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Analyses of tapered fgm beams with nonlocal theory

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Abstract. In the present article bending, buckling and vibration analyses of tapered beams using Eringen non-local elasticity theory are being carried out. The associated governing differential equations are solved employing Rayleigh-Ritz method. Both Euler-Bernoulli and Timoshenko beam theories are considered in the analyses. Present results are in good agreement with those reported in literature. Beam material is considered to be made up of functionally graded materials (fgms). Non-local analyses for tapered beam with simply supported - simply supported, clamped - simply supported and clamped - free boundary conditions are carried out and discussed. Further, effect of length to height ratio on maximum deflections, vibration frequencies and critical buckling loads are studied.

Keywords: non local theory; Rayleigh-Ritz method; tapered beam; bending; buckling; vibration and boundary conditions.

1. Introduction

Most classical continuum theories are based on hyper elastic constitutive relations which assume that the stress at a point are functions of strains at that point. On the other hand, the non-local continuum mechanics assumes that the stress at a point is a function of strains at all points in the continuum. Such theories contain information about the forces between atoms, and the internal length scale is introduced into the constitutive equations as a material parameter. The non-local theory of elasticity has been used to study lattice dispersion of elastic waves, wave propagation in composites, dislocation mechanics, fracture mechanics, surface tension on fluids, etc. Work on nonlocal elasticity is introduced by Eringen (1972), (1983) and (2002) and Eringen and Edelen (1972). Peddieson *et al.* (2003) employed nonlocal continuum model in nanotechnology. Pin *et al* (2007) employed nonlocal beam model in carbon nanotubes. Reddy (2007) applied nonlocal theories and reported bending, buckling and vibration results of beams. Wang and Liew (2007) applied nonlocal continuum mechanics and conducted static analysis of micro- and nano-structures. Heireche *et al.* (2008) employed nonlocal elasticity and studied sound wave propagation in single-walled carbon nanotubes. Zhou and Chung (2000) studied free vibration of tapered beams. Maalek (2004) investigated shear deflections of tapered Timoshenko beams. Ece *et al.* (2007) reported vibration response of variable cross section beam. Ganesan and Zabihollah (2007a) and (2007b) conducted parametric study on vibration of tapered beams. They employed higher order finite

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element analysis. Reddy *et al.* (2000) found relation between bending solutions of classical and shear deformation beam theories. Reddy and Wang (2000) also discussed relationship between solutions of the classical and shear deformation plate theories. Liew *et al.* (2004) studied free vibration and buckling of shear- deformable plates based on meshfree method. Brown and Stone (1997) employed Rayleigh-Ritz method and studied composite materials. Leissa (2005) reported historical bases of Rayleigh-Ritz method. Shames and Dym (2006) reported energy and finite element methods for various structural mechanics problems. Functionally graded material are found in applications of high temperature environment. These are discussed by Pradhan *et al.* (2000), Pradhan (2005), Pradhan (2008) and Murmu and Pradhan (2008).

The tapered beams are increasingly being used in engineering applications, such as turbine blades, helicopter blades and yokes, robot arms and satellite antennas. Here stiffness of the structure is varied along the length of the beam. Nonlocal analysis of tapered beams is important and little work are available in the literature. Thus in the present work authors have attempted to carry out nonlocal analyses of tapered fgm beams with various boundary conditions. This work includes bending, buckling and vibration of the beams.

2. Formulation

2.1 Nonlocal theory

The stress field at a point x in an elastic continuum depends on the strain field at the point (hyper elastic case) as well as strains at all other points of the body. Eringen 2002 attributed this fact to the atomic theory of lattice dynamics and experimental observations on phonon dispersion. Thus, the non-local stress tensor σ at point x is expressed as an integral form over the body

$$\boldsymbol{\sigma} = \int_V \boldsymbol{K}\left(\left|\boldsymbol{x}^{\prime} - \boldsymbol{x}\right|, \frac{\sqrt{\mu}}{\boldsymbol{l}}\right)\boldsymbol{t}(\boldsymbol{x}^{\prime})dv(\boldsymbol{x}^{\prime}) \tag{1}$$

where t is the classical, macroscopic stress tensor at point $x^{\prime}$ in the body and the nonlocal kernel function $\boldsymbol{K}(\left|\boldsymbol{x}^{\prime} - x\right|, \sqrt{\mu}/\boldsymbol{l})$ which brings the influence of strain at distant points $x^{\prime}$ to the stress at x. is the distance in Euclidean norm. μ is a material constant that depends on internal and external characteristic lengths such as the lattice spacing and wavelength, respectively. The macroscopic stress 't' at a point x in a Hookean solid is related to the strain ε at the point by the generalized Hook's law

$$t(x) = C(x) : \varepsilon(x) \tag{2}$$

where C is the fourth-order elasticity tensor and : denotes the double-dot product. The constitutive Eqs. (1) and (2) together define the non-local constitutive behaviour of a Hookean solid (Reddy 2007). Further, Eq. (1) represents the weighted average of the contributions of the strain field of all points in the body to the stress field at a point. This represents the integral constitutive relations in an equivalent differential form as

$$(1 - \mu\nabla^2)\sigma = \boldsymbol{t} \tag{3}$$

where $\mu = e_0^2 a^2$. e_0 is a material constant. a and l are the internal and external characteristic lengths, respectively. Using Eqs. (2) and (3), stress resultants are expressed in terms of the strains in different beam theories. In the local theory the relation of stress resultants and strains are represented as linear algebraic equations. While in non-local theory the relation of stress resultants and strains are represented as differential equations. For homogeneous isotropic beams the non-local behavior is assumed to be negligible in the thickness direction. The constitutive relation for macroscopic stress take the special relation for beams

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\varepsilon_{xx}, \quad \sigma_{xz} - \mu \frac{\partial^2 \sigma_{xz}}{\partial x^2} = 2G\varepsilon_{xz} \tag{4}$$

The axial force-strain relation is given by

$$N - \mu \frac{\partial^2 N}{\partial x^2} = E\varepsilon_{xx} \tag{5}$$

where $N = \int_A \sigma_{xx} dA$. The x-axis is considered along the geometric centroid of the beam. In Euler-Bernoulli beam theory, the constitutive relation is given by

$$M^E - \mu \frac{\partial^2 M^E}{\partial x^2} = EI\kappa^E \tag{6}$$

where $\kappa^E = -\partial^2 w^E / \partial x^2$ and $M^E = \int_A z\sigma_{xx} dA$.

The superscript 'E' denotes the quantities associated with Euler-Bernoulli beam theory. In case of the Timoshenko beam theory we have additional M^T and Q^T terms. The constitutive relation is given as

$$M^T - \mu \frac{\partial^2 M^T}{\partial x^2} = EI\kappa^T, \quad Q^T - \mu \frac{\partial^2 Q^T}{\partial x^2} = GAK_s\gamma^T \tag{7}$$

where $Q = \int_A \sigma_{xz} dA$. K_s denotes the shear correction factor. $\kappa^T = \partial\phi^T/\partial x$ and $\gamma^T = dw^T/dx + \phi^T$.

The superscript 'T' denotes the quantities associated with the Timoshenko beam theory (Reddy 2007).

In the present work deflections, natural frequencies and critical loads for uniform and tapered beams (Fig. 1) with various boundary condition are calculated. The moment of inertia and cross section area are changing along the beam axis.

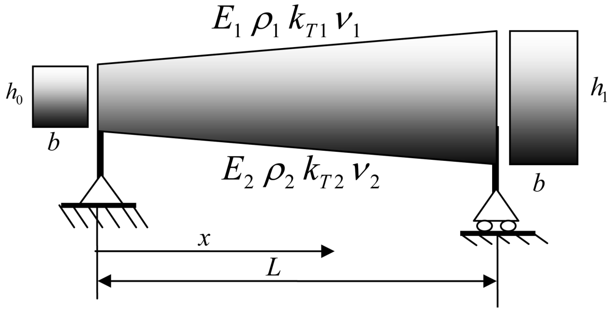

Fig. 1 Schematic of FGM Tapered beam with simply supported boundary condition

The moment equation for Euler-Bernoulli non-uniform beam theory is written as

$$M^E = -EI(x)\frac{\partial^2 w}{\partial x^2} + \mu\left[\frac{\partial}{\partial x}\left(N^E \frac{\partial w}{\partial x}\right) - q + m_0 \frac{\partial^2 w}{\partial t^2} - m_2 \frac{\partial^4 w}{\partial x^2 \partial t^2}\right] \tag{8}$$

while moment equation Timoshenko non-uniform beam theory is expressed as

$$M^T = EI(x)\frac{\partial \phi}{\partial x} + \mu\left[\frac{\partial}{\partial x}\left(N^T \frac{\partial w}{\partial x}\right) - q + m_0 \frac{\partial^2 w}{\partial t^2} - m_2 \frac{\partial^4 w}{\partial x^2 \partial t^2}\right] \tag{9}$$

Shear force is written as

$$Q^T = GA(x)K_s\left(\phi + \frac{\partial w}{\partial x}\right) + \mu \frac{\partial}{\partial x}\left[-q + \frac{\partial}{\partial x}\left(N \frac{\partial w}{\partial x}\right) + m_0 \frac{\partial^2 w}{\partial t^2}\right] \tag{10}$$

2.2 Bending

Flexural response of the beams are computed by employing Rayleigh-Ritz method. Strain energy for bending is expressed as

$$U = \frac{1}{2E}\int_0^L \frac{M^2}{I(x)} dx \tag{11}$$

Beam flexural equation is written as

$$M = -EI(x)\frac{d^2 w}{dx^2} \tag{12}$$

Putting (12) in (11)

$$U = \frac{1}{2}\int_0^L EI(x)\left(\frac{d^2 w}{dx^2}\right)^2 dx \tag{13}$$

Strain energy due to shear force is written as

$$V_S = \frac{1}{2G}\int_0^L \frac{Q^2}{A(x)} dx \tag{14}$$

Work done by uniformly distributed load (UDL) is expressed as

$$V_E = -\int_0^L q w dx \tag{15}$$

Total potential energy for the Euler-Bernoulli beam with UDL is expressed as

$$\Pi_p = U + V_E \tag{16}$$

where U is strain energy due to bending. V_E is the work done by external force. For purely bending analysis $N^E = 0$, $m_0 = 0$ and $m_2 = 0$ are incorporated in Eq. (8).

Eq. (8) is rewritten as

$$M^E = -EI(x)\frac{\partial^2 w}{\partial x^2} + \mu(-q) \tag{17}$$

Strain energy for bending

$$U = \frac{1}{2E}\int_0^L \frac{(M^E)^2}{I(x)}dx \tag{18}$$

Work done by UDL is expressed as

$$V_E = -\int_0^L qwdx \tag{19}$$

In Rayleigh-Ritz Method the component of approximate displacement w is approximated as functions containing a finite number of independent parameters, These parameters are determined such that the total potential energy computed on the basis of the approximate displacements is a minimum.

For a given structural system, w is assumed as

$$w = a_1u_1(x) + a_2u_2(x) + \ldots + a_nu_n(x) \tag{20}$$

where $a_1, a_2, \ldots, a_n$ are the linear independent parameters and $u_1, u_2, \ldots, u_n$ are the continuous functions of the co-ordinate x. $u_1, u_2, \ldots, u_n$ satisfy all the kinematics boundary conditions for all values of the constant $a_1, a_2, \ldots, a_n$. The total potential energy is a function of $a_1, a_2, \ldots, a_n$

When the system is in equilibrium,

$$\sum_{i=1}^{n} \frac{\partial\Pi}{\partial a_i}\partial a_i = 0 \tag{21}$$

Eq. (21) is satisfied only if

$$\frac{\partial\Pi}{\partial a_1} = 0 \quad \frac{\partial\Pi}{\partial a_2} = 0 \quad \ldots \quad \frac{\partial\Pi}{\partial a_n} = 0 \tag{22}$$

From Eqn (22) $a_1, a_2, \ldots, a_n$ are determined. Incorporating $a_1, a_2, \ldots, a_n$ in Eq. (20) approximate displacement w is determined.

Similarly beam bending deflections are computed by employing Rayleigh-Ritz method for the Timoshenko beam. Total potential energy for Timoshenko beam with UDL is expressed as

$$\Pi_p = U + V_S + V_E \tag{23}$$

For bending analysis of Timoshenko beam $N^T = 0$, $m_0 = 0$ and $m_2 = 0$ are put in Eq. (9) and we get

$$M^T = -EI(x)\frac{\partial\phi}{\partial x} + \mu(-q) \tag{24}$$

Strain energy for bending is

$$U = \frac{1}{2E}\int_0^L \frac{(M^T)}{I(x)}dx \tag{25}$$

For bending analysis $N = 0$ and $m_0 = 0$. q is independent of x. Putting these values in Eq. (10) we get

$$Q^T = GA(x)K_s\left(\phi + \frac{\partial w}{\partial x}\right) \tag{26}$$

Strain energy due to shear force is

$$V_S = \frac{1}{2G}\int_0^L \frac{(Q^T)^2}{A(x)}dx \tag{27}$$

work done by UDL is written as

$$V_E = -\int_0^L qw dx \tag{28}$$

In Timoshenko beam approximate displacement w and rotation ϕ are functions containing a finite number of independent parameters, These parameters are determined so that the total potential energy computed on the basis of the approximate displacements is a minimum. w and ϕ are expressed as

$$\begin{aligned} w &= a_1u_1(x) + a_2u_2(x) + \ldots + a_nu_n(x) \\ \phi &= b_1v_1(x) + b_2v_2(x) + \ldots + b_nv_n(x) \end{aligned} \tag{29}$$

where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are linear independent parameters and $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ are the continuous functions of the co-ordinate x. All the kinematics boundary conditions for all value of the constant $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are satisfied. The total potential energy is a function of $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$. System is in equilibrium implies

$$\sum_{i=1}^{n} \frac{\partial \Pi}{\partial a_i}\partial a_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \frac{\partial \Pi}{\partial b_i}\partial b_i = 0 \tag{30}$$

Eq. (30) is satisfied for arbitrary values of $\partial a_i, \partial b_i$ Thus

$$\begin{aligned} &\frac{\partial \Pi}{\partial a_1} = 0 \quad \frac{\partial \Pi}{\partial a_2} = 0 \quad \ldots \quad \frac{\partial \Pi}{\partial a_n} = 0 \\ &\frac{\partial \Pi}{\partial b_1} = 0 \quad \frac{\partial \Pi}{\partial b_2} = 0 \quad \ldots \quad \frac{\partial \Pi}{\partial b_n} = 0 \end{aligned} \tag{31}$$

From Eq. (31) $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are determined and putting these values in Eq. (29) we get approximate displacement w and rotation ϕ.

2.3 Buckling

Critical buckling loads are computed by employing Rayleigh-Ritz method. Total potential energy of the column with Euler-Bernaulli beam theory is expressed as

$$\Pi_p = V_{MAX} + V_P \tag{32}$$

where V_{MAX} is maximum strain energy for bending and V_P is work done due to external load.

For buckling analysis $q = 0$, $m_0 = 0$, $m_2 = 0$ are put in Eq. (8).

Eq. (8) is rewritten as

$$M^E = -EI(x)\frac{\partial^2 w}{\partial x^2} + \mu\left[\frac{\partial}{\partial x}\left(N^E\frac{\partial w}{\partial x}\right)\right] \tag{33}$$

Maximum strain energy for bending is expressed as

$$V_{\max} = \frac{1}{2E}\int_0^L \frac{(M^E)^2}{I(x)}dx \tag{34}$$

Work done due to external load is written as

$$V_P = -\frac{P}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx \tag{35}$$

In buckling analysis w is approximated with independent parameters satisfying kinematic boundary conditions. For maximum total potential energy

$$\frac{\partial \Pi}{\partial a_i} = 0 \tag{36}$$

After simplifying Eq. (36) we get

$$\sum_{i=1}^{n}(A_i - PD_i)a_i = 0 \tag{37}$$

where $A_i = \int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx$ and $D_i = \int_0^L \frac{1}{2}P\left(\frac{dw}{dx}\right)dx$.

We have a homogenous system of n number of eqns. For a nontrivial solution the determinant of the coefficients is equal to zero. Thus we get

$$|A_i - PD_i| = 0 \tag{38}$$

From Eq. (38) buckling load P is determined. Total potential energy of the column with Timoshenko beam theory is expressed as

$$\Pi_p = V_{MAX} + V_S + V_P \tag{39}$$

where V_{MAX}, V_S and V_P represent maximum strain energy for bending, strain energy due to shear force and work done by external load, respectively. For buckling analysis $q = 0$, $m_0 = 0$ and $m_2 = 0$ are incorporated in Eq. (9).

Eq. (9) is rewritten as

$$M^T = -EI(x)\frac{\partial \phi}{\partial x} + \mu\left[N\frac{\partial^2 w}{\partial t^2}\right] \tag{40}$$

Maximum strain energy for bending is written as

$$V_{MAX} = \frac{1}{2E}\int_0^L \frac{(M^T)^2}{I(x)}dx \tag{41}$$

For buckling analysis $q = 0$ and $m_0 = 0$ are incorporated in Eq. (10).
Eq. (10) is rewritten as

$$Q = GA(x)K_s\left(\phi + \frac{\partial w}{\partial x}\right) + \mu\left[N\frac{\partial^3 w}{\partial x^3}\right] \tag{42}$$

Strain energy due to shear force is written as

$$V_s = \frac{1}{2G}\int_0^L \frac{(Q^T)^2 dx}{A(x)} \tag{43}$$

Work done due to external load is expressed as

$$V_P = -\frac{P}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx \tag{44}$$

w and ϕ are approximate functions with independent parameters satisfying kinematic boundary conditions. For maximum total potential energy

$$\frac{\partial \Pi}{\partial a_i} = 0 \quad \text{and} \quad \frac{\partial \Pi}{\partial b_i} = 0 \tag{45}$$

After simplification, we get

$$\sum_{i=1}^{n} (A_i - PD_i)a_i = 0 \tag{46}$$

where $A_i = \int_0^L EI\left(\frac{d^2 w}{dx^2}\right)dx$ and $D_i = \int_0^L \frac{1}{2}P\left(\frac{dw}{dx}\right)dx$.

Thus we have a homogenous system of n number of eqns. For a nontrivial solution the determinant of the coefficients is equal to zero. Thus we get

$$|A_i - PD_i| = 0 \tag{47}$$

From Eq. (47) critical buckling load P is determined.

2.4 Vibration

Vibration frequencies of the beams are computed by employing Rayleigh-Ritz method. Total potential energy for Euler-Bernoulli beam is expressed as

$$\Pi_p = V_{MAX} - T_{MAX} \tag{48}$$

where V_{MAX} and T_{MAX} are total maximum strain energy due to bending and maximum kinetic energy, respectively. Kinetic energy is written as

$$T = \frac{1}{2}\int_0^l \left(\frac{dw(x,t)}{dt}\right)^2 dm = \frac{1}{2}\int_0^l \left(\frac{dw(x,t)}{dt}\right)^2 \rho A(x)dx \tag{49}$$

where $dm = \rho A(x)$. The maximum kinetic energy can be obtained by assuming a harmonic variation $w(x,t) = w(x)\cos\omega t$. Maximum kinetic energy is expressed as

$$T_{\max} = \frac{\omega^2}{2}\int_0^L \rho A(x) w^2(x) dx \tag{50}$$

For vibration analysis in Eq. (8) $q = 0$ and $N = 0$ are considered and Eq. (8) is rewritten as

$$M^E = -EI(x)\frac{\partial^2 w}{\partial x^2} + \mu\left[m_0 \frac{\partial^2 w}{\partial t^2}\right] \tag{51}$$

Maximum value of potential energy is expressed as

$$V_{\max} = \frac{1}{2E}\int_0^L \frac{(M^E)^2 dx}{I(x)} \tag{52}$$

In vibration analysis displacement w is expresses as an approximate function of independent parameters satisfying kinematic boundary conditions. For maximum total potential energy, we have

$$\frac{\partial \Pi}{\partial a_i} = 0 \tag{53}$$

After simplifying Eq. (53) we get

$$\sum_{i=1}^{n}(A_i - \omega^2 D_i)a_i = 0 \tag{54}$$

where $A_i = \int_0^L EI\left(\frac{d^2 w}{dx^2}\right)dx$ and $D_i = \int_0^L \frac{1}{2}\rho A w dx$

We have a homogenous system of n number of eqns. For a nontrivial solution the determinant of the coefficients is equal to zero. Thus we get

$$|A_i - \omega^w D_i| = 0 \tag{55}$$

From Eq. (55) frequency ω is determined. In a similar way frequency for Timoshenko beam is computed. Total potential energy for Timoshenko beam is expressed as

$$\Pi_p = V_{MAX} + V_S - T_{MAX} \tag{56}$$

V_{MAX}, V_S and T_{MAX} represent maximum strain energy due to bending, strain energy due to shear and maximum kinetic energy, respectively. Maximum kinetic energy T_{MAX} is expressed as in Eq. (50). For vibration analysis $q = 0$ and $N = 0$ are incorporated in Eq. (9).

Eq. (9) is rewritten as

$$M^T = -EI(x)\frac{\partial \phi}{\partial x} + \mu\left[m_0 \frac{\partial^2 w}{\partial t} + m_2 \frac{\partial^3 \phi}{\partial x \partial t^2}\right] \tag{57}$$

Maximum value of potential energy is

$$V_{MAX} = \frac{1}{2E}\int_0^L \frac{(M^T)^2 dx}{I(x)} \tag{58}$$

For vibration analysis $q = 0$ and $N = 0$ are incorporated in Eq. (10). Eq. (10) is rewritten as

$$Q^T = GAK_s\left(\phi + \frac{\partial w}{\partial x}\right) + \mu\left[m_0\frac{\partial^3 w}{\partial t^3}\right] \tag{59}$$

strain energy due to shear force is written as

$$V_S = \frac{1}{2G}\int_0^L \frac{(Q^T)^2 dx}{A(x)} \tag{60}$$

w and ϕ are approximate function with independent parameter that satisfy kinematic boundary conditions. For maximum total potential energy

$$\frac{\partial \Pi}{\partial a_i} = 0 \quad \text{and} \quad \frac{\partial \Pi}{\partial b_i} = 0 \tag{61}$$

After simplifying Eq. (61) we get

$$\sum_{i=1}^{n}(A_i - \omega^2 D_i)a_i = 0 \tag{62}$$

where $A_i = \int_0^L EI\left(\frac{d^2w}{dx^2}\right)dx$ and $D_i = \int_0^L \frac{1}{2}(\rho A w + \rho I\varphi)dx$

We have a homogenous system of n number of equations. For a nontrivial solution the determinant of the coefficients is equal to zero. Thus we get

$$|A_i - \omega^2 D_i| = 0 \tag{63}$$

From Eq. (63) frequency ω is determined.

2.5 FGM beam

FGM are made by mixing two or more different materials. Most of the FGM are being used in high temperature environment and their material properties are temperature dependent. A typical material property Pi is expressed as a function of the environment temperature $T(K)$

$$P_i = P_0(P_{-1}T^{-1} + 1 + P_1 T + P_2 T^2 + P_3 T^3) \tag{64}$$

where, P_0, P_{-1}, P_1, P_2 and P_3 are temperature coefficients and are unique to the constituent materials.

The material properties P_{fgm} of FGM are controlled by volume fractions V_{fi} and individual material properties P_i of the constituent materials.

$$P_{fgm} = \sum_{i=1}^{nm} P_i V_{fi} \tag{65}$$

In the present case two different materials are particle mixed to form the FGM material. Assuming there are no defects like voids and foreign particles in the FGM material, sum of the volume fractions of all the constituent materials is

$$\sum_{i=1}^{nm} V_{fi} = 1 \tag{66}$$

For example, metal and ceramic materials are mixed to form the FGM beam. Average volume fraction of the metal and ceramic materials is calculated by simple integration of the distribution over a domain. Different problems of interest have different expressions of volume fractions. For bending problems of beam, plates and shells the volume fractions of metal (V_m) and ceramic (V_C) materials are defined as

$$V_m = \left(\frac{h-2z}{2h}\right)^{Rn}; \quad V_c = 1 - V_m \tag{67}$$

where z is the thickness co-ordinate $(-h/2 \le z \le h/2)$ and h represents the beam thickness. R_n is the power law exponent $(0 \le R_n \le \infty)$. Here volume fraction of the metal material (V_m) varies from 100% to 0% as z varies from $-h/2$ to $h/2$. Similarly volume fraction of the ceramic material (V_C) varies from 0% to 100% as z varies from $-h/2$ to $h/2$. For various R_n values the average volume fractions of metal (V_m) and ceramic (V_C) materials are depicted .The Young's modulus and Poisson's ratio of a FGM beam made up of two different materials are expressed as

$$E_{fgm} = (E_2 - E_1)\left(\frac{2z+h}{2h}\right)^{Rn} + E_1 \tag{68}$$

3. Results and discussions

3.1 Bending of beam

In the nonlocal flexural beam analysis following configurations are considered (Reddy 2007). Length of beam L = 10.0 m, width b = 1.0 m, height h of 0.1 m, 1.0 m and 2.0 m, Young's modulus $E = 30 \times 10^6$ N/m^2, Poisson ratio $\nu = 0.3$, UDL per unit length q = 1 N/m, moment of inertia $I = (b \times h^3/12)$ m^4, cross section of beam $A = (b \times h)$ m^2, density ρ = 1 kg/m^3 and shear correction factor k = are considered. For tapered beam height h is assumed to be varying linearly along the beam length (Fig. 1). Moment of inertia and cross-section area are expressed as $I_1 = I_0(1+(x/L))$ m^4 and $A_1 = A_0(1+(x/L))$ m^2, respectively. Simply supported - simply supported (SS), clamped – simply supported (CS) and clamped - free (CF) boundary conditions are considered in the analyses.

Maximum deflection for SS, CS and CF beams are computed as mentioned in Eqs. (20, 29). Employing Euler-Bernoulli theory (EBT) and Timoshenko beam theory (TBT) for CF and SS beams results are listed in Tables 1, 2, respectively. From Table 1, one could observe that the present nonlocal results are exactly matching with those reported by Peddieson *et al.* (2003). Peddieson *et al.* (2003)'s nonlocal work is limited to Euler-Bernoulli theory. Further, in Table 2, it is observed that present results are in good agreement with those reported by Reddy (2007). Small difference in results is observed for higher values of nonlocal parameter and thick beams. This is attributed to the shear force effect in thick beams. In CF beam deflection is decreasing with increasing in non-local parameter. Thus in case of CF beam stiffness is directly proportional non-local parameter. However, in case of SS beam maximum deflection is increasing with increase in non-local parameter. Thus in case of SS beam the beam stiffness is inversely proportional non-local parameter.

Table 1 Non-dimensional maximum deflection at free end $[\hat{w} = w(EI/qL^4)]$ in clamped - free uniform beam

L/h	μ	Peddieson et al. 2003	Present result		% of difference
		EBT	EBT	TBT	EBT
100	0.0	0.1250	0.1250	0.1250	0.000
	0.5	0.1225	0.1225	0.1225	0.000
	1.0	0.1200	0.1200	0.1200	0.000
	1.5	0.1175	0.1175	0.1175	0.000
	2.0	0.1150	0.1150	0.1150	0.000
	2.5	0.1125	0.1125	0.1125	0.000
	3.0	0.1100	0.1100	0.1100	0.000
	3.5	0.1075	0.1075	0.1075	0.000
	4.0	0.1050	0.1050	0.1050	0.000
	4.5	0.1025	0.1025	0.1025	0.000
	5.0	0.1000	0.1000	0.1000	0.000

Table 2 Comparison of non-dimensional maximum center deflection $[\hat{w} = 10^2 \times w(EI/qL^4)]$ in simply upported - simply supported uniform beam

L/h	μ	(Reddy 2007)		Present result		% of difference	
		EBT	TBT	EBT	TBT	EBT	TBT
100	0.0	1.3130	1.3134	1.3021	1.3025	0.8317	0.8322
	0.5	1.3809	1.3813	1.3646	1.3650	1.1818	1.1822
	1.0	1.4487	1.4492	1.4271	1.4275	1.4924	1.4994
	1.5	1.5165	1.5170	1.4896	1.4900	1.7751	1.7818
	2.0	1.5844	1.5849	1.5521	1.5525	2.0399	2.0462
	2.5	1.6522	1.6528	1.6146	1.6150	2.2770	2.2888
	3.0	1.7201	1.7207	1.6771	1.6775	2.5010	2.5123
	3.5	1.7879	1.7886	1.7396	1.7400	2.7026	2.7189
	4.0	1.8558	1.8565	1.8021	1.8025	2.8947	2.9103
	4.5	1.9236	1.9244	1.8646	1.8650	3.0682	3.0882
	5.0	1.9914	1.9923	1.9271	1.9275	3.2299	3.2540
10	0.0	1.3130	1.3483	1.3021	1.3343	0.8317	1.0398
	0.5	1.3809	1.4210	1.3646	1.3968	1.1818	1.7044
	1.0	1.4487	1.4937	1.4271	1.4593	1.4924	2.3043
	1.5	1.5165	1.5664	1.4896	1.5218	1.7751	2.8486
	2.0	1.5844	1.6391	1.5521	1.5843	2.0399	3.3445
	2.5	1.6522	1.7118	1.6146	1.6468	2.2770	3.7983
	3.0	1.7201	1.7845	1.6771	1.7093	2.5010	4.2152
	3.5	1.7879	1.8572	1.7396	1.7718	2.7026	4.5994
	4.0	1.8558	1.9299	1.8021	1.8343	2.8947	4.9547
	4.5	1.9236	2.0026	1.8646	1.8968	3.0682	5.2841
	5.0	1.9914	2.0754	1.9271	1.9593	3.2299	5.5951

Table 3 Non-dimensional maximum deflection $[\hat{w} = 10^2 \times w(EI/qL^4)]$ in clamped - simply supported uniform beam

μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	0.5389	0.5421	0.5849	0.7094
0.5	0.5565	0.5596	0.6030	0.7288
1.0	0.5742	0.5771	0.6211	0.7482
1.5	0.5918	0.5946	0.6392	0.7676
2.0	0.6094	0.6121	0.6573	0.7870
2.5	0.6271	0.6296	0.6754	0.8063
3.0	0.6447	0.6471	0.6935	0.8257.
3.5	0.6623	0.6646	0.7116	0.8451
4.0	0.6800	0.6821	0.7297	0.8645
4.5	0.6976	0.6996	0.7478	0.8839
5.0	0.7152	0.7171	0.7659	0.9033

Non-dimensional maximum deflections are computed for the uniform beam with CS boundary condition and results are listed in Table 3. In case of CS uniform beam deflection is observed to be increasing with increase in non-local parameter and inclusion of Timoshenko beam theory. This is due to the fact that beam stiffness for CS boundary condition is inversely proportional to the nonlocal parameter. For the increase of nonlocal parameter from 0 to 5 there is an increase of 51 percent, increase of 34 percent and decrease of 20 percent in maximum deflections for SS, CS and CF uniform beams, respectively. Nonlocal effect is found to be in increasing order for CF, CS and SS boundary conditions.

3.2 Buckling of column

Nonlocal critical buckling loads for SS, CS and CF columns are computed as mentioned in Eqs. (38 and 47). Column configurations are assumed as mentioned in numerical example of sub section 3.1. The SS column results are listed Table 4. From this table one could observe that present results are in good agreement with those reported in Reddy (2007). Small difference in results is observed for higher values of nonlocal parameter and thick beams. This is attributed to the shear force effect in thick beams. In case of SS column critical buckling load is observed to be decreasing with increase in non-local parameter. Thus in case of SS beam the beam stiffness is found to be inversely

Table 4 Comparison of non-dimensional critical buckling loads $[P = P_{cr} \times (L^2/EI)]$ in simply supported - simply supported uniform beam

L/h	μ	(Reddy 2007)		Present result		% of Difference	
		EBT	TBT	EBT	TBT	EBT	TBT
100	0.0	9.8696	9.8671	9.8696	9.8621	0.0001	0.0511
	0.5	9.4055	9.4031	9.0094	9.0033	4.2110	4.2523
10	0.0	9.8696	9.6227	9.8696	9.6227	0.0001	0.0002
	0.5	9.4055	9.1701	9.0094	8.7789	4.2110	4.2658

proportional to non-local parameter.

Non-dimensional critical buckling loads are computed for the uniform beam with CS and CF boundary conditions. Results are listed in Tables 5, 6, respectively. In case of CS uniform beam critical buckling load is observed to be decreasing with increase in non-local parameter and inclusion of Timoshenko beam theory. This is due to the fact that beam stiffness for CS boundary condition is inversely proportional to the nonlocal parameter. While, in case of CF uniform beam critical buckling load is observed to be increasing with increase in non-local parameter and inclusion of Timoshenko beam theory. This is due to the fact that beam stiffness for CF boundary condition is directly proportional to the nonlocal parameter.

For the increase of nonlocal parameter from 0 to 5 there is an decrease of 63 percent and increase of 5 percent in critical buckling loads for CS and CF uniform columns, respectively. Nonlocal effect is found to be in increasing order for CF and CS boundary conditions.

Table 5 Non-dimensional critical buckling loads $[P = P_{cr} \times (L^2/EI)]$ in clamped-simply supported uniform column

μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	21.7050	21.5844	20.0970	16.7875
0.5	17.9782	17.8799	16.7247	14.0205
1.0	15.4984	15.4102	14.4570	12.1371
1.5	13.7005	13.6168	12.8026	10.7529
2.0	12.3238	12.2418	11.5307	9.6893
2.5	11.2289	11.1467	10.5161	8.8446
3.0	10.3332	10.2498	9.6845	8.1552
3.5	9.5844	9.4990	8.9881	7.5794
4.0	8.9475	8.8596	8.3949	7.0891
4.5	8.3980	8.3073	7.8827	6.6648
5.0	7.9183	7.8246	7.4353	6.2931

Table 6 Non-dimensional critical buckling loads $[P = P_{cr} \times (L^2/EI)]$ in clamped-free uniform column

μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	2.4897	2.4672	2.4488	2.3940
0.5	2.5481	2.5301	2.5104	2.4517
1.0	2.6105	2.5971	2.5759	2.5131
1.5	2.6774	2.6688	2.6460	2.5785
2.0	2.7493	2.7457	2.7211	2.6484
2.5	2.8269	2.8286	2.8019	2.7233
3.0	2.9111	2.9183	2.8892	2.8039
3.5	3.0027	3.0157	2.9840	2.8911
4.0	3.1029	3.1221	3.0873	2.9856
4.5	3.2132	3.2391	3.2006	3.0887
5.0	3.3354	3.3685	3.3258	3.2019

3.3 Vibration of beam

Nonlocal fundamental frequencies for SS, CS and CF beams are computed as mentioned in Eqs. (45 and 63). Beam configurations are assumed as mentioned in numerical example of sub section 3.1. The fundamental frequencies for SS beam are listed Table 7. From this table one could observe that present results are in good agreement with those reported in Reddy (2007). Small difference in results is observed for higher values of nonlocal parameter and thick beams. This is attributed to the shear force effect in thick beams. In case of SS beam natural frequency is observed to be decreasing with increase in non-local parameter. Thus in case of SS beam the beam stiffness is inversely proportional non-local parameter.

Non-dimensional fundamental frequencies are computed for the uniform beam with CS and CF boundary conditions. Results are listed in Tables 8, 9, respectively. In case of CS uniform beam fundamental frequency is observed to be decreasing with increase in non-local parameter and inclusion of Timoshenko beam theory. This is due to the fact that beam stiffness for CS boundary

Table 7 Comparison of non-dimensional fundamental natural frequencies [$\varpi = \omega_1 L^2\sqrt{\rho A/EI}$] in simply supported - simply supported uniform beam

L/h	μ	(Reddy 2007)		Present result		% of difference	
		EBT	TBT	EBT	TBT	EBT	TBT
100	0.0	9.8696	9.8683	9.8745	9.8706	0.05005	0.02331
	0.5	9.6347	9.6335	9.4297	9.4266	2.12752	2.14792
	1.0	9.4159	9.4147	9.0543	9.0518	3.84010	3.85472
	1.5	9.2113	9.2101	8.7306	8.7286	5.21848	5.22850
10	0.0	9.8696	9.7454	9.8745	9.7482	0.05005	0.02842
	0.5	9.6347	9.5135	9.4297	9.3279	2.12752	1.95081
	1.0	9.4159	9.2973	9.0543	8.9708	3.84010	3.51188
	1.5	9.2113	9.0953	8.7306	8.6612	5.21848	4.77279

Table 8 Non-dimensional fundamental frequencies [$\varpi = \omega_1 L^2\sqrt{\rho A/EI}$] in clamped - simply supported uniform beam

μ	EBT	TBT	TBT	TBT
	L/h = 100	*L/h* = 100	*L/h* = 10	*L/h* = 5
0.0	15.4252	15.4141	14.9345	13.7532
0.5	14.6405	14.6330	14.2450	13.2736
1.0	14.0137	14.0102	13.6871	12.8719
1.5	13.4968	13.4966	13.2223	12.5287
2.0	13.0674	13.0627	12.8268	12.2313
2.5	12.6798	12.6896	12.4848	11.9707
3.0	12.3650	12.3646	12.1856	11.7405
3.5	12.0804	12.0784	11.9213	11.3529
4.0	11.7997	11.5975	11.6861	11.1891
4.5	11.5673	11.3940	11.4758	11.0421
5.0	11.3579	11.2108	11.2869	10.9103

Table 9 Non-dimensional fundamental frequencies [$\varpi = \omega_1 L^2 \sqrt{\rho A/EI}$] in clamped - free uniform beam

μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	3.5168	3.5160	3.4981	3.4428
0.5	3.5322	3.5312	3.5137	3.4593
1.0	3.5479	3.5468	3.5297	3.4761
1.5	3.5639	3.5627	3.5461	3.4933
2.0	3.5803	3.5790	3.5629	3.5110
2.5	3.5972	3.5957	3.5800	3.5291
3.0	3.6144	3.6128	3.5976	3.5477
3.5	3.6321	3.6303	3.6157	3.5667
4.0	3.6503	3.6483	3.6342	3.5863
4.5	3.6689	3.6667	3.6532	3.6064
5.0	3.6880	3.6856	3.6727	3.6272

condition is inversely proportional to the nonlocal parameter. While, in case of CF uniform beam fundamental frequency is observed to be increasing with increase in non-local parameter and inclusion of Timoshenko beam theory. This is due to the fact that beam stiffness for CF boundary condition is directly proportional to the nonlocal parameter.

For the increase of nonlocal parameter from 0 to 5 there is decrease of 26 percent and increase of 5 percent in natural frequencies for CS and CF uniform beams, respectively. Nonlocal effect is found to be in increasing order for CF and CS boundary conditions.

3.4 Bending of fgm tapered beam

Following parameters are considered in the analyses of the fgm beam. Length L = 10.0 m, width b = 1.0 m and thickness h_0 = 0.1 m, 1.0 m and 2.0 m. Thickness $h = h_0(1 + (x/L))$ is assumed to be varying linearly along the beam length. Moment of inertia and cross-section area are expressed as $I_1 = I_0(1 + (x/L))$ m^4 and $A_1 = A_0(1 + (x/L))$ m^2, respectively. Shear correction factor k = 5/6 is considered in the analysis. Silicon Nitrate (Si_3N_4)as ceramic and stainless steel (SUS304) are assumed to constitute the fgm beam. Power index R_n = 1.0 and UDL q = 1 N/m are considered. Material properties of Silicon Nitrate and stainless steel are as follows

Properties of Silicon Nitride(Si_3N_4) and Stainless Steel(SUS304)

Materials	properties	P_0	P_{-1}	P_1	P_2	P_3
Si_3N_4	E (P_a)	348.43×10^9	0	-3.070×10^{-4}	2.160×10^{-7}	-8.946×10^{-11}
	ν	0.2400	0	0	0	0
	α (/K)	5.8723×10^{-6}	0	9.095×10^{-4}	0	0
	κ (W/mK)	13.723	0	-1.023×10^{-3}	5.466×10^{-7}	-7.876×10^{-11}
SUS304	E (P_a)	201.04×10^9	0	3.070×10^{-4}	-6.534×10^{-7}	0
	ν	0.3262	0	-2.002×10^{-4}	3.797×10^{-7}	0
	α (/K)	12.33×10^{-6}	0	8.086×10^{-4}	0	0
	κ (W/mK)	15.379	0	-1.264×10^{-4}	2.092×10^{-2}	-7.223×10^{-10}

Present flexural response computation is extended to the tapered fgm beams. Non-dimensional maximum deflections for SS, CS and CF boundary conditions are being computed and listed in Tables 10-12, respectively.

From Tables 10-12 following observations are made. In case of SS and CS tapered beam maximum deflection is observed to be increasing with increase in non-local parameter. When Timoshenko beam theory is included maximum deflection is found to increase with decrease in length to height ratio for SS, CS and CF boundary conditions. In case of CF boundary condition the deflection of the tapered beam is observed to be decreasing with increase in non-local parameter. For the increase of nonlocal parameter from 0 to 5 there is an increase of 48 percent, increase of 30 percent and decrease of 18 percent in maximum deflections of SS, CS and CF tapered beams,

Table 10 Non-dimensional maximum center deflection $[\hat{w} = 10^2 \times w(EI/qL^4)]$ in simply supported - simply supported fgm tapered beam

μ	EBT	TBT	TBT	TBT
	L/h = 100	*L/h* = 100	*L/h* = 10	*L/h* = 5
0.0	0.8810	0.8866	0.9086	0.9749
0.5	0.9235	0.9285	0.9504	1.0167
1.0	0.9659	0.9703	0.9922	1.0585
1.5	1.0084	1.0122	1.0340	1.1003
2.0	1.0509	1.0540	1.0758	1.1421
2.5	1.0933	1.0958	0.1176	1.1839
3.0	1.1358	1.1376	1.1295	1.2257
3.5	1.1783	1.1795	1.2013	1.2675
4.0	1.2208	1.2213	1.2431	1.3093
4.5	1.2631	1.2631	1.2849	1.3511
5.0	1.3047	1.3049	1.3267	1.3929

Table 11 Non-dimensional maximum deflection $[\hat{w} = w(EI/qL^4)]$ in clamped - simply supported fgm tapered beam

μ	EBT	TBT	TBT	TBT
	L/h = 100	*L/h* = 100	*L/h* = 10	*L/h* = 5
0.0	0.3853	0.3892	0.4118	0.4781
0.5	0.3972	0.4012	0.4242	0.4911
1.0	0.4089	0.4133	0.4365	0.5040
1.5	0.4207	0.4253	0.4488	0.5170
2.0	0.4324	0.4374	0.4612	0.5299
2.5	0.4441	0.4494	0.4735	0.5429
3.0	0.4558	0.4615	0.4858	0.5559
3.5	0.4675	0.4736	0.4982	0.5688
4.0	0.4793	0.4856	0.5105	0.5818
4.5	0.4910	0.4977	0.5229	0.5948
5.0	0.5027	0.5097	0.5352	0.6077

Table 12 Non-dimensional maximum deflection $[\hat{w} = w(EI/qL^4)]$ in clamped - free fgm tapered beam

μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	0.1059	0.1081	0.1090	0.1118
0.5	0.1039	0.1061	0.1070	0.1098
1.0	0.1020	0.1042	0.1051	0.1079
1.5	0.1001	0.1022	0.1032	0.1059
2.0	0.0982	0.1003	0.1012	0.1040
2.5	0.0962	0.0984	0.0993	0.1021
3.0	0.0943	0.9650	0.0974	0.1002
3.5	0.0924	0.0945	0.0955	0.0982
4.0	0.0904	0.0926	0.0935	0.0963
4.5	0.0885	0.0907	0.0916	0.0944
5.0	0.0866	0.0887	0.0897	0.0924

respectively. Nonlocal effect is found to be in increasing order for CF, CS and SS boundary conditions.

3.5 Buckling of fgm tapered column

Non-dimensional critical buckling loads for SS, CS and CF boundary conditions of the tapered fgm columns are being computed and listed in Tables 13-15, respectively.

From Tables 13-15 following observations are made. In case of SS and CS tapered columns critical buckling load is observed to be decreasing with increase in non-local parameter. This buckling load decreases with decreases in length to height ratio for SS, CS and CF boundary conditions. In case of CF tapered column critical buckling load is observed to be increasing with increase in non-local parameter. For the increase of nonlocal parameter from 0 to 5 there is decrease

Table 13 Non-dimensional critical buckling loads $[P = P_{cr} \times (L^2/EI)]$ in simply supported - simply supported fgm tapered column

μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	15.4037	14.7933	14.3475	13.1360
0.5	13.9459	13.4942	13.0887	11.9958
1.0	12.7826	12.4417	12.0713	11.0685
1.5	11.8268	11.5687	11.2271	10.2956
2.0	11.0238	10.8298	10.5122	9.6391
2.5	10.3374	10.1942	9.8969	9.0729
3.0	9.7423	9.6402	9.3605	8.5785
3.5	9.2203	9.1520	8.8877	8.1423
4.0	8.7579	8.7178	8.4671	7.7538
4.5	8.3448	8.3286	8.0899	7.4053
5.0	8.0149	7.9773	7.7495	7.0906

Table 14 Non-dimensional critical buckling loads $[P = P_{cr} \times (L^2/EI)]$ in clamped - simply supported fgm tapered column

μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	31.2600	29.4225	27.6641	23.4871
0.5	25.8038	24.5714	23.1572	19.6518
1.0	22.1765	21.2914	20.0991	16.9998
1.5	19.5491	18.8875	17.8553	15.0799
2.0	17.5394	17.0323	16.1240	13.6370
2.5	15.9426	15.5472	14.7398	12.5005
3.0	14. 6376	14.3258	13.6035	11.5664
3.5	13.5476	13.2998	12.6511	10.7767
4.0	12.6212	12.4234	11.8396	10.0971
4.5	11.8226	11.6644	11.1386	9.5050
5.0	11.1360	10.9995	10.5260	8.9840

Table 15 Non-dimensional Critical buckling loads $[P = P_{cr} \times (L^2/EI)]$ in clamped - free fgm tapered column

μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	4.0265	4.0023	3.5948	3.4568
0.5	4.1635	4.1102	3.6942	3.5482
1.0	4.3127	4.2266	3.8006	3.6458
1.5	4.4761	4.3526	3.9152	3.7503
2.0	4.6561	4.4896	4.0388	3.8626
2.5	4.8556	4.6394	4.1728	3.9835
3.0	5.0788	4.8040	4.3188	4.1144
3.5	5.3308	4.9861	4.4786	4.2565
4.0	5.6187	5.1892	4.6548	4.4114
4.5	5.9527	5.4177	4.8503	4.5811
5.0	6.3471	5.6776	5.0692	4.7682

of 48 percent, decrease of 64 percent and increase of 58 percent in critical buckling load for SS, CS and CF tapered columns, respectively. Nonlocal effect is found to be in increasing order for CF, SS and CS boundary conditions.

3.6 Vibration of fgm tapered beam

Present beam vibration computation is extended to tapered beams. Non-dimensional fundamental frequencies for SS, CS and CF boundary conditions are being computed for the fgm beam and results are listed in Tables 16-18, respectively.

From Tables 16-18 following observations are made. In case of SS and CS tapered beams vibration frequency is observed to be decreasing with increase in non-local parameter. In case of CF

Table 16 Non-dimensional fundamental frequencies $\varpi = \omega_1 L^2\sqrt{\rho A/EI}$ in simply supported - simply supported fgm tapered beam

μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	9.8747	9.8645	9.3385	8.0601
0.5	9.4293	9.4212	8.6444	7.8283
1.0	9.0539	9.0475	8.3638	7.6208
1.5	8.7295	8.7242	8.1151	7.4330
2.0	8.4455	8.4412	7.8921	7.2619
2.5	8.1934	8.1898	7.6904	7.1049
3.0	7.9671	7.9641	7.5065	6.9600
3.5	7.7622	7.7597	7.3379	6.8256
4.0	7.5753	7.5733	7.1822	6.7004
4.5	7.4038	7.4021	7.1822	6.5833
5.0	7.2455	7.2441	7.0379	6.4733

Table 17 Non-dimensional fundamental frequencies $\varpi = \omega_1 L^2\sqrt{\rho A/EI}$ in clamped - simply supported fgm tapered beam

μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	14.8665	14.6892	13.6500	11.2741
0.5	14.0837	13.9216	13.0414	10.9565
1.0	13.4570	13.3057	12.5391	10.6774
1.5	12.9378	12.7947	12.1136	10.4291
2.0	12.4970	12.3604	11.7463	10.2061
2.5	12.1159	11.9847	11.4244	10.0043
3.0	11.7816	11.6550	11.1390	9.8203
3.5	11.4852	11.3624	10.8835	9.6516
4.0	11.2197	11.1005	10.6530	9.4963
4.5	10.9803	10.8642	10.4436	9.3526
5.0	10.7630	10.6497	10.2524	9.2192

Table 18 Non-dimensional fundamental frequencies $\varpi = \omega_1 L^2\sqrt{\rho A/EI}$ in clamped - free fgm tapered beam

μ	EBT	TBT	TBT	TBT
	L/h = 100	L/h = 100	L/h = 10	L/h = 5
0.0	3.6461	3.6328	3.1088	2.6870
0.5	3.6643	3.6453	3.1189	2.6945
1.0	3.6830	3.6581	3.1292	2.7020
1.5	3.7021	3.6710	3.1396	2.7097
2.0	3.7218	3.6842	3.1502	2.7174
2.5	3.7420	3.6977	3.1610	2.7253
3.0	3.7627	3.7114	3.1720	2.7333
3.5	3.7841	3.7254	3.1832	2.7414
4.0	3.8061	3.7397	3.1946	2.7496
4.5	3.8288	3.7543	3.2061	2.7579
5.0	3.8523	3.7692	3.2179	2.7664

tapered beam vibration frequency is observed to be increasing with increase in non-local parameter. This vibration frequency decreases with decrease in length to height ratio for SS, CS and CF boundary conditions. For the increase of nonlocal parameter from 0 to 5 there is decrease of 27 percent, decrease of 28 percent and increase of 6 percent in vibration frequencies for SS, CS and CF tapered beams, respectively. Nonlocal effect is found to be in increasing order for CF, CS and SS boundary conditions.

4. Conclusions

Effect of nonlocal parameter on the structural response is sensitive to the applied boundary conditions and Timoshenko beam theory. In case of SS and CS beam deflection is observed to be increasing with increase in non-local parameter and inclusion of Timoshenko beam theory. In case of CF uniform beam deflection is observed to be decreasing with increase in non-local parameter and inclusion of Timoshenko beam theory.

In case of SS and CS columns critical buckling loads are observed to be decreasing with increase in non-local parameter and inclusion of Timoshenko beam theory. In case of CF column critical buckling load is observed to be increasing with increase in non-local parameter and inclusion of Timoshenko beam theory.

In case of SS and CS beams vibration frequencies are observed to be decreasing with increase in non-local parameter and inclusion of Timoshenko beam theory. In case of CF beam vibration frequency is observed to be increasing with increase in non-local parameter and inclusion of Timoshenko beam theory.

Effect of nonlocal parameter is larger on bending and buckling than in vibration of beams. Effect of nonlocal parameter in case of CF boundary condition is substantially less than those for SS and CS boundary conditions. Further, effect of nonlocal parameter in case of CF boundary condition is opposite in nature as compared to those for SS and CS boundary conditions.

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Notation

a_i : arbitrary constant
A : area of cross section
b : width of beam
E : modulus of elasticity
G : modulus of rigidity
h : height of beam
I : moment of inertia
k : shape factor
L : length of beam
M : bending moment
n_i : integer value
N : axial load
P : concentrated load

P_{cr} : critical buckling load
$T_{\max}$: maximum kinetic energy
u_i : displacement
U : strain energy due to bending
V_E : work done by external force
V_S : potential energy due to shear
w : transverse deflection of beam
ω : natural frequency
Π_P : total potential energy
ρ : density of material
P_i : material properties
V_{fi} : volume fraction
V_m : volume fraction of metal
V_c : volume fraction of ceramic
E_{fgm} : elastic modulus of fgm beam
ρ_{fgm} : density of fgm beam
γ_{fgm} : Poisson ratio of fgm beam
$T(K)$: temperature in Kelvin
P_{fgm} : material prorerty of fgm
σ_{xx} : bending stress
ε_{xx} : bending strain
τ_{xy} : shear stress
ε_{xy} : shear strain
$\varphi(x)$: rotation due to shear
μ : nonlocal parameter
m_2 : rotary inertia
κ_{fgm} : thermal conductivity of fgm beam
z : thickness of fgm beam
R_n : power law index of fgm beam