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Probabilistic analysis of buckling loads of structures via extended Koiter law

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Abstract. Initial imperfections, such as initial deflection or remaining stress, cause deterioration of buckling strength of structures. The Koiter imperfection sensitivity law has been extended to describe the mechanism of reduction for structures. The extension is twofold: (1) a number of imperfections are considered, and (2) the second order (minor) imperfections are implemented, in addition to the first order (major) imperfections considered in the Koiter law. Yet, in reality, the variation of external loads is dominant over that of imperfection. In this research, probabilistic evaluation of buckling loads against external loads subjected to probabilistic variation is conducted by extending the concept of imperfection sensitivity. A truss arch subjected to dead and live loads is considered as a numerical example. The mechanism of probabilistic variation of buckling strength of this arch is described by the proposed method, and its reliability is evaluated.

Keywords: buckling loads; imperfection sensitivity law; probabilistic analysis.

1. Introduction

Initial imperfections of structures often cause significant reduction in buckling strength. Since

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initial imperfections are subject to probabilistic variation, the study of initial imperfections must be combined with probabilistic treatment to make it practical. There are several methodologies to describe this variation:

- The Monte Carlo simulation came to be conducted to compute numerically the reliability of the buckling strength for measured or random initial imperfections (Edlund and Leopoldson 1975, Elishakoff 1978).
- Stochastic finite element method (SFEM) was employed to numerically tackle the probabilistic properties of structures (Astill *et al.* 1972, Nakagiri and Hisada 1980).
- The response surface approach was used to evaluate the reliability of structures (Faravelli 1989, Bucher and Bourgund 1990, Chryssanthopoulos 1998).
- The imperfection sensitivity law by Koiter 1945 was used as a transfer function from an initial imperfection to the deterministic critical load and, in turn, to obtain the probabilistic variation of critical load for an imperfection with a known probabilistic property (Bolotin 1958, Thompson 1967, Ikeda and Murota 1993, 2002).

Among these methodologies, the authors focused on the last method based on the sensitivity law, in favor of the explicit expression of the buckling load as a function of initial imperfections. This method has recently been extended to describe the mechanism of reduction for realistic structures (Ikeda and Ohsaki 2007, Ohsaki and Ikeda 2007). The extension is twofold: (1) a number of imperfections are considered, and (2) the second order (minor) imperfections are considered, in addition to the first order (major) imperfections considered in the Koiter law. Yet, in the evaluation of probabilistic buckling strength of realistic structures, the variation of external loads is predominant over that of imperfection.

In this research, the framework of the probabilistic method based on the concept of imperfection sensitivity is extended to evaluate the influence of the variation of external loads on buckling strength of a truss arch structure, modeling a bridge subjected to live and dead loads. The mechanism of probabilistic variation of buckling strength of this arch is described by the proposed method, and its reliability is evaluated.

2. Asymptotic theory

Asymptotic theory of nonlinear systems is summarized (Ikeda and Murota 2002, Ohsaki and Ikeda 2007).

2.1 Theoretical framework

We consider a system of nonlinear governing or equilibrium equations

$$\mathbf{F}(\mathbf{u}, f, \mathbf{v}) = \mathbf{0} \tag{1}$$

where $\mathbf{u}$ indicates an N-dimensional independent variable vector; f denotes a loading parameter; and $\mathbf{v}$ denotes a p-dimensional imperfection parameter vector.

At a critical point $(\mathbf{u}_c, f_c)$, such as a bifurcation point, the following condition of criticality is satisfied

$$\det\partial\mathbf{F}/\partial\mathbf{u} = 0 \tag{2}$$

Here $(\cdot)_c$ denotes a variable related to the critical point.

The imperfection parameter vector $\mathbf{v}$ is expressed as

$$\mathbf{v} = \mathbf{v}^0 + \varepsilon\mathbf{d} = \mathbf{v}^0 + \mathbf{e} \tag{3}$$

where $\mathbf{v}^0$ denotes the value of the imperfection parameter vector $\mathbf{v}$ for the perfect system, $\mathbf{d}$ is called the imperfection pattern vector which is normalized appropriately by the magnitude of initial imperfection ε; $\mathbf{e}$ is the imperfection vector, which plays a central role in this paper.

2.2 Imperfection sensitivity law

We focus on a simple-unstable-symmetric bifurcation point $(\mathbf{u}_c^0, f_c^0)$ for the perfect system with $\mathbf{v} = \mathbf{v}^0$, and consider a critical point $(\mathbf{u}_c, f_c)$ of an imperfect system with an initial imperfection $\mathbf{v} \neq \mathbf{v}^0$ $(\varepsilon\mathbf{d} \neq 0)$. We are interested in the change of buckling load between the perfect and imperfect systems, being defined as

$$\tilde{f}_c = f_c - f_c^0 \tag{4}$$

The dependency of $\tilde{f}_c$ on initial imperfections can be expressed by the Koiter imperfection sensitivity law

$$\tilde{f}_c = -(\mathbf{a}^T\mathbf{d}\varepsilon)^{2/3} + \text{h.o.t.} = -C\varepsilon^{2/3} + \text{h.o.t.} \tag{5}$$

where $C > 0$ is a constant and $\mathbf{a}$ is a constant vector. In the Koiter law (5), the imperfection pattern vector $\mathbf{d}$ is set constant and ε is chosen as an independent variable.

The law (5) was extended in Ikeda and Murota 2002 for a number of imperfection variables, expressed by an imperfection vector $\mathbf{e}(=\varepsilon\mathbf{d})$, namely

$$\tilde{f}_c = -(\mathbf{a}^T\mathbf{e})^{2/3} + \text{h.o.t.} \tag{6}$$

The law (6), as well as (5), captures the influence of the first order (major) imperfections of the power of 2/3rd, and is valid asymptotically for small imperfection magnitude $\|\mathbf{e}\| \sim 0$, but is inaccurate when it is large. In fact, it was pointed out that the minor imperfection can sometimes have larger effect than the major imperfection (Ohsaki 2002). In order to address this problem, the law (6) was extended to include the second order effect as follows (Ikeda and Ohsaki 2007, Ohsaki and Ikeda 2007)

$$\begin{aligned}\tilde{f}_c &= -(\mathbf{a}^T\mathbf{d}\varepsilon)^{2/3} + \mathbf{b}^T\mathbf{d}\varepsilon + \text{h.o.t.}\\ &= -(\mathbf{a}^T\mathbf{e})^{2/3} + \mathbf{b}^T\mathbf{e} + \text{h.o.t.}\\ &= -a^{2/3} + b + \text{h.o.t.}\end{aligned} \tag{7}$$

where $\mathbf{b}$ is a constant vector, and a and b are scalar variables to be defined as functions of $\mathbf{e}$ as

$$a = a(\mathbf{e}) = \mathbf{a}^T\mathbf{e}, \quad b = b(\mathbf{e}) = \mathbf{b}^T\mathbf{e} \tag{8}$$

3. Theory on probabilistic variation of buckling loads

Probabilistic variation of buckling loads is investigated when the imperfection vector $\mathbf{e}$ in the sensitivity law (7) is subjected to probabilistic variation.

Consider the probabilistic variation of the imperfection vector $\mathbf{e}$, and denote its average by $\bar{\mathbf{e}}$, and define its variance-covariance matrix by

$$W = \overline{(\mathbf{e}-\bar{\mathbf{e}})(\mathbf{e}-\bar{\mathbf{e}})^T} \tag{9}$$

where $\overline{\cdots}$ is the average of the associated variable.

3.1 General properties

Then probabilistic properties of $a = a(\mathbf{e})$ and $b = b(\mathbf{e})$ in (8) are:

- their averages are $\bar{a} = \mathbf{a}^T\bar{\mathbf{e}}$ and $\bar{b} = \mathbf{b}^T\bar{\mathbf{e}}$, respectively;
- their variance-covariance matrix is given by

$$W_{ab} = \begin{pmatrix} \mathbf{a}^T W\mathbf{a} & \mathbf{a}^T W\mathbf{b} \\ \mathbf{a}^T W\mathbf{b} & \mathbf{b}^T W\mathbf{b} \end{pmatrix} \tag{10}$$

For a general case, for which the covariance $\mathbf{a}^T W\mathbf{b}$ is non-zero, the evaluation of the probability density function of the buckling load reduction $\tilde{f}_c$ in (7) becomes cumbersome.

3.2 Simple procedure

A simple procedure suitable for practical application is presented on the basis of the following assumptions:

- The imperfection vector $\mathbf{e}$ is subjected to an normal distribution with mean $\bar{\mathbf{e}}$ and variance σ^2: $N(\bar{\mathbf{e}}, \sigma^2)$.
- The covariance satisfies

$$\mathbf{a}^T W\mathbf{b} = 0 \tag{11}$$

This assumption actually holds for the example truss arch studied in the next section.

Then $a = \mathbf{a}^T\mathbf{e}$ and $b = \mathbf{b}^T\mathbf{e}$ are statistically independent, and respectively subjected to normal distributions $N(\bar{a}, \sigma_a^2)$ and $N(\bar{b}, \sigma_b^2)$ with

$$\bar{a} = \mathbf{a}^T\bar{\mathbf{e}}, \quad \bar{b} = \mathbf{b}^T\bar{\mathbf{e}} \tag{12}$$

$$\sigma_a^2 = \mathbf{a}^T W\mathbf{a}, \quad \sigma_b^2 = \mathbf{b}^T W\mathbf{b} \tag{13}$$

Namely, the probability density functions of a and b respectively are given by

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This assumption actually holds for the example truss arch studied in the next section.

$$\phi_a = \frac{1}{\sqrt{2\pi}\sigma_a}\exp\left(-\frac{(a-\bar{a})^2}{2\sigma_a^2}\right) \quad (-\infty < a < \infty) \tag{14}$$

$$\phi_b = \frac{1}{\sqrt{2\pi}\sigma_b}\exp\left(-\frac{(b-\bar{b})^2}{2\sigma_b^2}\right) \quad (-\infty < b < \infty) \tag{15}$$

Then the probability density functions of the 2/3rd power $\hat{a} = a^{2/3}$ is to be obtained by transforming (14) as

$$\phi_{\hat{a}} = \frac{3|\hat{a}|^{1/2}}{\sqrt{2\pi}\sigma_a}\exp\left(-\frac{(|\hat{a}|^{3/2}-\bar{a})^2}{2\sigma_a^2}\right) \quad (-\infty < \hat{a} < \infty) \tag{16}$$

The probability density function of the buckling load reduction $\tilde{f}_c$ in (7) is given in an integral form

$$\phi_{\tilde{f}_c} = \int_{-\infty}^{0} \phi_{\hat{a}}(\hat{a})\,\phi_b(\tilde{f}_c - \hat{a})\,d\hat{a} \quad (-\infty < \tilde{f}_c < \infty) \tag{17}$$

Remark 1 The assumption of the normal distribution for **e** employed herein has simplified the evaluation of probability density function. It will be a topic in the future to evaluate the accuracy of this assumption in view of actual data of live loads by traffic. □

3.3 Vanishing of covariance for system with bilateral symmetry

The simplifying assumption (11) holds generically for unstable bifurcation of a system with bilateral symmetry (Ikeda and Fujisawa 2006). The numerical example in this paper actually corresponds to this case.

The equilibrium Eq. (1) satisfies the symmetry condition, called equivariance in nonlinear mathematics

$$T(\sigma)\mathbf{F}(\mathbf{u},f,\mathbf{v}) = \mathbf{F}(T(\sigma)\mathbf{u},f,S(\sigma)\mathbf{v}) \tag{18}$$

where σ denotes a bilateral reflection, $T(\sigma)$ is $N \times N$ matrix representing the action of the reflection σ on $\mathbf{u}$, and $S(\sigma)$ is $p \times p$ matrix representing the action of the reflection σ on $\mathbf{v}$. We also assume the objectivity of the variance-covariance matrix W, being expressed as

$$T(\sigma)W = WT(\sigma) \tag{19}$$

We consider the transformation that decomposes the imperfection vector **e** into symmetric and anti-symmetric components

$$\mathbf{e} = H\hat{\mathbf{e}} = (H^+, H^-)\begin{pmatrix}\hat{\mathbf{e}}^+\\ \hat{\mathbf{e}}^-\end{pmatrix} = H^+\hat{\mathbf{e}}^+ + H^-\hat{\mathbf{e}}^- \tag{20}$$

Here H is $p \times p$ transformation matrix, H^+ is a block matrix of H that consists of symmetric components, H^- is a block that consists of anti-symmetric components, and $\hat{\mathbf{e}}^+$ and $\hat{\mathbf{e}}^-$ are symmetric and anti-symmetric imperfection vectors, respectively.

In view of the symmetry condition (19), the variance-covariance matrix W can be put into a block diagonal form

$$H^T W H = \begin{pmatrix} H^{+T} W H^+ & O \\ O & H^{-T} W H^- \end{pmatrix} \tag{21}$$

With the use of the transformation (20), the sensitivity law (7) can be rewritten as

$$\tilde{f}_c = -(\mathbf{a}^T H^+ \hat{\mathbf{e}}^+ + \mathbf{a}^T H^- \hat{\mathbf{e}}^-)^{2/3} + \mathbf{b}^T H^+ \hat{\mathbf{e}}^+ + \mathbf{b}^T H^- \hat{\mathbf{e}}^- + \text{h.o.t.} \tag{22}$$

We refer to the fact that only anti-symmetric imperfections contribute to the 2/3rd order term in the sensitivity law (22), and symmetric ones to the linear term (cf., Ikeda and Fujisawa 2006). This fact entails

$$\mathbf{a}^T H^+ = \mathbf{0}^T \quad \mathbf{b}^T H^- = \mathbf{0}^T \tag{23}$$

and simplifies the sensitivity law in (22) as

$$\tilde{f}_c = -(\mathbf{a}^T H^- \hat{\mathbf{e}}^-)^{2/3} + \mathbf{b}^T H^+ \hat{\mathbf{e}}^+ + \text{h.o.t.} \tag{24}$$

With the use of (21) and (23), the covariance of the variables a and b can be shown to vanish

$$\begin{aligned} \mathbf{a}^T W \mathbf{b} &= \mathbf{a}^T H H^T W H H^T \mathbf{b} \\ &= (\mathbf{a}^T H^+, \mathbf{a}^T H^-) H^T W H \begin{pmatrix} H^{+T} \mathbf{b} \\ H^{-T} \mathbf{b} \end{pmatrix} \\ &= (O, \mathbf{a}^T H^-) \begin{pmatrix} H^{+T} W H^+ & O \\ O & H^{-T} W H^- \end{pmatrix} \begin{pmatrix} H^{+T} \mathbf{b} \\ O \end{pmatrix} \\ &= O \end{aligned} \tag{25}$$

Thus the simplifying assumption (11) holds generically for unstable bifurcation of a system with bilateral symmetry.

4. Numerical example

Consider the truss arch as shown in Fig. 1 as an example. All the truss members have the same linear elastic member properties: EA = 1.0, where E is the modulus of elasticity and A is the cross section.

4.1 Live and dead loads

This arch is subjected to the vertical loads $f \times \mathbf{f}$, where f is the loading parameter and $\mathbf{f} = (f_1, ..., f_5)^T$ is the load pattern vector. We consider dead loads and live loads, and set

$$\mathbf{f} = D\mathbf{f}_D + L_1 \mathbf{f}_{L1} + L_2 \mathbf{f}_{L2} \tag{26}$$

where D expresses the magnitude of the dead loads and L_1 and L_2 are the magnitudes of two types of live loads; the dead load pattern vector $\mathbf{f}_D$ and the live load pattern vectors $\mathbf{f}_{L1}$ and $\mathbf{f}_{L2}$ are set to be

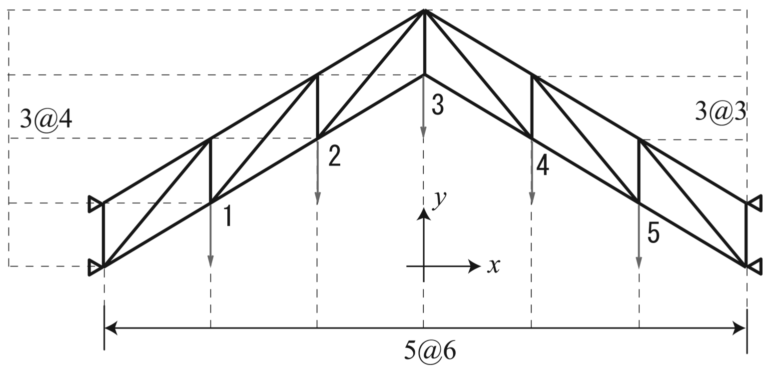

Fig. 1 Truss arch

$$\mathbf{f}_D = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{f}_{L1} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{f}_{L2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \tag{27}$$

Note that the live load $\mathbf{f}_{L1}$ acts on the left side of the arch, and the live load $\mathbf{f}_{L2}$ acts on its right side.

4.2 Perfect behavior for dead loads

The equilibrium paths of the arch as shown in Fig. 2 were computed for D = 5, L_1 = L_2 = 0, for which only dead loads are present. On the fundamental path, the bifurcation point shown by was found at f_c^0 = 0.0022, at which a bifurcated path emanated. Deformation modes of the arch are shown in Fig. 3, which displays in (a) that bilateral symmetry is preserved on the fundamental path,

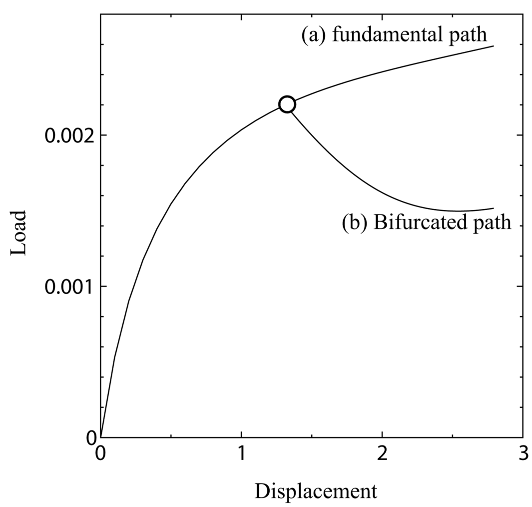

Fig. 2 Equilibrium paths of the arch

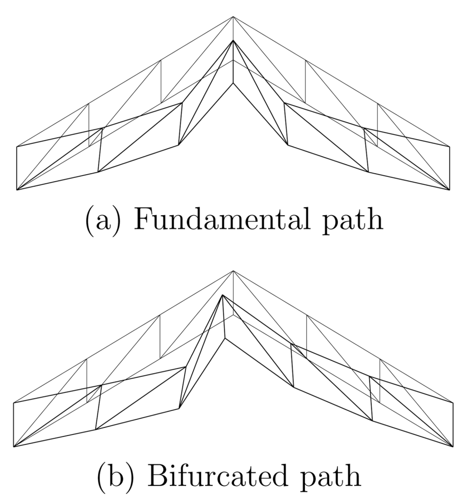

Fig. 3 Deformation modes of the arch

and in (b) that such symmetry is broken on the bifurcated path.

4.3 Interpretation of live loads as imperfections

We march on to consider the influence of live loads. For this purpose, the concept of imperfection sensitivity is extended to live loads. We set

$$\mathbf{e} = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \tag{28}$$

We consider a transformation matrix

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{29}$$

Then the imperfection vector e is expressed as (cf. (20))

$$\mathbf{e} = H^{+}\hat{e}^{+} + H^{-}\hat{e}^{-} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\hat{e}^{+} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\hat{e}^{-} \tag{30}$$

Or, inversely,

$$\hat{\mathbf{e}} = H^{T}\mathbf{e} \Rightarrow \hat{e}^{+} = \frac{1}{\sqrt{2}}(L_1 + L_2) \quad \hat{e}^{-} = \frac{1}{\sqrt{2}}(L_1 - L_2) \tag{31}$$

4.4 Imperfection sensitivity law for live loads

Carrying out several path tracing analyses of the arch for various values of $\hat{e}^{+}$, we obtained the imperfection sensitivity law for $\hat{e}^{+}$ $(\hat{e}^{-} = 0)$ as

$$\tilde{f}_c \sim C^{+}\hat{e}^{+} \approx -2.34 \times 10^{-4}\hat{e}^{+} \tag{32}$$

and the imperfection sensitivity law for $\hat{e}^{-}$ $(\hat{e}^{+} = 0)$ was obtained as

$$\tilde{f}_c \sim C^{-}(\hat{e}^{-})^{2/3} \approx -2.41 \times 10^{-4}(\hat{e}^{-})^{2/3} \tag{33}$$

In the derivation of these laws (32) and (33), the path-tracing analyses of approximately 10 imperfect structures were conducted. The linear law (32) and the 2/3rd power law (33) are combined, and are expressed in terms of $\mathbf{e} = (L_1, L_2)^T$ by (31) to arrive at

$$\begin{aligned}\tilde{f}_c &\sim C^{-}(\hat{e}^{-})^{2/3} + C^{+}\hat{e}^{+} \\ &= (C^{-}/2^{1/3})(L_1 - L_2)^{2/3} + (C^{+}/\sqrt{2})(L_1 + L_2) \\ &= -1.91 \times 10^{-4}(L_1 - L_2)^{2/3} - 1.66 \times 10^{-4}(L_1 + L_2)\end{aligned} \tag{34}$$

From (34), the vectors $\mathbf{a}$ and $\mathbf{b}$ in (24) are evaluated to

$$\mathbf{a} = 2.64\times10^{-6}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{b} = -1.66\times10^{-4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{35}$$

4.5 Probabilistic variation of live loads

Live loads $\mathbf{L} = (L_1, L_2)^T$ are assumed to be subjected to a multi-variate normal distribution with average $\overline{\mathbf{L}} = (\mu, \mu)^T$, and with variance-covariance matrix

$$W = \begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix} \tag{36}$$

The averages of the variables a and b in (8) are given by (12) as

$$\bar{a} = \mathbf{a}^T\bar{\mathbf{e}} = 0, \quad \bar{b} = \mathbf{b}^T\bar{\mathbf{e}} = -3.32\times10^{-4}\ \mu \tag{37}$$

The variance-covariance matrix (10) for the variables a and b in (8) reads

$$W_{ab} = \mathbf{a}^T W \mathbf{b} = \begin{pmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_b^2 \end{pmatrix} \tag{38}$$

with

$$\sigma_a^2 = (2.64\times10^{-6})^2\{2\sigma^2(1-\rho)\}, \quad \sigma_b^2 = (-1.66\times10^{-4})^2\{2\sigma^2(1+\rho)\} \tag{39}$$

The theoretical curve of the probability density function of the buckling load reduction $\tilde{f}_c$ in Fig. 4 is plotted by the formula (17) with the use of the averages of a and b in (37) and their variances in (39), and ρ =0.15, μ =0.4, σ =0.1. This theoretical curve correlates well with the histogram obtained by the Monte Carlo simulation, in which the set of buckling loads were obtained by the path-tracing for a series of random $\mathbf{e}$'s that were generated following the prescribed normal distribution. This assesses the validity of the present theory.

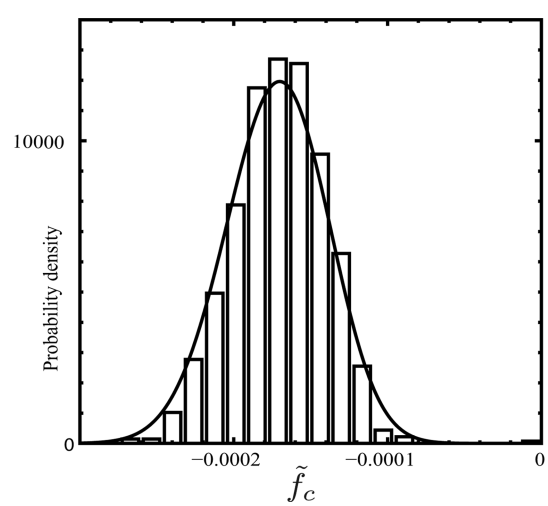

Fig. 4 Comparison of theoretical probability density function of $\tilde{f}_c$ with the histogram by Monte Carlo simulation

Remark 2 For this arch, the transformation matrix $T(\sigma)$ in (19) reads

$$T(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{40}$$

and the objectivity condition (19) is satisfied as follows

$$T(\sigma)W = WT(\sigma) = \begin{pmatrix} \rho\sigma^2 & \sigma^2 \\ \sigma^2 & \rho\sigma^2 \end{pmatrix} \tag{41}$$

The orthogonality condition

$$\mathbf{a}^T W\mathbf{b} = 2.64\times10^{-6}(1,-1)\begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix}\left\{-1.66\times10^{-4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} = 0 \tag{42}$$

is satisfied. □

4.6 Reliability of the arch

The reliability of the arch is investigated. Recall that the external loads were given by $f\times\mathbf{f} = f\times(f_1,\ldots,f_5)^T$, and the value f_c of loading parameter at buckling has been interpreted as the buckling load up to now.

In the evaluation of the reliability, we consider $\mathbf{f}$ as external loads and interpret $f_c = \gamma$ as the safety factor. Namely, the arch collapses for $\gamma \geq 1$ and remains safe for $\gamma < 1$.

The possibility of failure P_f can be expressed as

$$P_f = \int_{-\infty}^{1} \phi_\gamma(\gamma)d\gamma \tag{43}$$

We design the section rigidity EA for $P_f \simeq 1.0\times10^{-4}$ and for two different values of the

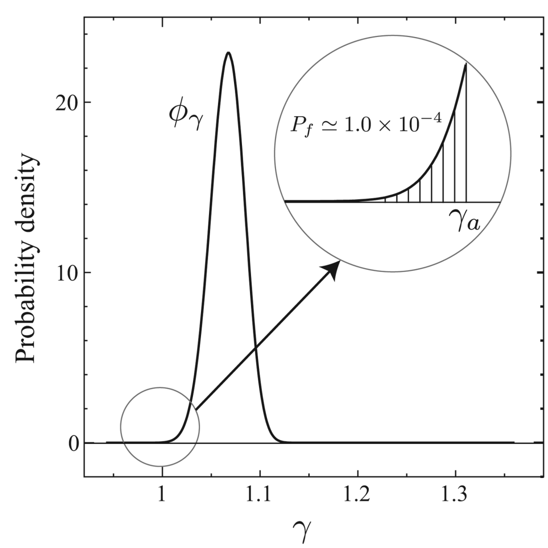

Fig. 5 The curve of probability density function for this set of values that fulfill $P_f \simeq 1.0\times10^{-4}$

correlation coefficient ρ = 0.0 and 0.5. The values of EA that achieve $P_f \simeq 1.0\times10^{-4}$, and associated values of $C^-/2^{1/3}$ and $C^+/\sqrt{2}$ in the extended sensitivity law (34) are as follows

$$\begin{cases} EA = 521.92, \quad C^-/2^{1/3} = -0.0995, \quad C^+/\sqrt{2} = -0.0803, \quad \text{for} \quad \rho = 0.0 \\ EA = 523.11, \quad C^-/2^{1/3} = -0.0998, \quad C^+/\sqrt{2} = -0.0806, \quad \text{for} \quad \rho = 0.5 \end{cases} \tag{44}$$

The curve of probability density function for the set of values for ρ = 0.0 that fulfill $P_f \simeq 1.0\times10^{-4}$ is shown in Fig. 5.

5. Conclusions

In this paper, the concept of imperfection sensitivity has been extended to the variation of external loads. By virtue of this extension, the probabilistic variation of buckling loads and the reliability of the arch has been simulated in a systematic manner. It is a topic in the future to apply the present method to more realistic structures with the use of the strategy proposed by Ikeda and Ohsaki (2007) that can deal with many imperfections.

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