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A two dimensional mixed boundary-value problem in a viscoelastic medium

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Abstract. A fundamental solution for the transient, quasi-static, plane problems of linear viscoelasticity is introduced for a specific material. An integral equation has been found for any problem as a result of dynamic reciprocal identity which is written between this fundamental solution and the problem to be solved. The formulation is valid for the first, second and mixed boundary-value problems. This integral equation has been solved by BEM and algorithm of the BEM solution is explained on a sample, mixed boundary-value problem. The forms of time-displacement curves coincide with literature while time-surface traction curves being quite different in the results. The formulation does not have any singularity. Generalized functions and the integrals of them are used in a different form.

Keywords: viscoelasticity; quasi-static solution; reciprocity theorem; boundary element method; integral equations; transient loads.

1. Introduction

The viscoelastic behaviour of the materials attracted a great interest as a consequence of advances in the materials science. In the presented study, the use of dynamic reciprocity theorem in transient, isothermal viscoelasticity problems is introduced for a specific material which is linear, homogeneous and isotropic. A simple mathematical model has been selected to represent the viscoelastic behaviour of the material. Poisson's ratio has been considered to be constant. This assumption leads identical viscoelastic behaviour in bulk and shear. The inertia terms have been neglected in the formulation. Depending upon this negligence, the definition of a quasi-linear viscoelastic state has been presented similar to the concept of elastodynamic state given by Gurtin and Sternberg (1962). The stress tensor is related to the displacement gradient history (see, Christensen 1971) for the considered material. An integral equation is obtained from the dynamic reciprocal theorem (see, Achenbach 1973, 2003), which relates two different viscoelastic states of the same body. The first viscoelastic state in the expression of the reciprocal theorem represents the problem to be solved whereas the second one expresses the displacement and stress fields in an unbounded medium due to a sudden application of a time-dependent point load. The second state is also named as a fundamental solution. Here, a quasi-static fundamental solution is also constructed

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for the selected, specific material in an infinite plane region using Laplace Transform. Simple mathematical model provides to determine the exact solution of this fundamental state. Lee *et al.* (1994) have used an approximation for inverse Laplace Transform. Similar expressions can be found in literature, for example, Sim and Kwak (1988), Banerjee and Butterfield (1981) but this state given here does not involve any constant term which means the infinite medium at rest before the application time of the point load. In transient problems, entire motion starts when time is greater than zero. The aforementioned integral equation turns out to be a summation of the boundary integrals and these boundary integrals have Riemann convolutions in their kernels. For plane problems, the boundary integrals are reduced to line integrals. The aim of the boundary element method is to reduce the resulting integral equation to a system of linear algebraic equations for any time. The construction and the unknowns of this system are different for the first, second and mixed boundary-value problems. The construction of this system and the unknowns have been explained on a sample, mixed boundary-value problem, which is a thick and wide concrete column under a transient singular and eccentric normal force. The displacement components on a part of the boundary and the surface tractions on a second part of the boundary have been determined as functions of time for the sample problem. Results coincide with those given by others (Banerjee and Butterfield 1981, Mesquita and Coda 2007a, 2007b) for the forms of time-displacement curves but the forms of time-surface traction curves are quite different.

2. Basic formulation

The definition of a quasi-linear viscoelastic state is summarized below:

A region B with interior volume V and boundary S is considered. The ordered triple $\mathcal{S}[\boldsymbol{u}(\boldsymbol{x},t),\ \boldsymbol{\tau}(\boldsymbol{x},t), \boldsymbol{f}(\boldsymbol{x},t)]$ defines a quasi-linear viscoelastic state on $(\overline{V}\times\mathcal{T})$, where $\overline{V}$ is the closure of V and $\mathcal{T}$ is an arbitrary interval of time. $\boldsymbol{u}(\boldsymbol{x}, t)$ is displacement vector and $\boldsymbol{x}$, t denote the position vector of a point and time, respectively. $\boldsymbol{\tau}(\boldsymbol{x}, t)$ is the stress tensor. $\boldsymbol{f}$ denotes body force. They satisfy the following relations

$$\tau_{kj,j}+f_i = 0 \tag{1}$$

$$\tau_{ij} = \int_{-\infty}^{t}\left\{\lambda(t-\xi)\frac{\partial\varepsilon_{kk}(\xi)}{\partial\xi}\delta_{ij}+2\mu(t-\xi)\frac{\partial\varepsilon_{ij}(\xi)}{\partial\xi}\right\}d\xi \tag{2}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{3}$$

$$\lambda(t) = \lambda_o\varphi(t),\quad \mu(t) = \mu_o\varphi(t),\quad \lambda_o = \frac{2\nu}{1-2\nu}\mu_o \tag{4}$$

where $\lambda(t)$ and $\mu(t)$ are time dependent coefficients and ν is the Poisson's ratio. It is assumed that time dependency, represented by $\varphi(t)$ function, is the same for both $\lambda(t)$ and $\mu(t)$ functions (Kadioglu *et al.* 2007). ε_{ij} is the strain tensor. δ_{ij} represents Kronecker's delta. λ_o and μ_o indicate Lamé's elastic coefficients in classical elasticity. The integral form of stress-strain constituve relations in Eq. (2) is named as Stieltjes convolution notation (Christensen 1971), and, this form of hereditary integral type has been used by many authors (Syngellakis 2003, Syngellakis and Wu

2004, Wang and Birgisson 2007) while some others (Lahellec and Suquet 2007, Mesquita and Coda 2001, 2007a, 2007b) have used the differential equation form of constituve equation.

The expression of the dynamic reciprocal identity which is written between two viscoelastic states $\mathcal{S}^*[\boldsymbol{u}^*(\boldsymbol{x},t), \boldsymbol{\tau}^*(\boldsymbol{x},t), \boldsymbol{f}^*(\boldsymbol{x},t)]$ and $\mathcal{S}[\boldsymbol{u}(\boldsymbol{x},t), \boldsymbol{\tau}(\boldsymbol{x},t), \boldsymbol{f}(\boldsymbol{x},t)]$ of the same body is (Achenbach 1973 2003)

$$\int_S \boldsymbol{T} * \boldsymbol{u}^* dS + \int_V \boldsymbol{f} * \boldsymbol{u}^* dV = \int_S \boldsymbol{T}^* * \boldsymbol{u} dS + \int_V \boldsymbol{f}^* * \boldsymbol{u} dV \tag{5}$$

$$T_i = \tau_{ij} n_j, \quad T_i^* = \tau_{ij}^* n_j^* \tag{6}$$

where $\boldsymbol{T}$ and $\boldsymbol{T}^*$ are surface traction vectors in two states, respectively. $\boldsymbol{n}$ is the outward normal of the surface S. Sign * represents Riemann convolution as follows

$$\boldsymbol{f}(\boldsymbol{x},t) * \boldsymbol{g}(\boldsymbol{x},t) = \int_0^t f_i(\boldsymbol{x}, t-\xi) g_i(\xi) d\xi \tag{7}$$

It will be considered that the viscoelastic state $\mathcal{S}[\boldsymbol{u}(\boldsymbol{x},t), \boldsymbol{\tau}(\boldsymbol{x},t), \boldsymbol{f}(\boldsymbol{x},t)]$ represents a problem to be solved on the region B of volume V bounded by surface S. This problem (Sokolnikoff 1956) can be a first, second or mixed boundary-value problem. The body force $\boldsymbol{f}$ will be neglected in the formulation. The second viscoelastic state $\mathcal{S}^*[\boldsymbol{u}^*(\boldsymbol{x},t), \tau^*(\boldsymbol{x},t), \boldsymbol{f}^*(\boldsymbol{x},t)]$ represents the displacement and stress fields in an unbounded viscoelastic medium due to a sudden application of a time dependent point load $\boldsymbol{f}^*$. The viscoelastic state $\mathcal{S}^k[\boldsymbol{u}^k(\boldsymbol{x},\boldsymbol{y},t), \boldsymbol{\tau}^k(\boldsymbol{x},\boldsymbol{y},t), \boldsymbol{f}^k(\boldsymbol{x},\boldsymbol{y},t)]$, which will have been constructed here, will be used as $\mathcal{S}^*[\boldsymbol{u}^*(\boldsymbol{x},t), \boldsymbol{\tau}^*(\boldsymbol{x},t), \boldsymbol{f}^*(\boldsymbol{x},t)]$ in Eq. (5) for the quasi-static solutions of plane viscoelasticity problems.

3. A singular quasi-static viscoelastic state for the solutions of plane viscoelasticity problems

A body force in an infinite viscoelastic medium having the same material with the problem to be solved is defined as

$$\boldsymbol{f}^k(\boldsymbol{x},\boldsymbol{y},t) = \boldsymbol{e}_k g(t) \delta(\boldsymbol{x}-\boldsymbol{y}) \tag{8}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ represent the position vectors of an arbitrary point and a specific point of volume V, respectively. $\boldsymbol{e}_k$ (k = 1, 2) represents a base vector in Cartesian coordinates. $\delta(\boldsymbol{x}-\boldsymbol{y})$ is a generalized function, which is known as Dirac delta function satisfying following property for an infinite volume V

$$\begin{aligned} \int_V h(\boldsymbol{x}) \delta(\boldsymbol{x}-\boldsymbol{y}) dV_x &= h(\boldsymbol{y}) \quad \text{for} \quad \boldsymbol{y} \in V \\ &= 0 \quad \text{for} \quad \boldsymbol{y} \notin V \end{aligned} \tag{9}$$

Besides $g(t)$ function is selected as

$$g(t) = \delta(t), \quad \frac{d^n \delta(t)}{dt^n} = 0 \quad (n = 0, 1, 2, \ldots) \quad \text{for } t \le 0 \tag{10}$$

Similar to Achenbach's (1973, 2003) algorithm, $\boldsymbol{f}^k$ and the displacement field $\boldsymbol{u}^k$ due to this body force can be represented as

$$\boldsymbol{f}^k(\boldsymbol{x},\boldsymbol{y},t) = \nabla^2\left[\frac{1}{2\pi}e_k\ln(r)g(t)\right] \tag{11}$$

$$\boldsymbol{u}^k(\boldsymbol{x},\boldsymbol{y},t) = \nabla\phi^k + \nabla\wedge\psi^k \tag{12}$$

$$r = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} \tag{13}$$

The lower boundaries of the integrals in Eq. (2) can be taken as zero since the medium at rest for $t\leq 0$. This reduces a Stieltjes convolution to a Riemann convolution without an additional term having initial values. This was done before by Wang and Birgisson (2007), but not Syngellakis (2003) and Syngellakis and Wu (2004). Then substituting Eqs. (2) to (4), Eq. (7), Eq. (11) and Eq. (12) in Eq. (1)

$$\nabla^2\left[\varphi(t) * \left\{\frac{2(1-\nu)}{1-2\nu}\nabla\dot{\phi}^k + \nabla\wedge\dot{\psi}^k\right\}\right] = \nabla^2\left[\frac{-1}{2\pi\mu_o}e_k\ln(r)g(t)\right] \tag{14}$$

is found. $\dot{f}$ represents the time derivative of f. And the Laplace transform of Eq. (14) over t variable becomes

$$\nabla^2\left[\bar{\varphi}(s)s\left\{\frac{2(1-\nu)}{1-2\nu}\nabla\bar{\phi}^k(s) + \nabla\wedge\bar{\psi}^k(s)\right\}\right] = \nabla^2\left[\frac{-1}{2\pi\mu_o}e_k\ln(r)\bar{g}(s)\right] \tag{15}$$

From now on, the Laplace transform of any function $f(t)$ over t variable will be represented by $\bar{f}(s)$. Following Malvern (1969), a new vector function $\bar{\boldsymbol{\Phi}}^k(s)$ can be defined as

$$\bar{\boldsymbol{\Phi}}^k(s) = \frac{2(1-\nu)}{1-2\nu}\nabla\bar{\phi}^k(s) + \nabla\wedge\bar{\psi}^k(s) \tag{16}$$

From Eq. (16) and Eq. (15) $\bar{\boldsymbol{\Phi}}^k(s)$ can be expressed as

$$\bar{\boldsymbol{\Phi}}^k(s) = \frac{-1}{2\pi\mu_o}\frac{\bar{g}(s)}{s\bar{\varphi}(s)}e_k\ln(r) \tag{17}$$

Again following Malvern (1969), $\bar{\phi}^k(s)$ and the curl of $\bar{\psi}^k(s)$ can also be determined as follows

$$\bar{\phi}^k(s) = -\frac{(1-2\nu)}{8\pi\mu_o(1-\nu)}\frac{\bar{g}(s)}{s\bar{\varphi}(s)}(x_k-y_k)\ln(r) \tag{18}$$

$$\nabla\wedge\bar{\psi}^k(s) = \frac{-1}{2\pi\mu_o}\frac{\bar{g}(s)}{s\bar{\varphi}(s)}\left[e_k\ln(r) - \frac{1}{2}\nabla[(x_k-y_k)\ln(r)]\right] \tag{19}$$

Evaluating Laplace transform of Eq. (12) and substituting Eq. (18) and Eq. (19) in this expression, the Laplace transform of $\boldsymbol{u}^k(\boldsymbol{x},\boldsymbol{y},t)$ and inverting this $\boldsymbol{u}^k(\boldsymbol{x},\boldsymbol{y},t)$ due to the point load given in Eq. (8) is found as below

$$\bar{u}_i^k(\boldsymbol{x},\boldsymbol{y},s) = \frac{-1}{8\pi\mu_o(1-\nu)}\frac{\bar{g}(s)}{s\bar{\varphi}(s)}\left[(3-4\nu)\delta_{ik}\ln(r) - \frac{x_k'x_i'}{r^2}\right] \tag{20}$$

$$u_i^k(\boldsymbol{x},\boldsymbol{y},t) = \frac{-1}{8\pi\mu_o(1-\nu)}f_1(t)\left[(3-4\nu)\delta_{ik}\ln(r) - \frac{x'_k x'_i}{r^2}\right] \tag{21}$$

where

$$x'_k = x_k - y_k \tag{22}$$

$$f_1(t) = L^{-1}\frac{\bar{\delta}(s)}{s\bar{\varphi}(s)} \tag{23}$$

Using Eq. (3), the *(ij)*th component of strain tensor can be written as

$$\varepsilon_{ij}^k(\boldsymbol{x},\boldsymbol{y},t) = \frac{-1}{8\pi\mu_o(1-\nu)}f_1(t)\left[-\frac{x'_k}{r^2}\delta_{ij} + (1-2\nu)\left(\frac{x'_j}{r^2}\delta_{ik} + \frac{x'_i}{r^2}\delta_{jk}\right) + 2\frac{x'_k x'_i x'_j}{r^4}\right] \tag{24}$$

And substituting Eq. (24) in Eq. (2), the (*ij*)th component of the stress tensor can also be obtained as

$$\tau_{ij}^k(\boldsymbol{x},\boldsymbol{y},t) = -\frac{f_2(t)}{4\pi(1-\nu)}\left[(1-2\nu)\left(-\frac{x'_k}{r^2}\delta_{ij} + \frac{x'_j}{r^2}\delta_{ik} + \frac{x'_i}{r^2}\delta_{kj}\right) + 2\frac{x'_i x'_j x'_k}{r^4}\right] \tag{25}$$

where

$$f_2(t) = \varphi(t) * \dot{f}_1(t) \tag{26}$$

The expression of dynamic reciprocal identity (Eq. (5)) which is written between $\mathcal{S}[\boldsymbol{u}(\boldsymbol{x},t), \boldsymbol{\tau}(\boldsymbol{x},t), \boldsymbol{f}(\boldsymbol{x},t)]$ and $\mathcal{S}^k[\boldsymbol{u}^k(\boldsymbol{x},\boldsymbol{y},t), \boldsymbol{\tau}^k(\boldsymbol{x},\boldsymbol{y},t), \boldsymbol{f}^k(\boldsymbol{x},\boldsymbol{y},t)]$ is reduced to the following form:

$$\begin{aligned}&\int_S u_i^k(\boldsymbol{x},\boldsymbol{y},t) * \boldsymbol{T}_i(\boldsymbol{x},t)dS + \int_V u_i^k(\boldsymbol{x},\boldsymbol{y},t) * f_i(\boldsymbol{x},t)dV \\ &- \int_S T_i^k(\boldsymbol{x},\boldsymbol{y},t) * u_i(\boldsymbol{x},t)dV = \begin{cases} u_k(\boldsymbol{y},t) & \boldsymbol{y}\in V \\ 0 & \boldsymbol{y}\notin V\end{cases}\end{aligned} \tag{27}$$

It is clear that if the boundary values of $\boldsymbol{T}(\boldsymbol{x}, t)$ and $\boldsymbol{u}(\boldsymbol{x}, t)$ are known on the boundary S, displacement vector at an inner point $\boldsymbol{y}$ can be determined using Eq. (27). Besides the stress components can also be calculated at this point using Eq. (27) and Eqs. (2) to (4). This expression is given below:

$$\tau_{kl}(\boldsymbol{y},t) = \int_S u_i^{kl}(\boldsymbol{x},\boldsymbol{y},t) * \bar{\boldsymbol{T}}_i(\boldsymbol{x},t)dS_x + \int_V u_i^{kl}(\boldsymbol{x},\boldsymbol{y},t) * f_i(\boldsymbol{x},t)dV_x - \int_S T_{ij}^{kl}(\boldsymbol{x},\boldsymbol{y},t)n_j * u_i(\boldsymbol{x},t)dS_x \tag{28}$$

where

$$u_i^{kl}(\boldsymbol{x},\boldsymbol{y},t) = \frac{f_2(t)}{4\pi(1-\nu)}\left[(1-2\nu)\left(\frac{x'_l}{r^2}\delta_{ik} + \frac{x'_k}{r^2}\delta_{il} - \frac{x'_i}{r^2}\delta_{kl}\right) + 2\frac{x'_i x'_k x'_l}{r^4}\right] \tag{29}$$

$$\tau_{ij}^{kl}(\boldsymbol{x},\boldsymbol{y},t) = \frac{\mu_o f_3(t)}{2\pi(1-\nu)}\left[\begin{array}{l} -(1-4\nu)\delta_{ij}\delta_{kl}\dfrac{1}{r^2}+(1-2\nu)\left(\dfrac{1}{r^2}\delta_{il}\delta_{jk}+\dfrac{1}{r^2}\delta_{ik}\delta_{jl}\right)+ \\ 2(1-2\nu)\left(\dfrac{x'_k x'_l}{r^4}\delta_{ij}+\dfrac{x'_i x'_j}{r^4}\delta_{kl}\right)+ \\ 2\nu\left(\dfrac{x'_j x'_l}{r^4}\delta_{ik}+\dfrac{x'_j x'_k}{r^4}\delta_{il}+\dfrac{x'_i x'_k}{r^4}\delta_{jl}+\dfrac{x'_i x'_l}{r^4}\delta_{jk}\right)-8\dfrac{x'_i x'_j x'_k x'_l}{r^6} \end{array}\right] \tag{30}$$

where

$$f_3(t) = \varphi(t) * \dot{f}_2(t) \tag{31}$$

4. The sample viscoelastic material

A simple compression test has been considered. And the time dependencies of strain components are modeled in Cartesian coordinates as

$$\tau_{11} = \tau_{33}, \quad \tau_{22} = -\sigma_o \quad \text{for} \quad t \geq 0 \tag{32}$$

$$\varepsilon_{22} = -\frac{\sigma_o}{E_o}(1-e^{t/t_1}), \quad \varepsilon_{11} = \varepsilon_{33} \tag{33}$$

Here E_o is a constant and σ_o denotes the constant compression applied to the specimen. t_1 is a constant. The Poissons's ratio ν is also accepted to be constant for this material and time dependent coefficients are defined as

$$\lambda(t) = \frac{E(t)}{(1+\nu)(1-2\nu)}, \quad \mu(t) = \frac{E(t)}{2(1+\nu)} \tag{34}$$

The results of experiments of Akbarov (2005) and approximation are given in Fig. 1.

Eq. (2) will be used to determine $E(t)$ function. Substitution of Eqs. (32) to (33) in Eq. (2) gives the following integral equation.

$$E(t) = \int_0^t E(t-\xi)\left[\frac{1}{t_1}e^{-(\xi/t_1)}\right]d\xi \tag{35}$$

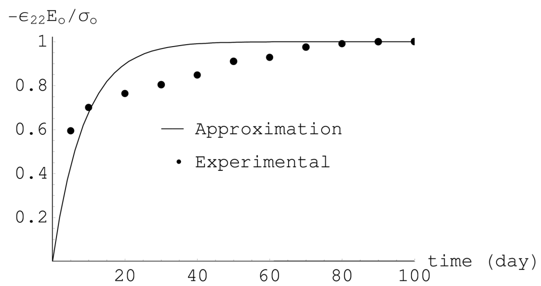

Fig. 1 Strain-time relations for the considered material

The solution of this integral equation (Kadioglu *et al.* (2007)) is

$$E(t) = E_o[1 + t_1 \delta(t)] \tag{36}$$

And the $\varphi(t)$ function mentioned in Eq. (4) and Laplace transform of it become

$$\varphi(t) = [1 + t_1 \delta(t)], \quad \bar{\varphi}(s) = \frac{1 + st_1}{s} \tag{37}$$

$f_1(t)$,$f_2(t)$ and $f_3(t)$ functions defined in Eqs. (23), (26) and (30) for this material can be calculated as follows

$$f_1(t) = H(t)\frac{1}{t_1}e^{-(t/t_1)}, \quad f_2(t) = \delta(t), \quad f_3(t) = \delta(t) + t_1 \dot{\delta}(t) \tag{38}$$

where $H(t)$ is Heaviside's unit step function and it is accepted that

$$H(t) = \begin{cases} 1 & (t > 0) \\ 0 & (t \le 0) \end{cases} \tag{39}$$

5. Sample problem

A vertical concrete column under an eccentrical normal force P = 1000 kN is considered acting at a point $\boldsymbol{y}_3$ as shown in Fig. 2. The body force will be neglected and the Poisson's ratio, ν is 0.2 and t_1 = 2.79 days. The problem is considered as a plane stress problem. The third dimension of the column is 0.4 m.

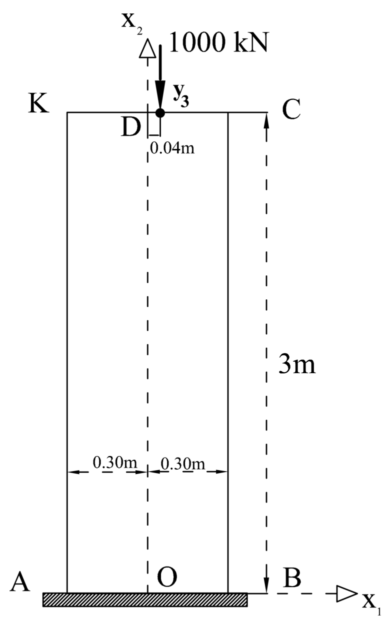

Fig. 2 Sample problem

The body force $\boldsymbol{f}$ will be defined as

$$\boldsymbol{f} = \boldsymbol{0} \tag{40}$$

And the boundary conditions of the problem are as follows:
The surface tractions on the $BCDKA$ part of the boundary can be defined as

$$\boldsymbol{T} = -PH(t)\,\delta(\boldsymbol{x}-\boldsymbol{y}_3)\boldsymbol{e}_2 \tag{41}$$

The displacement components on AB part of the boundary can be written as

$$u_1(\boldsymbol{x},t) = u_2(\boldsymbol{x},t) = 0 \quad \text{for} \quad x_2 = 0, \quad x_1 \in [-0.3, 0.3] \tag{42}$$

Since the problem is a plane problem, volume V and surface S came out to be a planar area and the summation of plane lines, respectively. And the integrals over boundary are reduced to line integrals. $BCKA$ and AB parts of the boundary are named as L_1 and L_2, respectively. The surface tractions are known on L_1 while displacements are known on L_2. Because of these, problem is a mixed boundary-value problem. From now on, $\mathcal{S}[\boldsymbol{u}(\boldsymbol{x},t), \boldsymbol{\tau}(\boldsymbol{x},t), \boldsymbol{0}]$ will represent the problem mentioned above. Substituting Eqs. (40) to (42) in Eq. (27), the following two integral equations given below are found

$$\begin{aligned} &-u_2^k(\boldsymbol{y}_3,\boldsymbol{y},t)P * H(t) + \int_{L_2} u_i^k(\boldsymbol{x},\boldsymbol{y},t) * T_i(\boldsymbol{x},t)dL_2 - \int_{L_1} T_i^k(\boldsymbol{x},\boldsymbol{y},t) * u_i(\boldsymbol{x},t)dL_1 \\ &= \begin{cases} u_k(\boldsymbol{y},t) & (\text{if } \boldsymbol{y} \text{ is an inner point of the } BCKA \text{ plane region}) \\ 0 & (\text{if } \boldsymbol{y} \text{ is outside of the } BCKA \text{ plane region}) \end{cases} \quad (k = 1,2) \end{aligned} \tag{43}$$

From now on, it is considered that point $\boldsymbol{y}$ is not an inner point of the planar surface, S. Then the unknowns of the problem become the surface traction vector $\boldsymbol{T}(\boldsymbol{x},t)$ on L_2 and the displacement vector $\boldsymbol{u}(\boldsymbol{x},t)$ on L_1. It must be emphasized that during integrations over the boundary, one must keep the region on the left. The procedure which will be used to solve these unknowns by Boundary Element Method has been explained below step by step:

The total boundary $(L_1 + L_2)$ is idealized as a collection of line segments which named as boundary elements. If the number of these line segments is N, the number of the end points, named as nodal points, is also N. The starting and end points of Jth element are $\boldsymbol{x}(J)$ and $\boldsymbol{x}(J+1)$. Besides time t is also divided to intervals with constant length, $l(J)$, the length of Jth line element. The starting point of the Ith interval is $t = t(I)$ and the end point is $t = t(I+1)$ with $t(1) = 0$. It is assumed that the variation of any displacement or stress component on the Jth line segment in the Kth time interval has the following form.

$$\begin{aligned} u_k(t,s) = &\left\{ u_k(K,J)\left[1-\frac{s}{l(J)}\right] + u_k(K,J+1)\left[\frac{s}{l(J)}\right]\right\}\left[1-\frac{t-t(K)}{t(K+1)-t(K)}\right] \\ &+\left\{ u_k(K+1,J)\left[1-\frac{s}{l(J)}\right] + u_k(K+1,J+1)\left[\frac{s}{l(J)}\right]\right\}\left[\frac{t-t(K)}{t(K+1)-t(K)}\right] \quad (k = 1,2) \end{aligned} \tag{44}$$

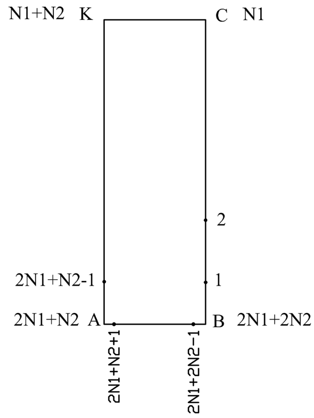

Fig. 3 Nodal points

$$T_k(t,s) = \left\{ T_k(K,J)\left[1-\frac{s}{l(J)}\right] + T_k(K,J+1)\left[\frac{s}{l(J)}\right]\right\}\left[1-\frac{t-t(K)}{t(K+1)-t(K)}\right]$$

$$+\left\{ T_k(K+1,J)\left[1-\frac{s}{l(J)}\right] + T_k(K+1,J+1)\left[\frac{s}{l(J)}\right]\right\}\left[\frac{t-t(K)}{t(K+1)-t(K)}\right] \quad (k=1,2) \tag{45}$$

where s is the distance from $\boldsymbol{x}(J)$ to any point between $\boldsymbol{x}(J)$ and $\boldsymbol{x}(J+1)$. A similar form to this variation has been used by Carrer and Mansur (2006) for an elastodynamic problem. After these definitions, the unknowns of the problem will be reduced to the nodal values of displacement components on L_1 and the surface traction vectors on L_2 at any time $t = t(K+1)$. Both BC and KA lines have been divided to $N1$ intervals while CK and AB divided to $N2$. Then the number of the nodal points becomes $N = 2N1 + 2N2$. And point B is selected as the last nodal point having the nod number N. After this selection the nodal numbers of C, K, A points become $N1$, $(N1 + N2)$ and $(2N1 + N2)$, respectively (Fig. 3).

Depending upon these, the numbers of the unknowns and their order can be expressed at any time $t(K+1)$, as follows:

First $(2N1 + N2 - 1)$ unknowns are the horizontal displacement components on nodal points on L_1 starting from $u_1(K+1, 1)$.

The following $(2N1 + N2 - 1)$ unknowns of the problem are the vertical displacement components on nodal points L_1 starting from $u_2(K+1, 1)$.

The third group of unknowns will be the horizontal component of the surface traction vector on nodal points on L_2. The values of this quantity are equal to zero for this problem at both A and B points. Then the first and last elements of this group, having $N2 - 1$ unknowns, will be $T_1(K+1, 2N1 + N2 + 1)$ and $T_1(K+1, 2N1 + 2N2 - 1)$. The last group of $N2 + 1$ unknowns are the vertical components of the surface traction vector on nodal points on L_2 starting from $T_2(K+1, 2N1 + N2)$.

And the total numbers of the unknowns becomes $M = 4N1 + 4N2 - 2$ at any time $t = t(K+1)$. To determine these unknowns M equations is necessary. Any of these equations can be written selecting loading point $\boldsymbol{y}$ to be any nodal point $\boldsymbol{x}(I)$ and k being equal to 1 or 2 in Eq. (43). But $\boldsymbol{x}(I)$ is a boundary point of the planar region. Because of this an artificial boundary including all of the

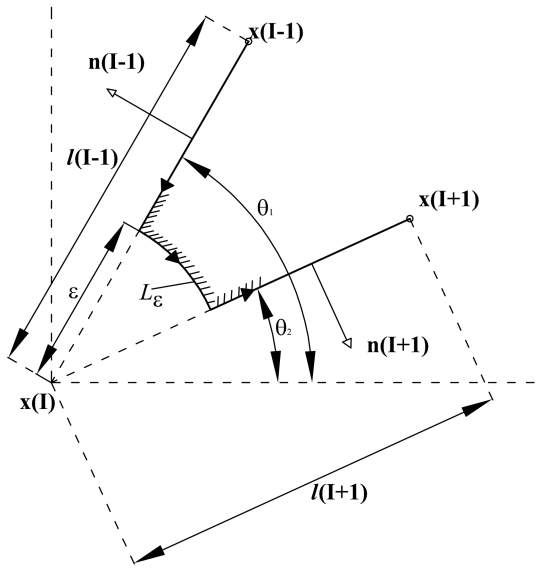

Fig. 4 Artificial boundary

line segments but not the nodal point $\boldsymbol{x}(I)$, will be defined for a singular loading on that nodal point. Around $\boldsymbol{x}(I)$ a small circular arc L_ε, with radius ε which leaves this nodal point outside the region (Fig. 4) is added to complete this artificial boundary (Kadioglu and Ataoglu 2007).

It is assumed that the variations of the components of the displacement and surface traction vectors on this circular arc will be represented by $u_k(t, \boldsymbol{x}(I))$ and $T_k(t, \boldsymbol{x}) = 0$ for $(k = 1, 2)$. As a consequence of the definition of the artificial boundary, when the loading point is $\boldsymbol{x}(I)$, right side of Eq. (43) becomes zero because $\boldsymbol{x}(I)$ is not a point in the region bounded by this artificial boundary. After necessary calculations, the radius ε will be shrunk to the nodal point $\boldsymbol{x}(I)$. The first assumption on circular arc, L_ε, means that any displacement component at a nodal point is single valued at any time t. The second assumption is that there is not a singular force acting at that nodal point. Then if a singular force exists at a point of the boundary this point must not be selected as a nodal point. After these, Eq. (43) takes the following form

$$\begin{aligned}&\int_{L_i} T_i^k(\boldsymbol{x},\boldsymbol{x}(I),t) * u_i(\boldsymbol{x},t)dL_1 + \lim_{\varepsilon\to 0}\left(\int_{L_\varepsilon} T_i^k(\boldsymbol{x},\boldsymbol{x}(I),t) * u_i(\boldsymbol{x}(I),t)dL_\varepsilon\right)\\&-\int_{L_2} u_i^k(\boldsymbol{x},\boldsymbol{x}(I),t) * T_i(\boldsymbol{x},t)dL_2 = -Pu_2^k(\boldsymbol{y}_3,\boldsymbol{x}(I),t) * H(t) \quad (k=1,2)\end{aligned} \tag{46}$$

where k represents the direction of the loading and when $k = 1$ this direction coincides with the direction of x_1 axis while $k = 2$ indicates the loading direction to be the direction of x_2 axis. As it is mentioned above, M equations, each of these corresponding to a singular loading at a nodal point in any direction, are necessary. The order of these loadings is as below:

For the first $(2N1 + N2 - 1)$ equations, loading points are the nodal points on L_1 and the loading index k is one and for the second $(2N1 + N2 - 1)$ equations, loading points are the same but index k is two. For the following $(N2 - 1)$ equations, k is one and the loading points are the nodal points on L_2 except A and B points. And in the last $(N2 + 1)$ equations, loading points are the nodal points on

L_2 either but including A and B points and k is two. Besides it must be emphasized that the convolutions in these equations, at any time $t = t(K+1)$ will be performed as

$$\boldsymbol{f}(\boldsymbol{x},t) * \boldsymbol{g}(\boldsymbol{x},t) = \int_0^{t(K+1)} f_i(\boldsymbol{x},\boldsymbol{t}(K+1)-\xi)g_i(\xi)d\xi \tag{47}$$

After writing the necessary M equations and substituting Eq. (6), Eq. (21), Eq. (25), the first two of Eqs. (38) and the Eqs. (44) to (45) in Eqs. (46), the following system of linear algebraic equations, given in partitioned form, is obtained for any time $t = t(K+1)$, $(K = 1, 2, 3...)$.

$$[\boldsymbol{A}^{M\times(4N1+2N2-2)}, \boldsymbol{B}^{M\times(2N2)} \cdot f12(K,K)]\boldsymbol{X}^{(4N1+4N2-2)\times 1}(K) = \mathbf{C}^{M\times 1}(K) + \boldsymbol{R}^{M\times 1}(K) \tag{48}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$ are constant matrices while $\boldsymbol{C}(K)$ and $\boldsymbol{R}(K)$ are dependent to time $t = t(K+1)$. The components of these matrices are given as follows

$$A(I,J) = \delta_{IJ}AD11(I) + \int_0^{l(J)}\left\{\sigma^1_{1i}(\boldsymbol{x},\boldsymbol{x}(I))n_i\left[1-\frac{s}{l(J)}\right]\right\}ds$$

$$+\int_0^{l(J-1)}\left\{\sigma^1_{1i}(\boldsymbol{x},\boldsymbol{x}(I))n_i\left[\frac{s}{l(J-1)}\right]\right\}ds$$

$$A(I,J+2N1+N2-1) = \delta_{IJ}AD12(I) + \int_0^{l(J)}\left\{\sigma^1_{2i}(\boldsymbol{x},\boldsymbol{x}(I))n_i\left[1-\frac{s}{l(J)}\right]\right\}ds$$

$$+\int_0^{l(J-1)}\left\{\sigma^1_{2i}(\boldsymbol{x},\boldsymbol{x}(I))n_i\left[\frac{s}{l(J-1)}\right]\right\}ds$$

$$A(I+2N1+N2-1,J) = \delta_{IJ}AD21(I) + \int_0^{l(J)}\left\{\sigma^2_{1i}(\boldsymbol{x},\boldsymbol{x}(I))n_i\left[1-\frac{s}{l(J)}\right]\right\}ds$$

$$+\int_0^{l(J-1)}\left\{\sigma^2_{1i}(\boldsymbol{x},\boldsymbol{x}(I))n_i\left[\frac{s}{l(J-1)}\right]\right\}ds$$

$$A(I+2N1+N2-1,J+2N1+N2-1) = \delta_{IJ}AD22(I) +$$

$$\int_0^{l(J)}\left\{\sigma^2_{2i}(\boldsymbol{x},\boldsymbol{x}(I))n_i\left[1-\frac{s}{l(J)}\right]\right\}ds + \int_0^{l(J-1)}\left\{\sigma^2_{2i}(\boldsymbol{x},\boldsymbol{x}(I))n_i\left[\frac{s}{l(J-1)}\right]\right\}ds$$

$$\text{for } (I=1 \text{ to } 2N1+N2-1, J=1 \text{ to } 2N1+N2-1) \tag{49}$$

where δ_{IJ} is the Kronecker's delta, and $\sigma^k_{ij}(\boldsymbol{x},\boldsymbol{y})$ is defined as

$$\sigma^k_{ij}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{f_2(t)}\tau^k_{ij}(\boldsymbol{x},\boldsymbol{y},t) \tag{50}$$

Additional matrices ***AD11***, ***AD12***, ***AD21*** and ***AD22*** which correspond to the second term in Eq. (49) can be expressed, in terms of θ_1 and θ_2 angles are shown in Fig. 4, as

$$
\begin{aligned}
AD11(I) &= -\frac{1}{4\pi(1-\nu)}[2(1-\nu)(\theta_2-\theta_1)-n_1(I)n_2(I)+n_1(I-1)n_2(I-1)] \\
AD12(I) &= -\frac{1}{4\pi(1-\nu)}[-n_2(I)n_2(I)+n_2(I-1)n_2(I-1)] \\
AD21(I) &= \frac{1}{4\pi(1-\nu)}[n_1(I)n_1(I)-n_1(I-1)n_1(I-1)] \\
AD22(I) &= -\frac{1}{4\pi(1-\nu)}[2(1-\nu)(\theta_2-\theta_1)+n_1(I)n_2(I)-n_1(I-1)n_2(I-1)]
\end{aligned}
\tag{51}
$$

The remaining terms of the matrix $\boldsymbol{A}$ are given as follows:

$$
\begin{aligned}
A(I+4N1+2N2-2,J) &= \int_0^{l(J)}\left\{\sigma_{1i}^{1}(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))n_i\left[1-\frac{s}{l(J)}\right]\right\}ds \\
&+\int_0^{l(J-1)}\left\{\sigma_{1i}^{1}(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))n_i\left[\frac{s}{l(J-1)}\right]\right\}ds \\
A(I+4N1+2N2-2,J+2N1+N2-1) &= \int_0^{l(J)}\left\{\sigma_{2i}^{1}(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))n_i\left[1-\frac{s}{l(J)}\right]\right\}ds \\
&+\int_0^{l(J-1)}\left\{\sigma_{2i}^{1}(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))n_i\left[\frac{s}{l(J-1)}\right]\right\}ds \\
&\text{for } (I=1 \text{ to } N2-1, J=1 \text{ to } 2N1+N2-1)
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
A(I+4N1+3N2-3,J) &= \int_0^{l(J)}\left\{\sigma_{1i}^{2}(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2-1))n_i\left[1-\frac{s}{l(J)}\right]\right\}ds \\
&+\int_0^{l(J-1)}\left\{\sigma_{1i}^{2}(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2-1))n_i\left[\frac{s}{l(J-1)}\right]\right\}ds \\
A(I+4N1+3N2-3,J+2N1+N2-1) &= \int_0^{l(J)}\left\{\sigma_{2i}^{2}(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2-1))n_i\left[1-\frac{s}{l(J)}\right]\right\}ds \\
&+\int_0^{l(J-1)}\left\{\sigma_{2i}^{2}(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2-1))n_i\left[\frac{s}{l(J-1)}\right]\right\}ds \\
&\text{for } (I=1 \text{ to } N2+1, J=1 \text{ to } 2N1+N2-1)
\end{aligned}
\tag{53}
$$

The elements of the matrix $\boldsymbol{B}$ are

$$B(I,J) = -\int_0^{l(J+2N1+N2)}\left\{\mathrm{v}_1^1(\boldsymbol{x},\boldsymbol{x}(I))\left[1-\frac{s}{l(J+2N1+N2)}\right]\right\}ds$$

$$-\int_0^{l(J+2N1+N2-1)}\left\{\mathrm{v}_1^1(\boldsymbol{x},\boldsymbol{x}(I))\left[\frac{s}{l(J+2N1+N2-1)}\right]\right\}ds$$

$$B(I+2N1+N2-1,J) = -\int_0^{l(J+2N1+N2)}\left\{\mathrm{v}_1^2(\boldsymbol{x},\boldsymbol{x}(I))\left[1-\frac{s}{l(J+2N1+N2)}\right]\right\}ds$$

$$-\int_0^{l(J+2N1+N2-1)}\left\{\mathrm{v}_1^2(\boldsymbol{x},\boldsymbol{x}(I))\left[\frac{s}{l(J+2N1+N2-1)}\right]\right\}ds$$

$$B(I,J+N2) = -\int_0^{l(J+2N1+N2)}\left\{\mathrm{v}_2^1(\boldsymbol{x},\boldsymbol{x}(I))\left[1-\frac{s}{l(J+2N1+N2)}\right]\right\}ds$$

$$-\int_0^{l(J+2N1+N2-1)}\left\{\mathrm{v}_2^1(\boldsymbol{x},\boldsymbol{x}(I))\left[\frac{s}{l(J+2N1+N2-1)}\right]\right\}ds$$

$$B(I+2N1+N2-1,J+N2) = -\int_0^{l(J+2N1+N2)}\left\{\mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(I))\left[1-\frac{s}{l(J+2N1+N2)}\right]\right\}ds$$

$$-\int_0^{l(J+2N1+N2-1)}\left\{\mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(I))\left[\frac{s}{l(J+2N1+N2-1)}\right]\right\}ds$$

$$\text{for } (I=1 \text{ to } 2N1+N2-1, J=1 \text{ to } N2-1) \tag{54}$$

$$B(I,N2) = -\int_0^{l(2N1+N2)}\left\{\mathrm{v}_2^1(\boldsymbol{x},\boldsymbol{x}(I))\left[1-\frac{s}{l(2N1+N2)}\right]\right\}ds$$

$$B(I+2N1+N2-1,N2) = -\int_0^{l(2N1+N2)}\left\{\mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(I))\left[1-\frac{s}{l(2N1+N2)}\right]\right\}ds$$

$$B(I,2N2) = -\int_0^{l(2N1+N2-1)}\left\{\mathrm{v}_2^1(\boldsymbol{x},\boldsymbol{x}(I))\left[\frac{s}{l(2N1+N2)}\right]\right\}ds$$

$$B(I+2N1+N2-1,2N2) = -\int_0^{l(2N1+2N2-1)}\left\{\mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(I))\left[\frac{s}{l(2N1+2N2-1)}\right]\right\}ds$$

$$\text{for } (I=1 \text{ to } 2N1+N2-1) \tag{55}$$

$$B(I+4N1+2N2-2,J) = -\int_0^{l(J+2N1+N2)} \left\{ \mathrm{v}_1^1(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))\left[1-\frac{s}{l(J+2N1+N2)}\right]\right\} ds$$

$$-\int_0^{l(J+2N1+N2-1)} \left\{ \mathrm{v}_1^1(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))\left[\frac{s}{l(J+2N1+N2-1)}\right]\right\} ds$$

$$B(I+4N1+3N2-2,J) = -\int_0^{l(J+2N1+N2)} \left\{ \mathrm{v}_1^2(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))\left[1-\frac{s}{l(J+2N1+N2)}\right]\right\} ds$$

$$-\int_0^{l(J+2N1+N2-1)} \left\{ \mathrm{v}_1^2(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))\left[\frac{s}{l(J+2N1+N2-1)}\right]\right\} ds$$

$$B(I+4N1+2N2-2,J+N2) = -\int_0^{l(J+2N1+N2)} \left\{ \mathrm{v}_2^1(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))\left[1-\frac{s}{l(J+2N1+N2)}\right]\right\} ds$$

$$-\int_0^{l(J+2N1+N2-1)} \left\{ \mathrm{v}_2^1(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))\left[\frac{s}{l(J+2N1+N2-1)}\right]\right\} ds$$

$$B(I+4N1+3N2-2,J+N2) = -\int_0^{l(J+2N1+N2)} \left\{ \mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))\left[1-\frac{s}{l(J+2N1+N2)}\right]\right\} ds$$

$$-\int_0^{l(J+2N1+N2-1)} \left\{ \mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))\left[\frac{s}{l(J+2N1+N2-1)}\right]\right\} ds$$

$$\text{for } (I=1 \text{ to } N2-1, J=1 \text{ to } (N2-1)) \tag{56}$$

$$B(I+4N1+2N2-2,N2) = -\int_0^{l(2N1+N2)} \left\{ \mathrm{v}_2^1(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))\left[1-\frac{s}{l(2N1+N2)}\right]\right\} ds$$

$$B(I+4N1+2N2-2,2N2) = -\int_0^{l(2N1+2N2-1)} \left\{ \mathrm{v}_2^1(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))\left[\frac{s}{l(2N1+2N2-1)}\right]\right\} ds$$

$$B(I+4N1+3N2-2,N2) = -\int_0^{l(2N1+N2)} \left\{ \mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))\left[1-\frac{s}{l(2N1+N2)}\right]\right\} ds$$

$$B(I+4N1+3N2-2,2N2) = -\int_0^{l(2N1+2N2-1)} \left\{ \mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(I+2N1+N2))\left[\frac{s}{l(2N1+2N2-1)}\right]\right\} ds$$

$$\text{for } (I=1 \text{ to } N2-1) \tag{57}$$

$$B(4N1+3N2-2,J) = -\int_0^{l(J+2N1+N2)} \left\{ \mathrm{v}_1^2(\boldsymbol{x},\boldsymbol{x}(2N1+N2)) \left[1 - \frac{s}{l(J+2N1+N2)} \right] \right\} ds$$

$$-\int_0^{l(J+2N1+N2-1)} \left\{ \mathrm{v}_1^2(\boldsymbol{x},\boldsymbol{x}(2N1+N2)) \left[\frac{s}{l(J+2N1+N2-1)} \right] \right\} ds$$

$$B(4N1+3N2-2,J+N2) = -\int_0^{l(J+2N1+N2)} \left\{ \mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(2N1+N2)) \left[1 - \frac{s}{l(J+2N1+N2)} \right] \right\} ds$$

$$-\int_0^{l(J+2N1+N2-1)} \left\{ \mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(2N1+N2)) \left[\frac{s}{l(J+2N1+N2-1)} \right] \right\} ds$$

$$B(4N1+4N2-2,J) = -\int_0^{l(J+2N1+N2)} \left\{ \mathrm{v}_1^2(\boldsymbol{x},\boldsymbol{x}(2N1+N2)) \left[1 - \frac{s}{l(J+2N1+N2)} \right] \right\} ds$$

$$-\int_0^{l(J+2N1+N2-1)} \left\{ \mathrm{v}_1^2(\boldsymbol{x},\boldsymbol{x}(2N1+2N2)) \left[\frac{s}{l(J+2N1+N2-1)} \right] \right\} ds$$

$$B(4N1+4N2-2,J+N2) = -\int_0^{l(J+2N1+N2)} \left\{ \mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(2N1+2N2)) \left[1 - \frac{s}{l(J+2N1+N2)} \right] \right\} ds$$

$$-\int_0^{l(J+2N1+N2-1)} \left\{ \mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(2N1+2N2)) \left[\frac{s}{l(J+2N1+N2-1)} \right] \right\} ds$$

$$\text{for } (J = 1 \text{ to } N2-1) \tag{58}$$

$$B(4N1+3N2-2,N2) = -\int_0^{l(2N1+N2)} \left\{ \mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(2N1+2N2)) \left[1 - \frac{s}{l(2N1+N2)} \right] \right\} ds \tag{59}$$

$$B(4N1+3N2-2,2N2) = -\int_0^{l(2N1+2N2-1)} \left\{ \mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(2N1+2N2)) \left[\frac{s}{l(2N1+2N2-1)} \right] \right\} ds \tag{60}$$

$$B(4N1+4N2-2,N2) = -\int_0^{l(2N1+N2)} \left\{ \mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(2N1+2N2)) \left[1 - \frac{s}{l(2N1+N2)} \right] \right\} ds$$

$$B(4N1+4N2-2,N2) = -\int_0^{l(2N1+2N2-1)} \left\{ \mathrm{v}_2^2(\boldsymbol{x},\boldsymbol{x}(2N1+2N2)) \left[\frac{s}{l(2N1+2N2-1)} \right] \right\} ds \tag{61}$$

where

$$\mathrm{v}_i^k(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{f_1(t)} u_i^k(\boldsymbol{x},\boldsymbol{y},t) \tag{62}$$

$$
\begin{aligned}
&\boldsymbol{X}(I) = u_1(K+1, I) \quad \text{for} \quad I = 1, 2N1+N2-1 \\
&\boldsymbol{X}(I+2N1+N2-1) = u_2(K+1, I) \quad \text{for} \quad I = 1, 2N1+N2-1 \\
&\boldsymbol{X}(I+4N1+2N2-2) = T_1(K+1, I+2N1+N2) \quad \text{for} \quad I = 1, N2-1 \\
&\boldsymbol{X}(I+4N1+3N2-3) = T_2(K+1, I+2N1+N2-1) \quad \text{for} \quad I = 1, N2+1
\end{aligned} \tag{63}
$$

$$
f12(K,K) = 1 - \frac{t_1}{t(K+1)-t(K)}[1-e^{-(t(K+1)-t(K))/t_1}] \tag{64}
$$

$$
\begin{aligned}
C(K)(I) &= -P\mathrm{v}_2^1(\boldsymbol{y}_3, \boldsymbol{x}(I))f0(K) \\
C(K)(I+2N1+N2-1) &= -P\mathrm{v}_2^2(\boldsymbol{y}_3, \boldsymbol{x}(I))f0(K) \quad \text{for} \quad I = 1, 2N1+N2-1
\end{aligned} \tag{65}
$$

where

$$
f0(K) = 1 - e^{-t(K+1)/t_1} \tag{66}
$$

$$
C(K)(I+4N1+2N2-2) = -P\mathrm{v}_2^1(\boldsymbol{y}_3, \boldsymbol{x}(I+2N1+N2))f0(K) \quad \text{for} \quad I = 1, N2-1 \tag{67}
$$

$$
C(K)(I+4N1+3N2-3) = -P\mathrm{v}_2^2(\boldsymbol{y}_3, \boldsymbol{x}(I+2N1+N2-1))f0(K) \quad \text{for} \quad I = 1, N2+1 \tag{68}
$$

$$
\begin{gathered}
R(K)(I) = -\textstyle\sum_{L=1}^{K-2}\{f11(K,L) \\
\{\textstyle\sum_{J=1}^{N2-1}[B(I,J)T_1(L, J+2N1+N2)] + \textstyle\sum_{J=1}^{N2+1}[B(I, J+N2-1)T_2(L, J+2N1+N2-1)]\} \\
+f12(K,L)\{\textstyle\sum_{J=1}^{N2-1}[B(I,J)T_1(L+1, J+2N1+N2)] + \\
\textstyle\sum_{J=1}^{N2+1}[B(I, J+N2-1)T_2(L+1, J+2N1+N2-1)]\}\} - f11(K,K) \\
\{\textstyle\sum_{J=1}^{N2-1}[B(I,J)T_1(K, J+2N1+N2)] + \\
\textstyle\sum_{J=1}^{N2+1}[B(I, J+N2-1)T_2(K, J+2N1+N2-1)]\} \quad \text{for } K = 2, 3, \ldots, \; I = 1, M \\
R(K)(I) = 0 \quad \text{for} \quad K = 1, \; I = 1, M
\end{gathered} \tag{69}
$$

where

$$
f11(K,L) = -\left(1 + \frac{t_1}{t(L+1)-t(L)}\right)e^{-(t(K+1)-t(L))/t_1} + \frac{t_1}{t(L+1)-t(L)}e^{-(t(K+1)-t(L+1))/t_1} \tag{70}
$$

$$
f12(K,L) = \frac{t_1}{t(L+1)-t(L)}e^{-(t(K+1)-t(L))/t_1} + \left(1 - \frac{t_1}{t(L+1)-t(L)}\right)e^{-(t(K+1)-t(L+1))/t_1} \tag{71}
$$

It is seen from Eq. (42) that

$$
\begin{aligned}
&T_1(1, J) = 0 \quad \text{for} \quad J = 2N1+N2+1, 2N1+2N2-1 \\
&T_2(1, J) = 0 \quad \text{for} \quad J = 2N1+N2, 2N1+2N2
\end{aligned} \tag{72}
$$

Substituting Eqs. (25) and (21) in Eqs. (50) and (62) $\mathrm{v}_i^k(\boldsymbol{x},\boldsymbol{y})$ and $\sigma_{ij}^k(\boldsymbol{x},\boldsymbol{y})$ functions are found. Using these expressions in Eqs. (49), (52), (53) and (54) to (61), performing the integrals, the entries of $\boldsymbol{A}$ and $\boldsymbol{B}$ constant matrices are calculated. But it must be indicated that there are singular terms during calculation of these integrals on the boundary elements having the numbers I-1 and I for a loading point $\boldsymbol{x}(I)$. Integrals have been calculated over these elements to eliminate these singularities analytically. The singularities which arise in the integrals of $\mathrm{v}_i^k(\boldsymbol{x},\boldsymbol{y})$ functions have the type of

$$S_1 = \lim_{s\to 0}\{s\ln(s)\} \tag{73}$$

which is zero. But the singularities arising in the integrals of $\sigma_{ij}^k(\boldsymbol{x},\boldsymbol{y})$ functions have the types of

$$S_2 = \lim_{s\to 0}\{\ln(s)\} \tag{74}$$

but the singular terms having this form eliminates each other during construction of the entries of the matrix $\boldsymbol{A}$. Substituting the constant matrices $\boldsymbol{A}$, $\boldsymbol{B}$, Eqs. (64), (65) to (68) and (69) in Eq. (48) a system of linear algebraic equations, whose unknowns have been defined in Eqs. (63), is found for

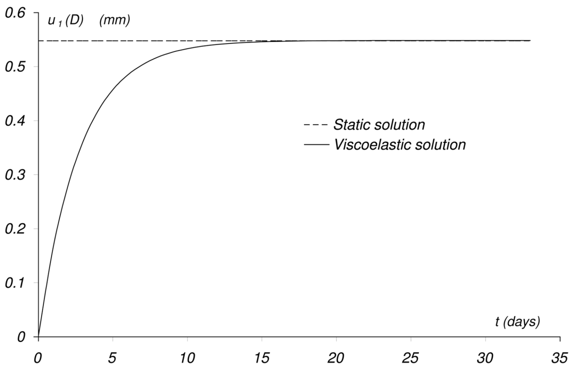

Fig. 5 Variation of the horizontal component of the displacement vector at point D versus time

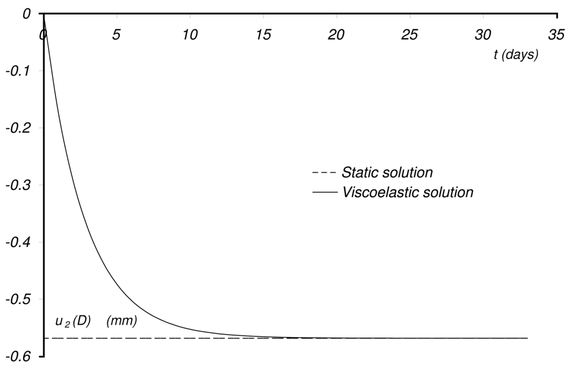

Fig. 6 Variation of the vertical component of the displacement vector at point D versus time

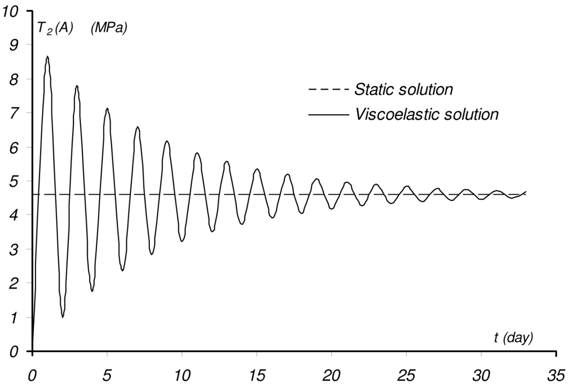

Fig. 7 Variation of the vertical component of the surface traction vector at point A versus time

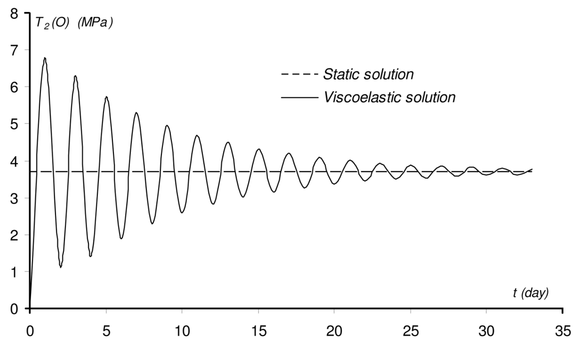

Fig. 8 Variation of the vertical component of the surface traction vector at point O versus time

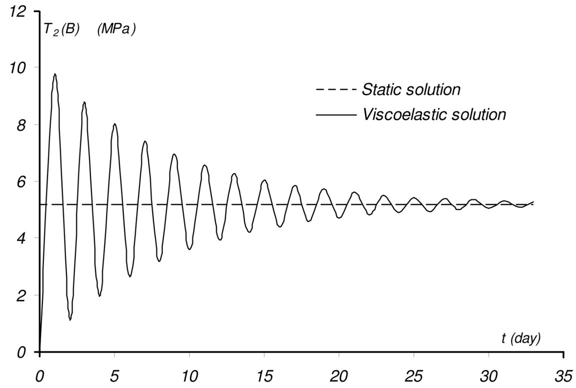

Fig. 9 Variation of the vertical component of the surface traction vector at point *B* versus time

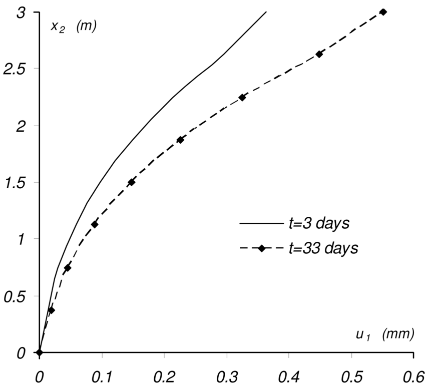

Fig. 10 Variation of the horizontal displacement component versus x_2 on *BC* line

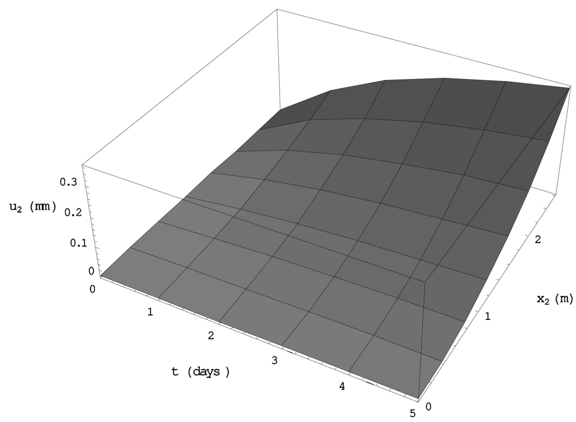

Fig. 11 Variation of the horizontal displacement component on *OD* line in time-x_2 plane

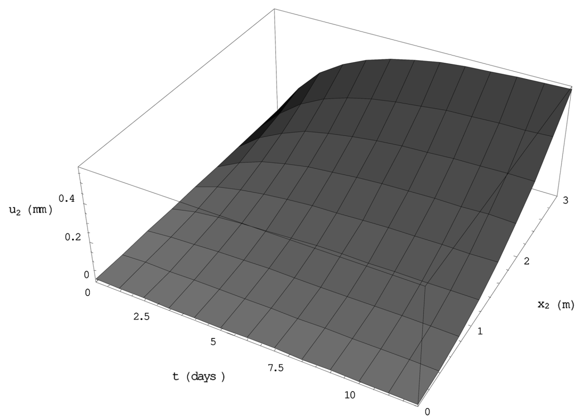

Fig. 12 Variation of the horizontal displacement component on *OD* line in time-x_2 plane

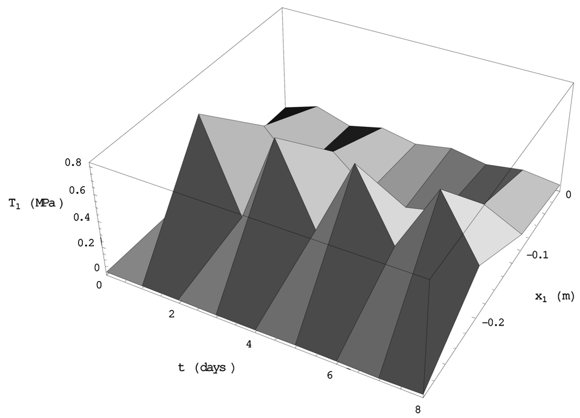

Fig. 13 Variation of the horizontal component of surface traction vector on *AO* line in time-x_1 plane

$t = t(K+1)$. Setting $K = 1$, the solution vector which corresponds to the nodal values of displacement and surface traction vectors at $t = t(2)$ is obtained. Then using these same unknowns can be calculated at $t = t(3)$. Here whole unknowns have been calculated till $t = t(34)$. Before starting dynamic problem, the static solution of the problem is solved writing $f12 = 1$, $\boldsymbol{R}(K) = \mathbf{0}$, $f0(K) = 1$ in Eq. (48). The variation of the horizontal and vertical components of displacement vector at point D versus time are given in Figs. 5 and 6, respectively.

Besides variations of the vertical component of the surface traction vector ($T_2 = -\tau_{22}$) at points A, O and B versus time are given in Figs. 7, 8 and 9, respectively.

And, the variation of the horizontal displacement component versus x_2 on BC line is also given for $t = 3$ days and $t = 33$ days in Fig. 10.

For a better demonstration of the results, the variation of the horizontal displacement component on OD line is also given in time-x_2 plane till $t = 5$ days and $t = 12$ days in Figs. 11 and 12, respectively.

And to be another demonstration, the variation of horizontal component of surface traction vector on AO line is given in time-x_1 plane till $t = 8$ days.

6. Conclusions

A solution method of plane problems of quasi-linear viscoelasticity has been explained on a sample mixed-boundary value problem for a specific material. It is known that inertia terms are neglected in quasi-static solutions. As a result of this negligence, the motion starts at $t = 0$ for every point of the region under a transient singular loading. This is not exactly true since the the starting times of the motion are different for the points having different positions relative to the point on which the transient load exists. But the arrival times are very small in comparison with the time scale of the problem. The time units have been given in days for the selected specific material. These arrival times are relatively very small for a problem having a very small time scale (nearly 30 milli-seconds) in nonviscous case either (Kadioglu and Ataoglu 2006). Therefore, for the specific concrete used here, this error can be neglected. The difference of the singular quasi-static state, which was given here, is that no constant terms exist in the expressions of both displacements $\boldsymbol{u}^k(\boldsymbol{x}, \boldsymbol{y}, t)$ and the stresses $\boldsymbol{\tau}^k(\boldsymbol{x}, \boldsymbol{y}, t)$. The initial values of any quantity have been hidden in $H(t)$ function which arising as a result of the convolutions in reciprocal identity. The viscoelastic solution of any problem must give the solution of the same problem for elastodynamic or elastostatic case as a limit if the viscous characteristics are eliminated in the formulation. In this study, to write $t_1 = 0$ is sufficient in Eq. (48) to achieve the elastostatic solution of the same problem. This fact can also be seen in Ref. (Kadioglu *et al.* 2007) which is the analytical solution of a viscoelasticity problem considering acceleration terms. The solution method, presented here, has been explained on the sample problem in details. Dynamic reciprocity theorem provides a relation between displacements, traction components and body forces for two loading states of the same body and this relation gives a boundary integral equation for unknown fields on the boundary, complementary to the applied fields. This integral equation has been solved numerically. The selected approximations for unknowns are linear in both time and space coordinates and the integral equation is reduced to a system of algebraic equations. Of course, higher order polynomials can be selected for a better approximation but it must be emphasized that dominant terms of the coefficients matrix are heavily dependent to the constant additional matrices ***AD11***, ***AD12***, ***AD21***, ***AD22***, and the constant terms in approximation polynomials while the effects of linear terms are secondary. The solution of the same problem for nonviscous case (elastostatic solution) has been found to check the accuracy of the formulation and the same elastostatic problem has also been solved by FEM (ANSYS 10.0) for a second control. Results are nearly the same for displacements while element numbers are quite different. In BEM, 28 elements are used while 38 elements (PLANE82) and 147 nodes in FEM. Relative errors are calculated using equilibrium equations. And the relative errors are 0.003868 and 0.00947 in horizontal and vertical directions respectively for BEM while 0.0086 and 0.0122 in FEM. The increment of the element number slightly affect the error after 28 in BEM. As an example, relative errors are 0.003499 and 0.00227 for 38 elements. Because of these, the number of boundary elements is also taken to be 28 for the viscoelastic problem. The forms of the variations of the displacement components by time coincides with the experimental results and the other numerical solutions [Sim and Kwak (1988), Mesquita and Coda (2007a, 2007b). But the forms of time-surface traction curves are clearly different. A similar formulation has been used by Wang and Birgisson (2007) and it is expected that their results must be also different from others. This cannot be easily seen in their results because they have selected to give the stress variation by time for jumping values of time ($t_1 = 0$ s, $t_2 = 20$ s, t = 100 s, $t_4 = 1000$ s). Here, the variation of any surface traction vector by time shows a damped vibration about nonviscous solution. The components of the displacement and surface traction vectors are dependent to histories of these quantities at any time $t = t(K + 1)$ and this fact has been represented by the matrix $\boldsymbol{R}(K)$ which exists in Eq. (48). A similar formulation has been presented before by Kadioglu and Ataoglu (2005, 2006) in time domain. And, another similar system has also been constructed for a different problem by Carrer and Mansur (2006). The presented results may help to the difficulties which arise during stress computations in experiments (Reddy and Ataoglu 2004).

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