into account. While classical elasticity theory is indifferent to scale effects. Peddieson (2003) proposed analysis of nanostructures based on Eringen's nonlocal elasticity theory. The nonlocal elasticity theory has been further applied to the static and dynamic analysis of single walled and multi walled CNTs (Wang *et al.* 2006, Wang and Varadan 2006, Aydogdu 2009, Artan and Tepe 2008, Aydogdu 2009, Pradhan and Sarkar 2009, Pradhan and Murmu 2009, Pin *et al.* 2007).

In all these above mentioned work, analysis of the homogeneous nanotubes (CNT) has been carried out. Seeman (1999) found that nonhomogeneous nanotubes are frequently encountered in DNA nanotechnology applications. He observed that different proteins are chemically glued to form nano-architectures of the nanotubes. Rothemund *et al.* (2004) reported that DNA nanotubes are similar in size and shape as carbon nanotubes. They suggested that DNA nanotubes could be easily modified and connected to other structures. Important applications of DNA nanotubes include nano-wires and nano-pipes. Nonhomogeneous nanotubes can be addressed similar to CNTs. To the authors' best knowledge no work has been addressed for the analysis of nonhomogeneous nanotubes employing continuum mechanics approach. Therefore in the present work bending, vibration and buckling analysis of nonhomogeneous nanotubes have been carried out and results are discussed.

## 2. Formulation

### 2.1 Nonhomogeneous nanotube

Non-homogeneity imparts additional complexity to the analysis of the nanotubes in the following ways. The material properties viz elastic modulus and density are functions of spatial coordinates. For the beam structure these variations are assumed be in the axial direction. The internal characteristic lengths are different for different materials. Thus variation of nonlocal parameter along the nanotube axial direction needs to be considered. Further different bond lengths will result in variation of the nanotube diameter. Therefore in the analysis of nonhomogeneous nanotubes variation of elastic modulus, density, nonlocal parameter and nanotube diameter along the axial direction are to be taken included. In the present study a nonhomogeneous nanotube is modeled by nonlocal elastic continuum Euler-Bernoulli beam of annular crossection where elastic modulus, density, scale factor and diameter of nanotube are assumed to be functions of axial coordinates. For modeling the double walled nonhomogeneous nanotube multi Euler-Bernoulli beam has been employed.

According to Eringen (1983) the nonlocal constitutive behavior of a Hookean solid can be represented by the following differential constitutive relation

$$(1 - \tau^2 \ell^2 \nabla^2)\sigma = \boldsymbol{t},\ \tau = \frac{\boldsymbol{e_0 a}}{\ell} \tag{1}$$

Here $e_0$ is a material constant, '$a$' and $\ell$ are external and internal characteristic lengths respectively. $\boldsymbol{t}$ is the macroscopic stress at a point which is related to strain by generalized Hooke's law

$$t(x) = C(x) : \varepsilon(x) \tag{2}$$

where $C$ is the fourth order elasticity tensor and ':' denotes the double dot product. The values of $e_0$ and '$a$' depend on the crystal structure in lattice dynamics and the nature of physics under investigation. Eringen (1983) proposed the value of $e_0$ to be 0.39 based on a study on the comparison of Rayleigh surface wave via nonlocal continuum mechanics and lattice dynamics. Wang and Hu (2005) proposed the value of $e_0$ to be 0.288 in the determination of the dispersion waves via elastic beam theories and the MD method. '$a$' being the internal characteristic length, have been assumed to be has a value of 0.14 nm (length of the C-C bond) by most of the researchers (Aydogdu 2008, 2009, Lu *et al.* 2007, Pradhan and Murmu 2009, Wang and Varadan 2006). For further information of the determination of $e_0$ and '$a$', please refer the work done by Wand and Wang (2007). It is assumed that nonlocal behavior is significant in axial direction of the nanotube. Thus, nonlocal constitutive relation mentioned in Eq. (1) takes the following form for an isotropic Euler-Bernoulli beam.

$$\sigma(x) - \mu \frac{d^2 \sigma(x)}{dx^2} = E\varepsilon(x) \tag{3}$$

Here, $\mu = (e_0 a)^2$ is the scale factor. $E$ is the elastic modulus. As it is a differential relation, non-homogeneity can be incorporated in this equation. For nonhomogeneous case this differential relation (3) takes the following form

$$\sigma(x) - \mu(x) \frac{d^2 \sigma}{dx^2} = E(x)\varepsilon(x) \tag{4}$$

From the definition of resulting bending moment and strain displacement relation in Euler-Bernoulli beam

$$E = \int_A z\sigma dA \tag{5}$$

$$\varepsilon = -z \frac{\partial^2 w}{\partial x^2} \tag{6}$$

Using Eqs. (4), (5) and (6) we get following moment-displacement relation

$$M(x) - \mu(x) \frac{d^2 M(x)}{dx^2} = E(x) I(x) \frac{d^2 w}{dx^2} \tag{7}$$

### *2.2 Bending Analysis of Single Walled Nanotube (SWNT)*

For an Euler-Bernoulli beam acted by distributed load $q(x)$, the equilibrium equation is expressed as

$$q(x) = \frac{d^2M(x)}{dx^2} \tag{8}$$

Differentiating twice Eq. (7) and substituting from Eq. (8), we get the following governing equation for bending of nonhomogeneous beam

$$\Omega_5^{bn}\frac{d^4w}{dx^4} + \Omega_4^{bn}\frac{d^3w}{dx^3} + \Omega_3^{bn}\frac{d^2w}{dx^2} + \Omega_0^{bn} = 0 \tag{9}$$

$\Omega_0^{bn}, \Omega_3^{bn}, \Omega_4^{bn}$ and $\Omega_5^{bn}$ are defined in Appendix.

### 2.3 Bending Analysis of Double Walled Nanotube (DWNT)

In the analysis of multi-walled carbon nanotubes multi beam models have been used by various researchers (Yoon *et al.* 2003, Ru 2000). Here all concentric single walled carbon nanotubes are modeled by an individual elastic beam. Displacements of the adjacent tubes are coupled due to van der Waals forces. The van der Waals forces have been modeled by Winkler-type elastic foundations whose elastic coefficient are determined by the following expression

$$c_j = \frac{320 \times (2R_j) erg/cm^2}{0.16\Delta^2} \qquad j = 1, 2, 3, \ldots, N-1 \tag{10}$$

$R_j$ is the center line radius for the $j^{\text{th}}$ tube and $\Delta$ is the length of Carbon-Carbon bond.

In the case of nonhomogeneous nanotube $R_j$ and $\Delta$ are no longer constants but are functions of $x$ coordinate. So here a variable Winkler elastic foundation modulus $C(x)$ has been used to develop the governing differential equations for DWNT. Due to the presence of van der Waals forces the effective vertical distributed load on the tubes can be written as

$$q_n(x) = q(x) - C(x) \times (w_2 - w_1) \tag{11}$$

$q_n(x)$, $w_1$, and $w_2$ denote new effective distributed load, displacement for the first nanotube and displacement for the second nanotube respectively. $C(x)$ represents variable elastic foundation modulus. Replacing $q(x)$ by $q_n(x)$ in Eq. (9) we find the governing equation for tube corresponding to the displacement $w_1$

$$\Phi1_5^{bn}\frac{d^4w_1}{dx^4} + \Phi1_4^{bn}\frac{d^3w_1}{dx^3} + \Phi1_3^{bn}\frac{d^2w_1}{dx^2} + \Phi1_2^{bn}\frac{dw_1}{dx} + \Phi1_1^{bn}w_1 + \Phi1_0^{bn} + \Phi2_3^{bn}\frac{d^2w_2}{dx^2} + \Phi2_2^{bn}\frac{dw_2}{dx}$$
$$+ \Phi2_1^{bn}w_2 = 0 \tag{12}$$

$\Phi1_0^{bn}, \Phi1_1^{bn}, \Phi1_2^{bn}, \Phi1_3^{bn}, \Phi1_5^{bn}, \Phi2_1^{bn}, \Phi2_2^{bn}$ and $\Phi2_3^{bn}$ are defined in the Appendix.

The governing equation for the second tube can be obtained by interchanging $w_1$ and $w_2$ in the previous equation and setting $q(x)$ and all its derivatives equal to zero. It should be noted that $I(x)$ should be modified for the second nano-tube according to its diameter.

### 2.4 Vibration analysis SWNT

For free vibration we have the following equation for equilibrium

$$\frac{\partial^2 M}{\partial x^2} = \rho(x)A(x)\frac{\partial^2 w}{\partial t^2} \tag{13}$$

$\rho(x)$ and $A(x)$ denote density of the material and area of cross section, respectively. Differentiating twice Eq. (7) and substituting in Eq. (13), we get the following governing equation for vibration of nonhomogeneous beam

$$\Upsilon_5^{vb}\frac{\partial^4 w}{\partial x^4} + \Upsilon_4^{vb}\frac{\partial^3 w}{\partial x^3} + \Upsilon_3^{vb}\frac{\partial^2 w}{\partial x^2} + \Theta_5^{vb}\frac{\partial^4 w}{\partial x^2 \partial t^2} + \Theta_4^{vb}\frac{\partial^3 w}{\partial x \partial t^2} + \Theta_3^{vb}\frac{\partial^2 w}{\partial t^2} = 0 \tag{14}$$

$\Upsilon_3^{vb}, \Upsilon_4^{vb}, \Upsilon_5^{vb}, \Theta_3^{vb}, \Theta_4^{vb}$ and $\Theta_5^{vb}$ are defined in the Appendix. The above equation is converted to an eigenvalue problem by assuming the periodic function

$$w(x,t) = w(x)e^{i\omega t} \tag{15}$$

Substituting Eq. (15) into Eq. (14), we have

$$\Omega_5^{vb}\frac{\partial^4 w}{\partial x^4} + \Omega_4^{vb}\frac{\partial^3 w}{\partial x^3} + \Omega_3^{vb}\frac{\partial^2 w}{\partial x^2} + \Omega_2^{vb}\frac{\partial w}{\partial x} + \Omega_1^{vb} w = 0 \tag{16}$$

$\Omega_1^{vb}, \Omega_2^{vb}, \Omega_3^{vb}, \Omega_4^{vb}$ and $\Omega_5^{vb}$ are defined in the Appendix.

### *2.5 Vibration analysis of DWNT*

With the similar argument as in the case of the bending, the new inertia force due to the presence of the Winkler elastic foundation representing van der Waals interaction, becomes

$$F^n = \rho(x)A(x)\frac{\partial^2 w}{\partial t^2} - C(x)(w_2 - w_1) \tag{17}$$

Replacing the old inertia force by this new one and using Eqs. (7) and (13), we obtain the governing equation for vibration of DWNT

$$\Upsilon 1_5^{vb}\frac{\partial^4 w_1}{\partial x^4} + \Upsilon 1_4^{vb}\frac{\partial^3 w_1}{\partial x^3} + \Upsilon 1_3^{vb}\frac{\partial^2 w_1}{\partial x^2} + \Upsilon 1_2^{vb}\frac{\partial w_1}{\partial x} + \Upsilon 1_1^{vb} w_1 + \Upsilon 2_3^{vb}\frac{\partial^2 w_2}{\partial x^2} + \Upsilon 2_2^{vb}\frac{\partial w_2}{\partial x}$$
$$+ \Upsilon 2_1^{vb} w_2 + \Theta_5^{vb}\frac{\partial^4 w_1}{\partial x^2 \partial t^2} + \Theta_4^{vb}\frac{\partial^3 w_1}{\partial x \partial t^2} + \Theta_3^{vb}\frac{\partial^2 w_1}{\partial t^2} = 0 \tag{18}$$

$\Upsilon 1_1^{vb}, \Upsilon 1_2^{vb}, \Upsilon 1_3^{vb}, \Upsilon 1_4^{vb}, \Upsilon 1_5^{vb}, \Upsilon 2_1^{vb}, \Upsilon 2_2^{vb}, \Upsilon 2_3^{vb}, \Upsilon 2_4^{vb}$ and $\Upsilon 2_5^{vb}$ are defined in Appendix. Eq. (18) represents governing equation for one tube corresponding to the displacement $w_1$. For the other tube governing equation can be obtained by interchanging $w_1$ and $w_2$ in Eq. (18) and modifying $A(x)$ and $I(x)$ according to its diameter. Considering the periodic relation mentioned in Eq. (15) we obtain the corresponding eigen-value equation for Eq. (18)

$$\Phi 1_5^{vb}\frac{\partial^4 w_1}{\partial x^4} + \Phi 1_4^{vb}\frac{\partial^3 w_1}{\partial x^3} + \Phi 1_3^{vb}\frac{\partial^2 w_1}{\partial x^2} + \Phi 1_2^{vb}\frac{\partial w_1}{\partial x} + \Phi 1_1^{vb} w_1 + \Phi 2_3^{vb}\frac{\partial^2 w_2}{\partial x^2} + \Phi 2_2^{vb}\frac{\partial w_2}{\partial x} + \Phi 2_1^{vb} w_2 = 0 \tag{19}$$

where $\Phi 1_1^{vb}, \Phi 1_2^{vb}, \Phi 1_3^{vb}, \Phi 1_4^{vb}, \Phi 1_5^{vb}, \Phi 2_1^{vb}, \Phi 2_2^{vb}$ and $\Phi 2_3^{vb}$ are defined in Appendix. For the other tube corresponding equation can be obtained by interchanging $w_1$ and $w_2$ in Eq. (19) and modifying $A(x)$ and $I(x)$ according to its diameter.

### 2.6 Buckling analysis SWNT

The equilibrium equation for buckling of an Euler-Bernoulli beam under axial compressive load $P$ is given by

$$\frac{d^2M}{dx^2} = P\frac{d^2w}{dx^2} \tag{20}$$

Differentiating twice Eq. (7) and substituting from Eq. (20) we get governing equation for buckling of nonhomogeneous nanotube

$$\Omega_5^{bk}\frac{d^4w}{dx^4} + \Omega_4^{bk}\frac{d^3w}{dx^3} + \Omega_3^{bk}\frac{d^2w}{dx^2} = 0 \tag{21}$$

$\Omega_3^{bk}\Omega_4^{bk}$ and $\Omega_5^{bk}$ are defined in the Appendix.

### 2.7 Buckling analysis DWNT

For buckling of double walled nanotube the equilibrium equation including the effect of van der Waals forces becomes

$$\frac{d^2M}{dx^2} = P\frac{d^2w}{dx^2} - C(x)(w_2 - w_1) \tag{22}$$

Using Eqs. (7) and (22) the governing differential equation corresponding to tube with displacement $w_1$ becomes

$$\Phi 1_5^{bk}\frac{d^4w_1}{dx^4} + \Phi 1_4^{bk}\frac{d^3w_1}{dx^3} + \Phi 1_3^{bk}\frac{d^2w_1}{dx^2} + \Phi 1_2^{bk}\frac{dw_1}{dx} + \Phi 1_1^{bk}w_1\Phi 2_3^{bk}\frac{d^2w_2}{dx^2} + \Phi 2_2^{bk}\frac{dw_2}{dx} + \Phi 2_1^{bk}w_2 = 0 \tag{23}$$

$\Phi 1_1^{bk}, \Phi 1_2^{bk}, \Phi 1_3^{bk}, \Phi 1_4^{bk}, \Phi 1_5^{bk}, \Phi 2_1^{bk}, \Phi 2_2^{bk}$ and $\Phi 2_3^{bk}$ are defined in the Appendix. Equation corresponding to other tube can be obtained by interchanging $w_1$ and $w_2$ in Eq. (23) and modifying $I(x)$ according to its diameter.

### 2.8 Boundary conditions

All four classical boundary conditions have been considered in the analysis. These are (i) Simply supported - Simply supported (S-S) (ii) Clamped – Free (C-F) (iii) Clamped – Simply supported (C-S) and (iv) Clamped – Clamped (C-C). As we are solving fourth order differential equations (Eqs. 9, 16, 21) on $w$ we need four boundary conditions in $w$ in each case. The displacement and stress boundary conditions associated with S-S, C-F, C-S and C-C cases are listed in Table 1.

Here $L$ denotes the length of the beam. To get four boundary conditions on $w$ we must convert the stress boundary conditions to corresponding displacement boundary conditions. However for C-C cases we have four inherent displacement boundary conditions and no stress boundary conditions,

Table 1 Displacement and Stress boundary conditions for various classical end conditions

| End boundary condition | Displacement Boundary condition | Stress Boundary condition |
|---|---|---|
| S-S | $w(x)\|_{x=0} = 0,\ w(x)\|_{x=L} = 0$ | $M(x)\|_{x=0} = 0, M(x)\|_{x=L} = 0$ |
| C-F | $w(x)\|_{x=0} = 0,\ \left.\frac{dw(x)}{dx}\right\|_{x=0} = 0$ | $M(x)\|_{x=L} = 0,$ $\left.\frac{dM(x)}{dx}\right\|_{x=L} = 0$ |
| C-S | $w(x)\|_{x=0} = 0,\ \left.\frac{dw(x)}{dx}\right\|_{x=0} = 0,$ $w(x)\|_{x=L} = 0$ | $M(x)\|_{x=L} = 0$ |
| C-C | $w(x)\|_{x=0} = 0,\ \left.\frac{dw(x)}{dx}\right\|_{x=0} = 0,$ $w(x)\|_{x=L} = 0,\ \left.\frac{dw(x)}{dx}\right\|_{x=L} = 0,$ | |

so no need of any conversion. The conversion from stress to displacement boundary conditions must be done by suitable stress-strain relationship given by nonlocal elasticity theory. We can use the moment-displacement relationship of Eq. (7) to get these nonlocal boundary conditions.

Conversion of $M(x)|_{x=0} = 0$ for S-S boundary condition:

Using Eqs. (7) and (8) and putting $x = 0$,

$$\left.\frac{d^2w(x)}{dx^2}\right|_{x=0} = \frac{\mu(0)q(0)}{E(0)I(0)} \tag{24}$$

Conversion of $M(x)|_{x=L} = 0$ for S-S, C-F, C-S boundary conditions:

Using Eqs. (7) and (8) and putting $x = L$,

$$\left.\frac{d^2w}{dx^2}\right|_{x=L} = \frac{\mu(L)q(L)}{E(L)I(L)} \tag{25}$$

Conversion of $\left.\frac{dM(x)}{dx}\right|_{x=L}$ for C-F boundary condition:

Differentiating Eq. (7), substituting in Eq. (8) and putting $x = L$ we obtain,

$$\left.\frac{d^3w}{dx^3}\right|_{x=L} = -\frac{1}{E(L)I(L)}\left.\frac{d(\mu(x)q(x))}{dx}\right|_{x=L} - \mu(L)q(L)\left.\frac{d(1/E(x)I(x))}{dx}\right|_{x=L} \tag{26}$$

Thus we get four boundary conditions on $w$ for all the boundary conditions considered in the present analysis.

## 3. GDQ Method

The governing equations for bending, vibration and buckling of nonhomogeneous SWNT are presented in Eqs. (9), (16) and (21), respectively. Further, the governing equations for bending, vibration and buckling of nonhomogeneous DWNT are presented in Eqs. (12), (19) and (23), respectively. These equations have been solved by the differential quadrature method (DQM) as introduced by Bellman *et al.* (1972). The DQ method has been proved to be an efficient numerical technique for the solution of initial and boundary value problems. Bert *et al.* (1988) first employed this method to solve structural mechanics problems. This method has also been applied successfully to a variety of structural problems (Bert and Malik 1996, Shu 2000). The fundamental concept of DQ method is to approximate the partial derivative of a function with respect to a space variable at a grid point by the weighted linear sum of the function values at all grid points in the whole domain. In the present case the computational domain for the problem is $0 \le x \le L$. So we have

$$\left.\frac{d^n g}{dx^n}\right|_{x = x_i} = \sum_{j=1}^{N} A_{ij}^n g(x_j) \tag{27}$$

$N$ is the number of grid points. $g$ is the function to be approximated. $A_{ij}^n$ are DQ weighting coefficients which can be calculated from the coordinates of the grid points as follows

$$A_{ij}^1 = \frac{M(x_i)}{(x_i - x_j)M(x_j)}, \quad \text{for} \quad i \ne j \tag{28}$$

$$A_{ii}^1 = -\sum_{j=1, j\ne i}^{N} A_{ij}^1 \tag{29}$$

$$A_{ij}^m = m\left(A_{ii}^{m-1} - \frac{A_{ij}^{m-1}}{x_i - x_j}\right), \quad \text{for} \quad i \ne j \tag{30}$$

$$A_{ii}^m = -\sum_{j=1, j\ne i}^{N} A_{ij}^m \tag{31}$$

for $\begin{matrix} i,j = 1, 2, 3, \ldots, N \\ m = 2, 3, 4, \ldots, N-1 \end{matrix}$

Here

$$M(x_i) = \prod_{k=1, k\ne i}^{N} (x_i - x_k) \tag{32}$$

To choose grid point distribution the well accepted mesh governed by the following rule for calculating interpolation points has been adopted

$$x_i = \frac{1}{2}\left[1 - \cos\left(\frac{\pi i}{n}\right)\right], \quad i = 0, \ldots, n \tag{33}$$

By applying DQ rule to Eqs. (9), (16) and (21) we obtain following discretized formulation for Eqs. (9), (16) and (21), respectively.

$$\Omega_5^{bn}(X_i)\sum_{k=1}^{N}A_{ik}^4 w_i+\Omega_4^{bn}(X_i)\sum_{k=1}^{N}A_{ik}^3 w_i+\Omega_3^{bn}(X_i)\sum_{k=1}^{N}A_{ik}^2 w_i+\Omega_0^{bn}(X_i) = 0 \quad (34)$$

$$\Omega_5^{vb}(X_i)\sum_{k=1}^{N}A_{ik}^4 w_i+\Omega_4^{vb}(X_i)\sum_{k=1}^{N}A_{ik}^3 w_i+\Omega_3^{vb}(X_i)\sum_{k=1}^{N}A_{ik}^2 w_i+\Omega_2^{vb}(X_i)\sum_{k=1}^{N}A_{ik}^1 w_i+\Omega_0^{vb}(X_i)w_i = 0 \quad (35)$$

$$\Omega_5^{bk}(X_i)\sum_{k=1}^{N}A_{ik}^4 w_i+\Omega_4^{bk}(X_i)\sum_{k=1}^{N}A_{ik}^3 w_i+\Omega_3^{bk}(X_i)\sum_{k=1}^{N}A_{ik}^2 w_i = 0 \quad (36)$$

$\Omega_5^{bn}(X_i), \Omega_4^{bn}(X_i), \Omega_5^{vb}(X_i), \Omega_5^{bk}(X_i)$ etc. denote values of $\Omega_5^{bn}, \Omega_4^{bn}, \Omega_5^{vb}, \Omega_5^{bk}$ … at the grid coordinate $(X_i)$. These contain first or higher order derivatives of elastic modulus, scale coefficient etc. which can be computed numerically by applying DQ approximation for derivatives. It should be noted that the discretized eigenvalue Eqs. (35) and (36) have been reduced to general eigenvalue problems. The GDQM technique presented by Shu (2000) could efficiently implement the four classical boundary conditions. The systematic algorithm adopted here for implementing the boundary conditions for bending analysis is as follows

Step 1. Apply discretized governing equations at internal (N-4) grid points (leaving two leftmost and two rightmost grid points) only.

Step 2. Apply discretized boundary conditions on leftmost and rightmost grid points. This will give four equations corresponding to four boundary conditions.

Step 3. Express displacements at two leftmost and two rightmost grid points in terms of other displacements at internal (N-4) grid points, using the four equations obtained in step 2.

Step 4. Substitute for these four displacements of step 3 in the discretized equations obtained in step 1.

Step 5. Solve these (N-4) equations of step 4 to compute displacements at internal (N-4) grid points.

Step 6. Update the displacements of two leftmost and two rightmost grid points using the computed displacements in step 5 and expressions obtained in step 3.

For vibration and buckling analyses one needs to solve the general eigen-value problem arising in Step 4.

## 4. Results and discussions

Convergence study of differential quadrature method with various grid points is conducted and results are shown in Fig. 1. In this particular example bending analysis has been carried out. Cubic variation of elastic modulus, nonlocal parameter, density and diameter of the nanotube are considered in this convergence study. From Fig. 1, one could observe that ten grid points are good enough for reasonably accurate results. The initial fluctuation in the graph is due to numerical instability. This is probably due to truncation error in the computation. But it can be seen that after nine grid points the solutions become stable. Also it was observed that fifteen grid points are enough to achieve reasonably accurate vibration and buckling results. So in all the following

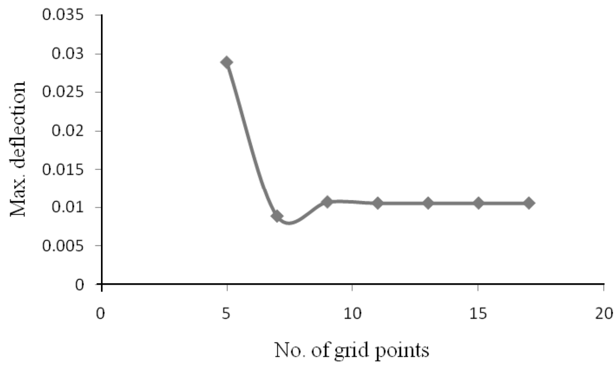


Fig. 1 Convergence study of differential quadrature method with various grid points

computations fifteen grid points are employed. This also reveals the efficiency of DQ method in analyzing nonhomogeneous nanotubes.

At first, bending, vibration and buckling results are obtained by employing local elasticity theory. These results are compared with those obtained by employing distributed transfer function method (DTFM) (Yang 2005) for all four boundary conditions. After this validation, bending, vibration and buckling results are obtained employing nonlocal elasticity theory. These results are also compared with corresponding results available in literature. Further, nonhomogeneous solutions with nonlocal elasticity theory are obtained. Various variations of elastic modulus, nonlocal parameter, density and diameter of nanotube along the axial direction are included in the investigation.

### *4.1 Validation of beam results with local theory*

Beam with following parameters is considered for the analysis. $E$ = 1 N/m$^2$, $L$ = 1 m, $A$ = 1 m$^2$, $q$ = 1 N/m, $\rho$ = 1 kg/m$^3$ and $I$ = 1 m$^4$. Maximum deflections, natural frequencies and

$$\hat{w} = w\frac{EI}{qL^4},\quad \hat{f} = fL^2\sqrt{\frac{\rho A}{EI}}\quad \text{and}\quad \hat{P}_{cr} = P_{cr}\frac{L^2}{EI} \tag{37}$$

Employing the classical local elasticity theory bending, vibration and buckling results are obtained. The non-dimensional maximum deflection ($\hat{w}$), natural frequencies ($\hat{f}$) and critical buckling loads ($\hat{P}_{cr}$) are listed in Tables 2-4, respectively. These results are compared with DTFM (Yang 2005) results. It is observed that present DQM results are in good agreement with results obtained employing DTFM.

Table 2 Non dimensional deflection ($\hat{w}$) of beams

| Boundary condition | Non dimensional max. deflection ($\hat{w}$) (Yang 2005) | Non dimensional max. deflection ($\hat{w}$) (Present) |
|---|---|---|
| S-S | 0.0130 | 0.0130 |
| C-F | 0.1240 | 0.1249 |
| C-S | 0.0054 | 0.0054 |
| C-C | 0.0026 | 0.0026 |

Table 3 Non dimensional natural frequencies ($\hat{f}$) of beam

| Boundary condition | Mode no. | Non-dimensional natural freq. ($\hat{f}$) (Yang 2005) | Non-dimensional natural freq. ($\hat{f}$) (Present) |
|---|---|---|---|
| S-S | 1 | 9.8690 | 9.8696 |
| | 2 | 39.5570 | 39.4784 |
| | 3 | 89.0040 | 88.8249 |
| C-F | 1 | 3.5231 | 3.5160 |
| | 2 | 22.0340 | 22.0345 |
| | 3 | 61.8208 | 61.6999 |
| C-S | 1 | 15.4490 | 15.4182 |
| | 2 | 50.0650 | 49.9648 |
| | 3 | 104.4560 | 104.2471 |
| C-C | 1 | 22.3730 | 22.3733 |
| | 2 | 61.7960 | 61.6728 |
| | 3 | 121.1450 | 120.9021 |

Table 4 Non-dimensional critical buckling loads ($\hat{P}_{cr}$)of beam

| Boundary condition | Non-dimensional critical buckling load ($\hat{P}_{cr}$) (Yang 2005) | Non-dimensional critical buckling load ($\hat{P}_{cr}$) (Present) |
|---|---|---|
| S-S | 9.8696 | 9.86960 |
| C-F | 2.4670 | 2.47490 |
| C-S | 20.1900 | 20.1907 |
| C-C | 39.4800 | 39.4784 |

### *4.2 Validation of beam results with nonlocal theory*

Reddy (2007) obtained bending, vibration and buckling solutions of simply supported beams with nonlocal elasticity theory. Peddieson *et al.* (2003) obtained bending solution for cantilever beam employing nonlocal elasticity theory. Wang *et al.* (2006) employed nonlocal elasticity theory and obtained buckling solutions for columns with S-S, C-F and C-C boundary conditions. All these above mentioned solutions (Reddy 2007, Peddieson *et al.* 2003, Wang *et al.* 2006) are analytical in nature. But in the present analysis DQ method is employed because of complexity of the governing differential equations for nonhomogeneous naotubes. Material properties and geometrical dimensions of the beams are assumed to be same as mentioned above mentioned researchers. Present results of bending, vibration and buckling for all classical boundary conditions are listed in Tables 5-7, respectively. In these tables present results are compared with those available in the literature. From these tables one could observe that the present results are in good agreement with corresponding results reported in the literature. Present deflection results for CS and CC boundary conditions, frequency results for CS, CC and CF boundary conditions and critical loads for CS boundary conditions are new. These results for the above specific boundary conditions are not available in literature. From Table 5 it is observed that the maximum deflections for CC, CS, SS

Table 5 Non dimensional deflections ($\hat{w}$) of the beam with non nonlocal theory of elasticity

| Nonlocal parameter ($\mu$) | Boundary condition | Non-dimensional max. deflection ($\hat{w}$) (Reddy 2007) | Non-dimensional max. deflection ($\hat{w}$) (Peddieson *et al.* 2003) | Non-dimensional max. deflection $\hat{w}$ [Present] |
|---|---|---|---|---|
| 0.5 | S-S | 0.0756 | ---- | 0.0755 |
| | C-F | ---- | 0.1250 | 0.1249 |
| | C-S | ---- | ---- | 0.0237 |
| | C-C | ---- | ---- | 0.0026 |
| 1.0 | S-S | 0.1382 | ---- | 0.1380 |
| | C-F | ---- | 0.3750 | 0.3749 |
| | C-S | ---- | ---- | 0.0422 |
| | C-C | ---- | ---- | 0.0026 |

Table 6 Non-dimensional natural frequencies ($\hat{f}$) of the beam with non nonlocal theory of elasticity

| Nonlocal parameter | Boundary condition | Mode no. | Non-dimensional natural frequency ($\hat{f}$) (Reddy 2007) | Non-dimensional natural frequency ($\hat{f}$) (Present) |
|---|---|---|---|---|
| 0.5 | S-S | 1 | 4.0489 | 4.0513 |
| | | 2 | 8.6643 | 8.6689 |
| | | 3 | 13.1743 | 13.1809 |
| | C-F | 1 | ---- | 2.2498 |
| | | 2 | ---- | 11.9413 |
| | | 3 | ---- | 49.8439 |
| | C-S | 1 | ---- | 5.8851 |
| | | 2 | ---- | 10.6399 |
| | | 3 | ---- | 15.2169 |
| | C-C | 1 | ---- | 8.2756 |
| | | 2 | ---- | 12.4545 |
| | | 3 | ---- | 17.4647 |
| 1.0 | S-S | 1 | 2.9919 | 2.9936 |
| | | 2 | 6.2019 | 6.2051 |
| | | 3 | 9.3674 | 9.3720 |
| | C-F | 1 | ---- | 1.1236 |
| | | 2 | ---- | 7.1276 |
| | | 3 | ---- | 25.7857 |
| | C-S | 1 | ---- | 4.3182 |
| | | 2 | ---- | 7.6211 |
| | | 3 | ---- | 10.8302 |
| | C-C | 1 | ---- | 6.0566 |
| | | 2 | ---- | 8.895400 |
| | | 3 | ---- | 12.45250 |

Table 7 Non-dimensional critical buckling loads ($\hat{P}_{cr}$) of the beam with non nonlocal theory of elasticity

| Nonlocal parameter | Boundary condition | Non-dimensional critical buckling load ($\hat{P}_{cr}$) ( Wang *et al.* 2006) | Non-dimensional critical buckling load ($\hat{P}_{cr}$) (Present) |
|---|---|---|---|
| 0.5 | S-S | 1.6627 | 1.6630 |
| | C-F | 1.1041 | 1.1052 |
| | C-S | -------- | 1.8197 |
| | C-C | 1.9035 | 1.9036 |
| 1.0 | S-S | 0.9079 | 0.9080 |
| | C-F | 0.7114 | 0.7115 |
| | C-S | -------- | 0.9528 |
| | C-C | 0.9753 | 0.9753 |

and CF boundary conditions are in increasing order. From Table 6 it is observed that the natural frequencies for CF, SS, CS and CC boundary conditions are in increasing order. From Table 7 it is observed that the critical loads for CF, SS, CS and CC boundary conditions are in increasing order. Further from Tables 5-7 it is observed that increase in nonlocal parameter leads to increase in deflection and decrease in natural frequency and critical buckling load. This is attributed to the fact that increase in nonlocal parameter decreases the effective stiffness of the nanotube.

### *4.3 Nonhomogeneous nanotubes*

Young's modulus, nonlocal parameter, density and diameter of the nanotubes are assumed to vary along the axial direction. These parameters are expressed as

$$E = E_0\left\{1+k\left(\frac{x}{L}\right)^{\alpha}\right\},\quad \mu=\mu_0\left\{1+k\left(\frac{x}{L}\right)^{\alpha}\right\},\quad \rho=\rho_0\left\{1+k\left(\frac{x}{L}\right)^{\alpha}\right\},\quad d=d_0\left\{1+k\left(\frac{x}{L}\right)^{\alpha}\right\} \tag{38}$$

$E, \mu_0, \rho_0$ and $d_0$ are assumed to be the values of elastic modulus, nonlocal parameter, density and diameter at left end ($x$ = 0). $E, \mu_0, \rho_0$ and $d_0$ are considered to be 1 TPa, 0.0136 nm$^2$ and 2.3 gm/cm$^3$ and 0.7 nm, respectively. Length ($L$) and wall thickness ($t$) of the nanotubes are considered to be 35 nm and 0.35 nm, respectively. In Eq. (38) $\alpha$ equals to 1, 2 and 3 represent linear, quadratic and cubic variations of the parameters, respectively. Eighty per cent variation of diameter ($d$), Young's modulus ($E$), nonlocal parameter ($\mu$) and density ($\rho$) has been assumed in the nonhomogeneous analysis. SS boundary condition is considered for the nanotubes. Non-dimensional maximum deflections, natural frequencies and critical buckling loads are defined as follows

$$\bar{w} = w\frac{E_0 I_0}{qL^4},\quad \bar{f} = fL^2\sqrt{\frac{\rho_0 A_0}{E_0 I_0}}\quad \text{and}\quad \bar{P}_{cr} = P_{cr}\frac{L^2}{E_0 I_0} \tag{39}$$

These non-dimensional parameters are computed employing DQ method for S-S boundary condition. Effect of linear, quadratic and cubic variation of individual parameters has been investigated. Inter relation of these parameters are also investigated. Effect of linear, quadratic and cubic variation of elastic modulus for bending, vibration and buckling are shown in Figs. 2-4, respectively. From these Figs. it is observed that maximum deflection decreases with increase in nonhomogeneous parameter. While natural frequency and critical load increase with increase in

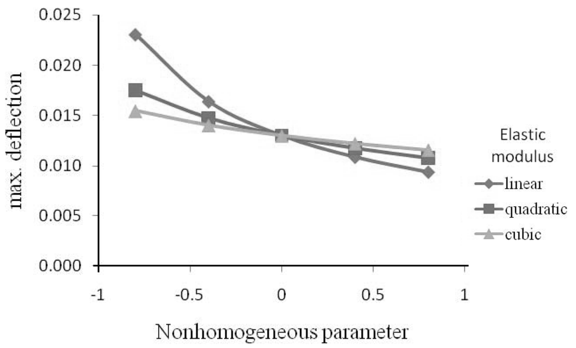


Fig. 2 Variations of non-dimensional maximum deflections ($\overline{w}$) with nonhomogeneous parameter ($k$) for linear, quadratic and cubic variations of elastic modulus

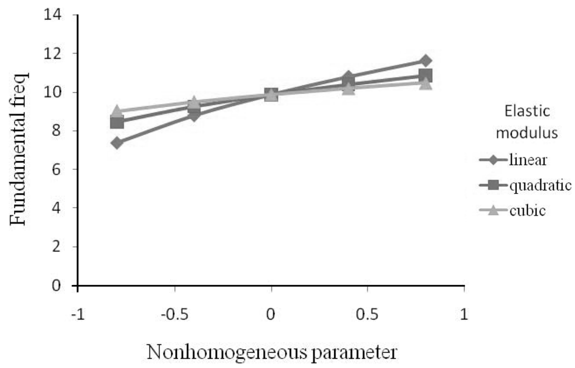


Fig. 3 Variation of non-dimensional fundamental natural frequencies ($\overline{f}$) with nonhomogeneous parameter ($k$) for linear, quadratic and cubic variations of elastic modulus

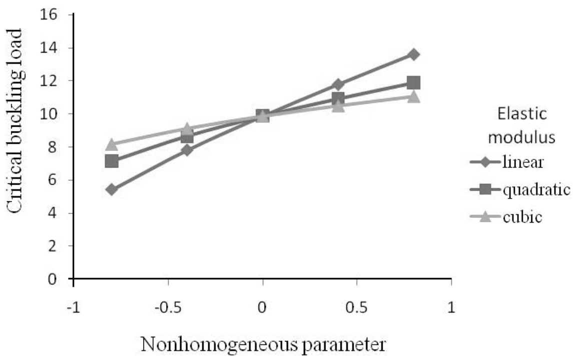


Fig. 4 Variation of non-dimensional critical buckling loads ($\overline{P}_{cr}$) with nonhomogeneous parameter ($k$) for linear, quadratic and cubic variations of elastic modulus

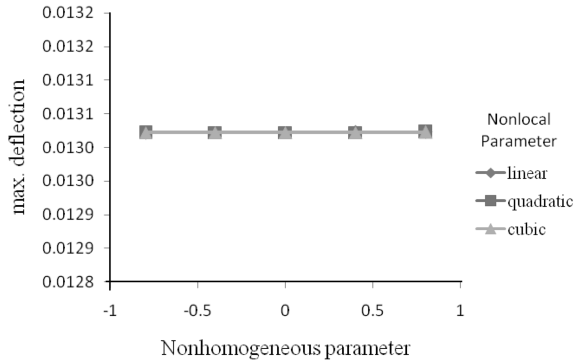


Fig. 5 Variations of non-dimensional maximum deflections ($\overline{w}$) with nonhomogeneous parameter ($k$) for linear, quadratic and cubic variations of nonlocal parameter

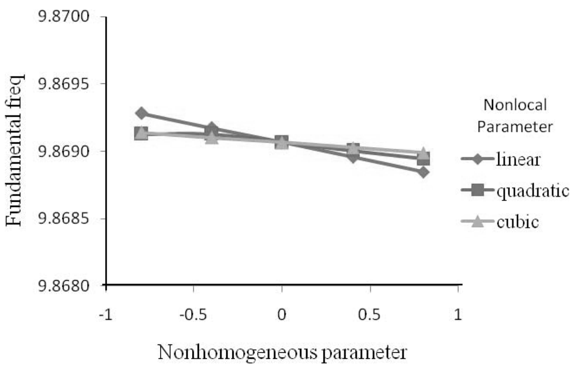


Fig. 6 Variations of non-dimensional fundamental natural frequencies ($\overline{f}$) with nonhomogeneous parameter ($k$) for linear, quadratic and cubic variations of nonlocal parameter

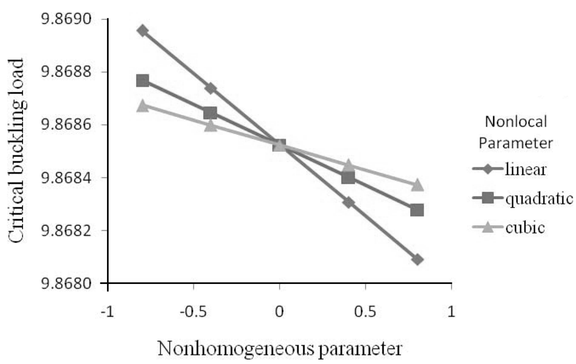


Fig. 7 Variations of non-dimensional maximum critical buckling loads ($\overline{P}_{cr}$) with nonhomogeneous parameter ($k$) for linear, quadratic and cubic variations nonlocal parameter

nonhomogeneous parameter. Further maximum deflection, natural frequency and critical load show greater rate of change for linear variation of nonhomogeneous parameter as compared to quadratic and cubic variations.

Similarly effect of linear, quadratic and cubic variation of nonlocal parameter for bending, vibration and buckling are shown in Figs. 5-7, respectively. From Figs. 5-7, one could observe that variation of nonlocal parameter has little effect on bending, vibration and buckling of nonhomogeneous nanotubes. So it can be concluded that though nonlocal parameter is considered to be an important factor for analysis of nano-structures, for nonhomogeneous nanotubes an average constant value of nonlocal parameter can be considered for various nanotube applications and designs. This could reduce substantially the complexity of the formulation and computational effort. In Fig. 8 effect of density on the vibration response of nanotubes has been shown. In this Fig. natural frequency shows greater rate of change for linear variation of nonhomogeneous parameter as compared to quadratic and cubic variations.

Effects of diameter of the nanotube on bending, vibration and buckling are shown in Figs. 9-11,

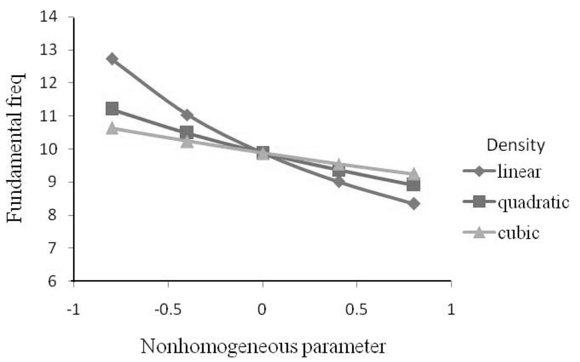


Fig. 8 Variations of non-dimensional fundamental natural frequencies ($\bar{f}$) with nonhomogeneous parameter ($k$) for linear, quadratic and cubic variations of density

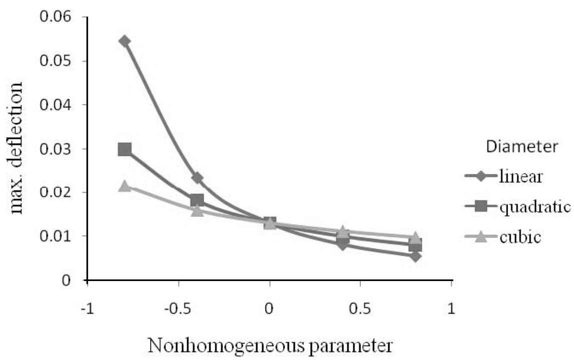


Fig. 9 Variations of non-dimensional maximum deflections ($\overline{w}$) with nonhomogeneous parameter ($k$) for linear, quadratic and cubic variations of diameter

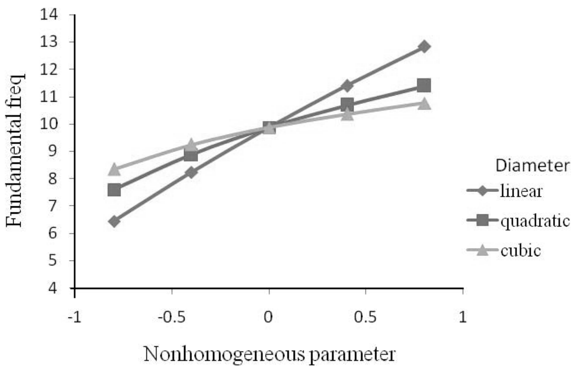


Fig. 10 Variations of non-dimensional fundamental natural frequencies ($\bar{f}$) with nonhomogeneous parameter ($k$) for linear, quadratic and cubic variations of diameter

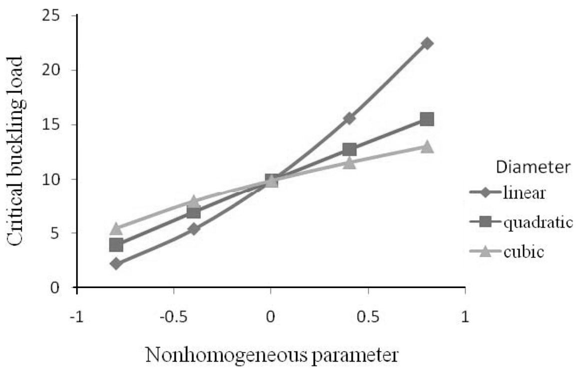


Fig. 11 Variations of non-dimensional critical buckling loads ($\bar{P}_{cr}$) with nonhomogeneous parameter ($k$) for linear, quadratic and cubic variations of diameter

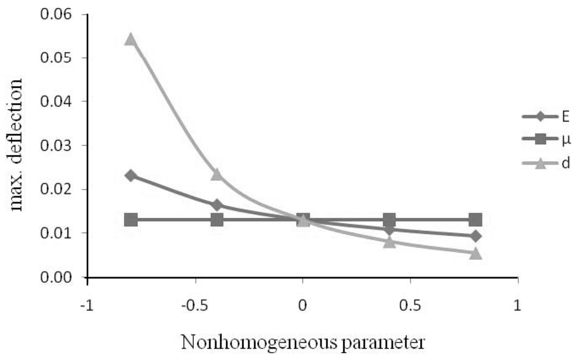


Fig. 12 Variations of non-dimensional maximum deflections ($\overline{w}$) with nonhomogeneous parameter ($k$) for linear variation of elastic modulus ($E$), nonlocal parameter ($\mu$) and diameter ($d$)

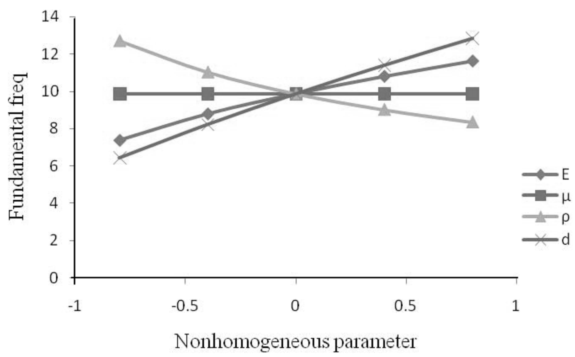


Fig. 13 Variations of non-dimensional fundamental natural frequencies ($\bar{f}$) with nonhomogeneous parameter ($k$) for linear variation of elastic modulus ($E$), nonlocal parameter ($\mu$), density ($\rho$) and diameter ($d$)

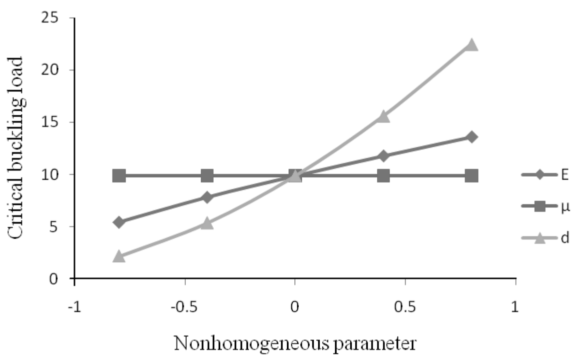


Fig. 14 Variations of non-dimensional critical buckling loads ($\overline{P}_{cr}$) with nonhomogeneous parameter ($k$) for linear variation of elastic modulus ($E$), nonlocal parameter ($\mu$) and diameter ($d$)

respectively. From these Figs. it is observed that maximum deflection decreases with increase in nonhomogeneous parameter. While natural frequency and critical load increase with increase in nonhomogeneous parameter. Further maximum deflection, natural frequency and critical load show greater rate of change for linear variation of nonhomogeneous parameter as compared to quadratic and cubic variations. Effects of linear variation of elastic modulus, nonlocal parameter and diameter on bending, vibration and buckling has been shown in Figs. 12-14, respectively. From Fig. 12 it is observed that maximum deflection decreases with increase in nonhomogeneous parameter for linear variation of elastic modulus and nanotube diameter. Further it can be found that linear variation of diameter has stronger influence on the maximum deflection than the linear variation of elastic modulus.

From Fig. 13 it is observed that natural frequency increases with increase in nonhomogeneous parameter for linear variation of elastic modulus and nanotube diameter. While natural frequency decreases with increase in nonhomogeneous parameter for linear variation of density of nanotube. Further it can be found that linear variation of diameter has stronger influence on the natural frequency than the linear variation of elastic modulus. From Fig. 14 it is observed that critical buckling load increases with increase in nonhomogeneous parameter for linear variation of elastic modulus and nanotube diameter. Further it can be found that linear variation of diameter has stronger influence on the critical buckling load than the linear variation of elastic modulus. Maximum deflection, natural frequency and critical buckling load are observed to be most sensitive to change in nanotube diameter, while these are observed to be insensitive to the change in nonlocal parameter.

## 5. Conclusions

In this work, formulation and solutions methods are developed for nonhomogeneous single walled and double walled nanotubes. Nonlocal theory has been implemented to take the scale effect into account. Present results are validated with the results available in the literature for homogeneous nanotubes. Effect of linear, quadratic and cubic variations of nanotube Young's modulus, nonlocal parameter, density and diameter on the structural response of the nonhomogeneous nanotubes is studied. It is observed that maximum deflection decreases with increase in nonhomogeneous parameter. While critical load increases with increase in nonhomogeneous parameter. Further maximum deflection, natural frequency and critical load show greater rate of change for linear variation of nonhomogeneous parameter as compared to quadratic and cubic variations. It has been observed that the variation of nonlocal parameter along the length has little effect on the structural response of nonhomogeneous nanotubes. While variations of diameter, elastic modulus and density of the nanotubes have substantial effect on the response of the nanotubes. The formulation, solution methods and numerical solutions presented in this paper will be helpful for engineers who are designing nano-electromechanical devices containing nonhomogeneous nanotubes. Extension of the present research work to incorporate shear deformation theories is under development.

## Acknowledgements

Authors are grateful to Professor J N Reddy of Texas A & M University, United States for the technical discussions about the Eringen's nonlocal elasticity theory.

## References

Artan, R. and Tepe, A. (2008), "The initial values method for buckling of nonlocal bars with application in nanotechnology", *Eur. J. Mech. A - Solid*, **27**, 469-477.

Aydogdu, M. (2008), "Vibration of multiwalled carbon nanotubes by generalized shear deformation theory", *Int. J. Mech. Sci.,* **50**, 837-844.

Aydogdu, M. (2009), "Axial vibration of nanorods with the nonlocal continuum rod model", *Physica E,* **41**, 861-864.

Aydogdu, M. (2009), "A general nonlocal beam theory: Its application to nanobeam bending, buckling and vibration", *Physica E,* (in press), Available online.

Ball, P. (2001), "Roll up for the revolution", *Nature* (London), **414**, 142-144.

Baughman, R.H., Zakhidov, A.A. and de Heer, W.A. (2002), "Carbon Nanotubes – The route toward applications", *Science,* **297**, 787-792.

Bellman, R.E., Kashef, B.G. and Casti, J. (1972), "Differential quadrature: A technique for the rapid solution of nonlinear partial differential equations", *J. Comput. Phys.,* **10**, 40.

Bert, C.W. and Malik, M. (1996), "Differential quadrature in computational mechanics: A review", *Appl. Mech.*

*Rev.*, **49**, 1-27.
Bert, C.W., Jang, S.K. and Striz, A.G. (1988), "Two new approximate methods for analyzing free vibration of structural components", *AIAA J.,* **26**, 612-618.
Bodily, B.H. and Sun, C.T. (2003), "Structural and equivalent continuum properties of single-walled carbon nanotubes", *J. Mater. Product Tech.,* **18**(4/5/6), 381-397.
Eringen, A.C. (1983), "On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves", *J. Appl. Phy.,* **54**, 4703-4710.
Eringen, A.C. (2002), *Nonlocal Continuum Field Theories,* Springer-Verlag NewYork
Gibson, R.F., Ayorinde, O.E. and Yuan-Feng Wen (2007), "Vibration of carbon nanotubes and there composites: A review", *Compos. Sci. Tech.,* **67**, 1-28.
Iijima, S. (1991), "Helical microtubules of graphitic carbon", *Nature,* **354**, 56-58.
Li, C. and Chou, T.W. (2003), "A structural mechanics approach for the analysis of carbon nanotubes", *Int. J. Solids Struct.,* **40**, 2487-2499.
Li, C. and Chou, T.W. (2003), "Single-walled nanotubes as ultrahigh frequency nanomechanical oscillators", *Phys. Rev. B,* **68**, art no: 073405.
Lu, P., Lee, H.P., Lu, C. and Zhang, P.Q. (2007), "Application of nonlocal beam models for carbon nanotubes", *Int. J. Solids Struct.,* **44**, 5289-5300.
Peddieson, J., Buchanan, G.R. and McNitt, R.P. (2003), "Application of nonlocal continuum models to nanotechnology", *Int. J. Eng. Sci.,* **41**, 305-312.
Pradhan, S.C. and Murmu, T. (2009), "Small scale effect on vibration analysis of single-walled carbon nanotubes embedded in an elastic medium using nonlocal elasticity theory", *J. Appl. Phys.,* **105**, 124306.
Pradhan, S.C. and Sarkar, A. (2009), "Analyses of tapered fgm beams with nonlocal theory", *Struct. Eng. Mech.*, (in press).
Reddy, J.N. (2007), "Nonlocal theories for bending buckling and vibration of beams", *Int. J. Eng. Sci.,* **45**, 288-307.
Rothemund Paul, W.K., Ekani-nkodo, A.P., Nick Kumar, Ashish Fygenson, Deborah Kuchnir and Winfree Erik (2004), "Design and characterization of programmable DNA nanotubes", *J. Am. Soc.,* **126**, 16344-16352.
Ru, C.Q. (2000), "Column buckling of multiwalled carbon nanotubes with interlayer radial displacements", *Phys. Rev. B,* **62**, 16962-16967.
Seeman, N.C. (1999), "DNA engineering and its application to nanotechnology", *Trends Biotechnol.,* **17**(11), 437-43.
Shu, C. (2000), *Differential Quadrature and Its Application in Engineering,* Springer Berlin.
Thostenson, E.T., Ren, Z. and Chou, T.W. (2001), "Advances in science and technology of carbon nanotubes and there composites: a review", *Compos. Sci. Tech.,* **61**, 1899-1912.
Wang, L. and Hu, H.Y. (2005), "Flexural wave propagation in single-walled carbon nanotube", *Phys. Rev. B*, **71**, 195412.
Wang, C.M., Tan, V.B.C. and Zhang, Y.Y. (2006), "Timoshenko Beam model for vibration analysis of multi-walled carbon nanotubes", *J. Sound Vib.,* **294**, 1060-1072.
Wang, Q. and Varadan, V.K. (2006), "Vibration of Carbon Nanotubes studied using nonlocal continuum mechanics", *Smart Mater. Struct.,* **15**, 659-666.
Wang, Q. and Vardan, V.K. (2005), "Wave characteristics of carbon nanotubes international", *Int. J. Solids Struct.,* **43**, 254-265.
Wang, Q., Varadan, V.K. and Quekc, S.T. (2006), "Small scale effect on elastic buckling of carbon nanotubes with nonlocal continuum models", *Phy. Letters A,* **357**, 130-135.
Wang, Q. and Wang, C.M. (2007), "The constitutive relation and small scale parameter of nonlocal continuum mechanics for modeling carbon nanotubes", *Nanotechnology*, **18**, 075702.
Wang, Q., Zhou, G.Y. and Lin, K.C. (2006), "Scale effect on wave propagation of double walled carbon nanotubes", *Int. J. Solids Struct.,* **43**, 6071-6084.
Yang, B. (2005), *Stress Strain and Structural Dynamics,* Elsevier Science and Technology Publishers.
Yoon, J., Ru, C.Q. and Mioduchowski, A. (2003), "A vibration of embedded multiwall carbon nanotubes", *Compos. Sci. Tech.,* **63**, 1533-1542.
Yoon, J., Ru, C.Q. and Mioduchowski, A. (2002), "A non-coaxial resonance of an isolated multiwall carbon nanotube", *Phys. Rev. B,* **66**, art no: 233402.

## Appendix

$$\Omega_0^{bn} = -q(x)\left\{1 - \frac{d^2\mu(x)}{dx^2}\right\} + 2\frac{d\mu(x)}{dx}\frac{dq(x)}{dx} + \mu(x)\frac{d^2q(x)}{dx^2}$$

$$\Omega_3^{bn} = \frac{d^2(E(x)I(x))}{dx^2}$$

$$\Omega_4^{bn} = 2\frac{d(E(x)I(x))}{dx}$$

$$\Omega_5^{bn} = E(x)I(x)$$

$$\Omega_3^{bk} = P\frac{d^2\mu(x)}{dx^2} - P + \frac{d^2(E(x)I(x))}{dx^2}$$

$$\Omega_4^{bk} = 2\left\{P\frac{d\mu(x)}{dx} + \frac{d(E(x)I(x))}{dx}\right\}$$

$$\Omega_5^{bk} = E(x)I(x) + P\mu(x)$$

$$\Omega_1^{vb} = -\left[\rho(x)A(x)\left\{1 - \frac{d^2\mu(x)}{dx^2}\right\} - 2\frac{d(\rho(x)A(x))}{dx}\frac{d\mu(x)}{dx} - \mu(x)\frac{d^2(\rho(x)A(x))}{dx^2}\right]\omega^2$$

$$\Omega_2^{vb} = 2\left\{\mu(x)\frac{d(\rho(x)A(x))}{dx} + \rho(x)A(x)\frac{d\mu(x)}{dx}\right\}\omega^2$$

$$\Omega_3^{vb} = \frac{d^2(E(x)I(x))}{dx^2} + \omega^2\mu(x)\rho(x)A(x)$$

$$\Omega_4^{vb} = 2\frac{d(E(x)I(x))}{dx}$$

$$\Omega_5^{vb} = E(x)I(x)$$

$$\Phi1_0^{bn} = -q(x)\left\{1 - \frac{d^2\mu(x)}{dx^2}\right\} + 2\frac{d\mu(x)}{dx}\frac{dq(x)}{dx} + \mu(x)\frac{d^2q(x)}{dx^2}$$

$$\Phi1_1^{bn} = -\Phi2_1^{bn} = \mu(x)\frac{d^2C(x)}{dx^2} + 2\frac{d\mu(x)}{dx}\frac{dC(x)}{dx} + C(x)\left(\frac{d^2\mu(x)}{dx^2} - 1\right)$$

$$\Phi1_2^{bn} = -\Phi2_2^{bn} = 2\left(\mu(x)\frac{dC(x)}{dx} + C(x)\frac{d\mu(x)}{dx}\right)$$

$$\Phi1_3^{bn} = \frac{d^2(E(x)I(x))}{dx^2} + C(x)\mu(x)$$

$$\Phi 1_4^{bn} = 2\frac{d(E(x)I(x))}{dx}$$

$$\Phi 1_5^{bn} = E(x)I(x)$$

$$\Phi 2_3^{bn} = -C(x)\mu(x)$$

$$\Phi 1_3^{bk} = P\frac{d^2\mu(x)}{dx^2} - P + \frac{d^2(E(x)I(x))}{dx^2} + \mu(x)C(x)$$

$$\Phi 1_4^{bk} = 2\left\{P\frac{d\mu(x)}{dx} + \frac{d(E(x)I(x))}{dx}\right\}$$

$$\Phi 1_5^{bk} = E(x)I(x) + P\mu(x)$$

$$\Phi 1_5^{vb} = E(x)I(x)$$

$$\Phi 1_4^{vb} = 2\frac{d(E(x)I(x))}{dx}$$

$$\Phi 1_3^{vb} = \frac{d^2(E(x)I(x))}{dx^2} + \mu(x)C(x) + \omega^2\mu(x)\rho(x)A(x)$$

$$\Phi 1_2^{vb} = 2\left\{\mu(x)\frac{dC(x)}{dx} + C(x)\frac{d\mu(x)}{dx}\right\} + 2\omega^2\left\{\mu(x)\frac{d(\rho(x)A(x))}{dx} + \rho(x)A(x)\frac{d\mu(x)}{dx}\right\}$$

$$\Phi 1_1^{vb} = \mu(x)\frac{d^2C(x)}{dx^2} + 2\frac{d\mu(x)}{dx}\frac{dC(x)}{dx} + C(x)\left(\frac{d^2\mu(x)}{dx^2} - 1\right)$$

$$-\omega^2\left\{\rho(x)A(x)(1 - \frac{d^2\mu(x)}{dx^2}) - 2\frac{d(\rho(x)A(x))}{dx}\frac{d\mu(x)}{dx} - \mu(x)\frac{d^2(\rho(x)A(x))}{dx^2}\right\}$$

$$\Phi 2_1^{vb} = -\mu(x)\frac{d^2C(x)}{dx^2} + 2\frac{d\mu(x)}{dx}\frac{dC(x)}{dx} + C(x)\left(\frac{d^2\mu(x)}{dx^2} - 1\right)$$

$$\Phi 2_2^{vb} = -2\left\{\mu(x)\frac{dC(x)}{dx} + C(x)\frac{d\mu(x)}{dx}\right\}$$

$$\Phi 2_3^{vb} = -\mu(x)C(x)$$

$$\Theta_4^{vb} = 2\left\{\mu(x)\frac{d(\rho(x)A(x))}{dx} + \rho(x)A(x)\frac{d\mu(x)}{dx}\right\}$$

$$\Theta_3^{vb} = -\rho(x)A(x)\left\{1 - \frac{d^2\mu(x)}{dx^2}\right\} + 2\frac{d(\rho(x)A(x))}{dx}\frac{d\mu(x)}{dx} + \mu(x)\frac{d^2(\rho(x)A(x))}{dx^2}$$

$$\Theta_5^{vb} = \mu(x)\rho(x)A(x)$$

$$\Upsilon_3^{vb} = \frac{d^2(E(x)I(x))}{dx^2}$$

$$\Upsilon_4^{vb} = 2\frac{d(E(x)I(x))}{dx}$$

$$\Upsilon_5^{vb} = E(x)I(x)$$

$$\Upsilon1_1^{vb} = -\Upsilon2_1^{vb} = \mu(x)\frac{d^2C(x)}{dx^2} + 2\frac{d\mu(x)}{dx}\frac{dC(x)}{dx} + C(x)\left(\frac{d^2\mu(x)}{dx^2} - 1\right)$$

$$\Upsilon1_2^{vb} = -\Upsilon2_2^{vb} = 2\left\{\mu(x)\frac{dC(x)}{dx} + C(x)\frac{d\mu(x)}{dx}\right\}$$

$$\Upsilon1_3^{vb} = \frac{d^2(E(x)I(x))}{dx^2} + \mu(x)C(x)$$

$$\Upsilon1_4^{vb} = 2\frac{d(E(x)I(x))}{dx}$$

$$\Upsilon1_5^{vb} = E(x)I(x)$$

$$\Upsilon2_3^{vb} = -\mu(x)C(x)$$