Structural Engineering and Mechanics, Vol. 32, No. 1 (2009) 127-145 127
DOI: http://dx.doi.org/10.12989/sem.2009.32.1.127

Approximation of reliability constraints by estimating quantile functions

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(Received May, 25, 2008, Accepted November 7, 2008)

Abstract. A novel approach is proposed to effectively estimate the quantile functions of normalized performance indices of reliability constraints in a reliability-based optimization (RBO) problem. These quantile functions are not only estimated as functions of exceedance probabilities but also as functions of the design variables of the target RBO problem. Once these quantile functions are obtained, all reliability constraints in the target RBO problem can be transformed into non-probabilistic ordinary ones, and the RBO problem can be solved as if it is an ordinary optimization problem. Two numerical examples are investigated to verify the proposed novel approach. The results show that the approach may be capable of finding approximate solutions that are close to the actual solution of the target RBO problem.

Keywords: reliability; reliability-based optimization; quantile function; stochastic simulation.

1. Introduction

Reliability-based optimization (RBO) (Enevoldsen and Sørensen 1994, Gasser and Schueller 1997, Papadrakakis and Lagaros 2002, Royset *et al.* 2001, Jensen 2005) has recently become an important research area because of the need of making decisions under uncertainties in many engineering applications. One of the difficulties encountered for RBO is in the reliability constraints, to directly ensure which during the optimization algorithm may require numerous reliability analyses. The required computational cost of doing so can be unacceptable, rendering many realistic RBO problems computationally intractable. One possible solution is to convert these reliability constraints into ordinary ones by first estimating failure probability as a function of the design variables. This approach was taken in Gasser and Schüeller (1997) and Jensen (2005), where the logarithm of such a function is assumed to be either linear or quadratic in the design variables. The similar approach was also taken with response surface methods or surrogate-based methods (Igusa and Wan 2003, Eldred *et al.* 2002).

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This paper presents a completely different and yet efficient approach of converting reliability constraints into ordinary ones. This novel approach is a generalization of a theory of equivalence between reliability and safety factor proposed by the authors in a previous work (Ching and Hsu 2008). The novel approach is able to estimate the quantile functions of the "normalized" performance indices in the reliability constraints. Moreover, such quantile functions can be obtained as functions of the design variables. It will be shown that once these quantile functions are obtained, all reliability constraints can be converted into ordinary ones, and the RBO problem can be solved approximately using any suitable deterministic optimization algorithm to find the approximate solution of the target RBO problem.

The structure of the paper is as follows: First, we define the problem of reliability-based optimization, then we introduce the theory of equivalence between reliability and safety factor and present the novel approach of estimating quantile functions. After that we describe the framework of the new RBO approach and demonstrate numerical examples.

2. Definition of RBO problems

Let Z be the uncertain variables of the target system and θ be the design parameters. Given the design parameters θ, the probability of failure of the target system is

$$P_F(\theta) = P(F|\theta) = \int_{\Omega_F} p(Z|\theta)dZ \tag{1}$$

where F denotes the failure event: $F \equiv \{R[Z, \theta] > 1\}$, where $R[Z, \theta]$ is called the performance index, and Ω_F is the failure domain in the Z space. The performance index does not necessarily define the complete collapse of the system but the performance of the system, e.g., serviceability and ultimate capacity. Throughout the paper, it is assumed without loss of generality that $R[Z, \theta]$ is positive and that the probability density function (PDF) of the uncertainty Z conditioning on θ, denoted by $p(Z|\theta)$, is known. A reliability-based optimization problem is to solve the following problem

$$\min_{\theta} c_0(\theta) \quad s.t. \quad P_{F,j}(\theta) \le P^*_{F,j} \quad j = 1, \ldots, M \quad c_l(\theta) \le 0 \quad l = 1, \ldots, L \tag{2}$$

where $c_0(\theta)$ is the objective function; $\{c_l(\theta) \le 0 : l = 1, \ldots, L\}$ are deterministic constraints of θ; while $\{P_{F,j}(\theta) \le P^*_{F,j} : j = 1, \ldots, M\}$ are the reliability constraints, where $P^*_{F,j}$ is the target probability of failure for the j-th reliability constraint. Note that there is another class of RBO problems where the failure probability is in the objective function. This class of RBO problems is not the focus of this paper.

The RBO problem in (2) cannot be easily solved using common optimization algorithms because of the reliability constraints. An obvious way of solving the RBO problem is to conduct a search, which may or may not require evaluating the gradients and Hessians of the functions in (2), in the θ space. This approach was adopted by Papadrakakis and Lagaros (2002), Tsompanakis and Papadrakakis (2004), Youn *et al.* (2004), etc. On the other hand, if the failure probability functions $\{P_{F,j}(\theta) : j = 1, \ldots, M\}$ can be obtained beforehand, the reliability constraints can then be transformed into ordinary constraints, so the RBO problem can be turned into an ordinary optimization problem that can be solved using suitable optimization algorithms. This approach was taken by Gasser and Schüeller (1997) and Jensen (2005).

3. Equivalence between reliability and factor of safety

In this paper, the proposed novel approach is able to transform all reliability constraints into ordinary ones by first estimating the quantile functions of the "normalized" performance indices, $R[Z,\theta]/\bar{R}(\theta)$, where $\bar{R}(\theta)$ is a "nominal" performance index. An example of $\bar{R}(\theta)$ is to take $R[Z,\theta]$ but fix Z at certain chosen nominal values, e.g., their mean values. Quantile functions are inverse functions of the cumulative density functions (CDF)

$$Q(p,\theta) = F^{-1}(p,\theta) \tag{3}$$

where F^{-1} is the inverse of the CDF, and p is the probability of interest. If $Q_{R/\bar{R}}(p,\theta)$ is the quantile function of the normalized performance index $R[Z,\theta]/\bar{R}(\theta)$, we have

$$P(R[Z,\theta]/\bar{R}(\theta) \le Q_{R/\bar{R}}(p,\theta)|\theta) = p \quad \text{or} \quad P(R[Z,\theta]/\bar{R}(\theta) > Q_{R/\bar{R}}(p,\theta)|\theta) = 1-p \tag{4}$$

For the convenience of later discussions, the normalized performance index $R[Z,\theta]/\bar{R}(\theta)$ will be denoted by $G(Z,\theta)$.

Based on the theorem developed in Ching and Hsu (2008), it turns out that the $1-P_F^*$ quantile of a normalized performance index $G(Z,\theta)$, namely $Q_{R/\bar{R}}(1-P_F^*,\theta)$, is exactly the equivalent safety factor if the target failure probability is P_F^*, i.e.: the reliability constraint is the same as a safety factor constraint with the safety factor equal to $Q_{R/\bar{R}}(1-P_F^*,\theta)$. In the next paragraph, the theorem will be briefly reviewed. A new theorem will be later proposed to generalize the existing theorem.

3.1 Theorem developed by Ching and Hsu (2008)

Let D be the prescribed allowable design region in the θ space. The safety-factor approach of design is to enforce the following constraint

$$\eta^*(\theta)\cdot\bar{R}(\theta) \le 1 \tag{5}$$

where $\eta^*(\theta)$ is the designated safety factor; in general, it may depend on θ. On the other hand, the reliability-based design approach is to enforce the following constraint during the design process

$$P(R[Z,\theta] > 1|\theta) \le P_F^* \tag{6}$$

The theorem states that the two constraints in (5) and (6) are equivalent if the safety factor $\eta^*(\theta)$ and P_F^* satisfy the following relation

$$P(G(Z,\theta) > \eta^*(\theta)\big|\theta) = P_F^* \tag{7}$$

Proof: It is desirable to show that if the safety factor $\eta^*(\theta)$ and P_F^* satisfy (7), the two statements in (5) and (6) are equivalent. To demonstrate the equivalence between (5) and (6), we need to show that under (7), (5) implies (6), and also the negation of (5) implies the negation of (6). Proof that (5) implies (6): If (5) is true, there must exist $\varepsilon \ge 0$ such that $\eta^*(\theta)\bar{R}(\theta)+\varepsilon = 1$. Since

$P(G(Z,\theta) > \eta^*(\theta)|\theta) = P_F^*$ and also $P(R[Z,\theta] > x|\theta)$ is a decreasing function of x

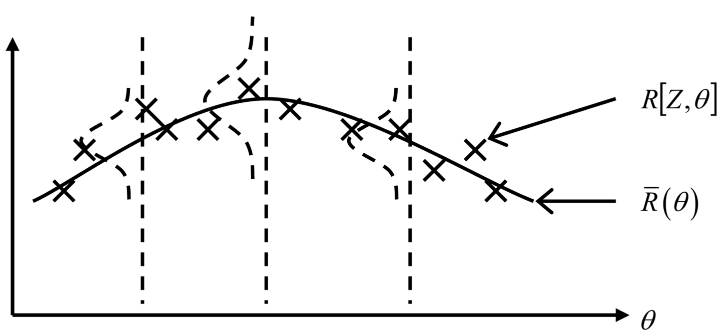
$$P(R[Z,\theta] > 1|\theta) = P(R[Z,\theta] > \eta^*(\theta)\overline{R}(\theta) + \varepsilon|\theta) = P(G(Z,\theta) > \eta^*(\theta) + \varepsilon/\overline{R}(\theta)|\theta) \le P_F^* \quad (8)$$

Therefore, (5) implies (6). The proof for the other direction is skipped since it is similar.

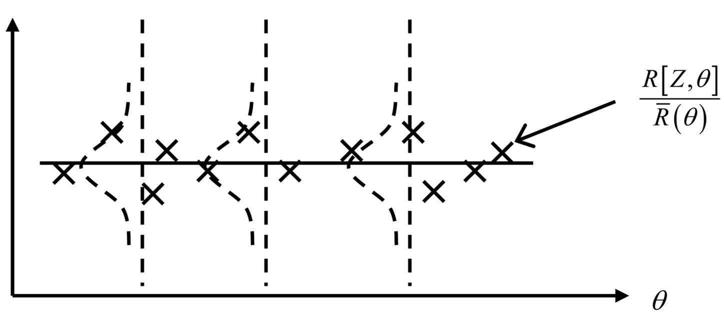
According to (7), it is clear that the safety factor $\eta^*(\theta)$ for target failure probability P_F^* is exactly the $1-P_F^*$ quantile of $G(Z,\theta)$, i.e.: $Q_{R/\overline{R}}(1-P_F^*,\theta)$. If the quantile $Q_{R/\overline{R}}(1-P_F^*,\theta)$ can be known beforehand, the reliability constraint in (6) is exactly the same as the safety-factor constraint in (5) where the safety factor $\eta^*(\theta)$ is replaced by $Q_{R/\overline{R}}(1-P_F^*,\theta)$. Therefore, the reliability constraint (6) can be transformed in an ordinary constraint (5) so that the RBO problem can be solved as an ordinary optimization problem.

Unfortunately, finding or estimating the quantile function $Q_{R/\overline{R}}(p,\theta)$ can be extremely challenging except the following special case: if the distribution of the normalized performance index $G(Z,\theta)$ does not vary with θ, its quantile $Q_{R/\overline{R}}(1-P_F^*,\theta)$ should be independent of θ, so $Q_{R/\overline{R}}(1-P_F^*,\theta)$ becomes $Q_{R/\overline{R}}(1-P_F^*)$. In this case, $Q_{R/\overline{R}}(1-P_F^*)$ can be easily found because it is simply the $1-P_F^*$ quantile of $G(Z,\theta)$ where θ is treated as random and uniformly distributed over D. Therefore, the property that the distribution of $G(Z,\theta)$ does not vary with θ is essential for the theorem to be practically useful.

The key to make the distribution of $G(Z,\theta)$ invariant in D is a proper choice of the nominal performance function $\overline{R}(\theta)$. In Ching and Hsu (2008), it is argued that finding a nominal function $\overline{R}(\theta)$ such that the distribution of $G(Z,\theta)$ is roughly invariant over θ is usually not a difficult task.



(a) Illustration of the distribution of $R[Z,\theta]$



(b) Illustration of the distribution of $R[Z,\theta]/\overline{R}(\theta)$

Fig. 1 Illustration of the distribution of $R[Z,\theta]$ and $R[Z,\theta]/\overline{R}(\theta)$

In their opinion, both $\bar{R}(\theta) = R[E(Z), \theta]$ or $\bar{R}(\theta) = E_Z(R[Z, \theta])$ may be acceptable. The rationale is as follows: although the distribution of $R[Z, \theta]$ may change dramatically with θ (see Fig. 1(a)), the distribution of $R[Z, \theta]/R[E(Z), \theta]$ or $R[Z, \theta]/E_Z(R[Z, \theta])$ usually does not (see Fig. 1(b)) due to the cancellation effect between $R[Z, \theta]$ and $R[E(Z), \theta]$ (or $E_Z(R[Z, \theta])$).

However, there are some cases where the above two choices of $\bar{R}(\theta)$ are not proper, i.e.: the distribution of the resulting $G(Z, \theta)$ significantly varies in D. For these cases, the quantile $Q_{R/\bar{R}}(1-P_F^*, \theta)$ varies with θ, and it is extremely challenging to find such a varying quantile function. If the procedure for invariant quantile is employed for these cases, the conversion from reliability constraints to safety-factor constraints may be misleading.

4. New theorem

A more generalized theorem of equivalence between safety factor and reliability is proposed in this paper to resolve the aforementioned difficulty. The new theorem is in fact a generalization of the old theorem proposed by Ching and Hsu (2008): the new theorem does not require the distribution of $G(Z, \theta)$ to be invariant; instead, it requires a weaker condition: there exists a monotonically increasing mapping L^θ such that the distribution of $L^\theta[G(Z, \theta)]$ is invariant over θ. The response of such a L^θ mapping may vary with θ, and its varying response somehow counteracts the varying behavior of $G(Z, \theta)$ so that the distribution of $L^\theta[G(Z, \theta)]$ becomes invariant over θ.

4.1 Theorem (equivalence between reliability and safety factor)

If there exists a monotonically increasing function $L^\theta: R^+ \to R^+$ such that the distribution of $L^\theta[G(Z, \theta)]$ is invariant over θ in D, the following two constraints are equivalent

$$L^{\theta^{-1}}(\eta^*) \cdot \bar{R}(\theta) \le 1 \tag{9}$$

and

$$P(R[Z, \theta] > 1 | \theta) \le P_F^* \tag{10}$$

where $L^{\theta^{-1}}$ is the inverse function of L^θ. Moreover, the functional relationship between the pair $[\eta^*, P_F^*]$ is as follows

$$P(L^\theta[G(Z, \theta)] > \eta^*) = P_F^* \tag{11}$$

where θ is treated as random and uniformly distributed over D.

Let us prove the theorem as follows. First note that if the distribution of $L^\theta[G(Z, \theta)]$ is invariant over θ in D, (11) is equivalent to

$$P(L^\theta[G(Z, \theta)] > \eta^* | \theta) = P_F^* \qquad \forall \theta \in D \tag{12}$$

Proof of the new theorem: To demonstrate the equivalence between (9) and (10) under the premise, we need to show that under (12), (9) implies (10), and also the negation of (9) implies the negation of (10). Proof that (9) implies (10): If (9) is true, there must exist $\varepsilon \geq 0$ such that $L^{\theta^{-1}}(\eta^*)\bar{R}(\theta) + \varepsilon = 1$. Since $P(L^\theta[G(Z,\theta)] > \eta^*) = P_F^*$ and also $P(R(Z,\theta) > x|\theta)$ is a decreasing function of x

$$\begin{aligned} P(R(Z,\theta) > 1|\theta) &= P(R[Z,\theta] > L^{\theta^{-1}}(\eta^*)\bar{R}(\theta) + \varepsilon|\theta) \\ &= P(L^\theta[G(Z,\theta)] > \eta^* + L^\theta(\varepsilon/\bar{R}(\theta))|\theta) \leq P_F^* \end{aligned} \tag{13}$$

Proof that the negation of (9) implies the negation of (10): If (9) is false, there must exist $\varepsilon > 0$ such that $L^{\theta^{-1}}(\eta^*)\bar{R}(\theta) - \varepsilon = 1$. Since $P(L^\theta[G(Z,\theta)] > \eta^*) = P_F^*$ and also $P(R[Z,\theta] > x|\theta)$ is a decreasing function of x, it is clear that

$$\begin{aligned} P(R(Z,\theta) > 1|\theta) &= P(R[Z,\theta] > L^{\theta^{-1}}(\eta^*)\bar{R}(\theta) - \varepsilon|\theta) \\ &= P(L^\theta[G(Z,\theta)] > \eta^* - L^\theta(\varepsilon/\bar{R}(\theta))|\theta) > P_F^* \end{aligned} \tag{14}$$

End of the proof. Note that this theorem does not assume any distribution type for $R[Z,\theta]$.

If the L^θ function is an identity map, i.e.: $L^\theta(x) = x$, the new theorem reduces to the old one. The inclusion of the L^θ function in the new theorem adds a new degree of freedom into the old theorem so that the new theorem may now be applied to handle the cases where the distribution of $G(Z,\theta)$ varies with θ. Also notice that in the new theorem, η^* is now the $1-P_F^*$ quantile of $L^\theta[G(Z,\theta)]$, denoted by $Q_{L(G)}(1-P_F^*)$, where θ is treated as random and uniformly distributed over D. Moreover, this quantile $Q_{L(G)}(1-P_F^*)$ does not depend on θ. Also, $L^{\theta^{-1}}(\eta^*) = L^{\theta^{-1}}[Q_{L(G)}(1-P_F^*)]$ is now the safety factor corresponding to the target failure probability P_F^*. That is to say, the reliability constraint (10) is equivalent to the safety factor constraint $L^{\theta^{-1}}[Q_{L(G)}(1-P_F^*)] \cdot \bar{R}(\theta) \leq 1$, where the quantile $Q_{L(G)}(1-P_F^*)$ can be found by solving (11) for η^*.

4.2 Choice of the L^θ function

A key to implement this new theorem is to find a suitable monotonically increasing L^θ function such that the distribution of $L^\theta[G(Z,\theta)]$ is invariant over θ in D. As discussed previously, for most cases the choice of $L^\theta(x) = x$, i.e.: the old theorem, works reasonably well. However, for some special cases, $L^\theta(x) = x$ works less satisfactorily: these are the cases where the distribution of $G(Z,\theta)$ varies over θ. For these cases, it would be beneficial to use a non-identity L_θ mapping.

In this study, it is proposed to first estimate the CDF of $G(Z,\theta)$ and take the estimated CDF as L^θ. This choice of L^θ will definitely work well because any random variable after being mapped by its CDF will be distributed uniformly over the [0,1] interval. Therefore, under this choice $L^\theta[G(Z,\theta)]$ is always uniformly distributed over [0,1] regardless the value of θ, hence the distribution of $L^\theta[G(Z,\theta)]$ is obviously invariant over θ. Furthermore, it is more important to estimate the right tail of the CDF for $G(Z,\theta)$ accurately, i.e.: the region where $G(Z,\theta)$ value is large, rather than the main body of the CDF or the left tail. This is because a large $G(Z,\theta)$ value corresponds to small failure probability, which is of interest to engineering designs. Therefore, a better strategy is to only estimate the CDF of $G(Z,\theta)$ for its right tail.

In particular, the generalized Pareto distribution (GPD) is proposed to fit the right tail of $G(Z, \theta)$. Pickands (1975) found that for many distributions, their tails can be well approximated by GPD. The CDF of the GPD is as follows

$$L^{\theta}(x) \equiv F_{GPD}(x) = P(X \leq x | X \geq \mu) = 1 - \left(1 + \frac{\xi(\theta)(x-\mu)}{\sigma(\theta)}\right)^{-1/\xi(\theta)} \tag{15}$$

where μ is the threshold, $\xi(\theta)$ is the shape parameter that depends on θ, and $\sigma(\theta) > 0$ is the scale parameter that also depends on θ. Note that x must be greater than μ; if $\xi < 0$, x must be smaller than $\xi - \sigma/\xi$. Its PDF is

$$f_{GPD}(x) = \frac{1}{\sigma(\theta)}\left(1 + \frac{\xi(\theta)(x-\mu)}{\sigma(\theta)}\right)^{-1-1/\xi(\theta)} \tag{16}$$

To find the estimate of the L^{θ} function, i.e.: the right tail of $G(Z, \theta)$, the following steps are taken:

(1) Monte Carlo simulation samples of $G(Z, \theta)$, denoted by $G^{(i)} \equiv G(Z^{(i)}, \theta^{(i)})$, are taken, where $Z^{(i)}$ is the *i*-th sample drawn from $p(Z|\theta)$, and $\theta^{(i)}$ is the *i*-th sample drawn from a uniform distribution over the region *D*. Drawing such samples for *N* times will obtain $\{G^{(i)}: i = 1, \ldots, N\}$.

(2) Only keep the G samples exceeding a chosen threshold μ and denote those samples by $\{G_{\mu}^{(i)}: i = 1, \ldots, N_{\mu}\}$, where N_{μ} is the number of exceeding samples. In this paper, the threshold μ is adaptively chosen so that there are 20% of the N samples exceeding the threshold, i.e.: $N_{\mu} = 0.2N$.

(3) Find the maximum likelihood estimate for $\sigma(\theta)$ and $\xi(\theta)$ by maximizing the following likelihood function:

$$f_{GPD} = \prod_{i=1}^{N_{\mu}} \frac{1}{\sigma(\theta^{(i)})}\left(1 + \frac{\xi(\theta^{(i)})(G_{\mu}^{(i)} - \mu)}{\sigma(\theta^{(i)})}\right)^{-1-1/\xi(\theta^{(i)})} \tag{17}$$

or equivalently, by maximizing the logarithm of the likelihood function

$$\log[f_{GPD}] = \sum_{i=1}^{N_{\mu}}\left(-\log[\sigma(\theta^{(i)})] - [1 - 1/\xi(\theta^{(i)})]\log\left(1 + \frac{\xi(\theta^{(i)})(G_{\mu}^{(i)} - \mu)}{\sigma(\theta^{(i)})}\right)\right) \tag{18}$$

In this study, $\sigma(\theta)$ and $\xi(\theta)$ are taken to be linear functions of θ due to simplicity, i.e.: $\sigma(\theta, a) = a_0 + a_1\theta_1 + \ldots + a_m\theta_m$ and $\xi(\theta, b) = b_0 + b_1\theta_1 + \ldots + b_m\theta_m$ (*m* is the dimension of θ). Under this setting, finding the maximum likelihood estimate for $\sigma(\theta)$ and $\xi(\theta)$ is equivalent to finding the $\{a, b\}$ that maximizes the following log-likelihood

$$\sum_{i=1}^{N_{\mu}}\left(-\log[a_0 + \ldots + a_m\theta_m^{(i)}] - [1 - 1/(b_0 + \ldots + b_m\theta_m^{(i)})]\log\left(1 + \frac{(b_0 + \ldots + b_m\theta_m^{(i)})(G_{\mu}^{(i)} - \mu)}{a_0 + \ldots + a_m\theta_m^{(i)}}\right)\right) \tag{19}$$

(4) Denote the maximum likelihood estimate for $\{a, b\}$ by $\{a^*, b^*\}$. Then the desirable L^{θ} is

$$L^{\theta}(G) = 1 - \left(1 + \frac{(b_0^* + \ldots + b_m^*\theta_m)(G - \mu)}{a_0^* + \ldots + a_m^*\theta_m}\right)^{-1/(b_0^* + \ldots + b_m^*\theta_m)} \tag{20}$$

5. Converting reliability constraints into ordinary ones under the new theorem

Once the L^θ function is obtained through the procedure described in the previous section, the distribution of $L^\theta(G(Z,\theta))$ should be roughly invariant over θ in D even though the distribution of $G(Z,\theta)$ may vary. Let us now consider multiple performance indices $\{R_j[Z,\theta]:j=1,\dots,M\}$, where M is the total number of the considered performance indices, same as the total number of reliability constraints. Also let $\{P^*_{F,j}:j=1,\dots,M\}$ be the corresponding target failure probabilities. For the j-th performance index, the $(1-P^*_{F,j})$ quantile of $L_j^\theta[G_j(Z,\theta)]$, i.e.: $Q_{L(G),j}(1-P^*_{F,j})$, can be found by solving η_j^* in the following equation

$$P(L_j^\theta[G_j(Z,\theta)]>\eta_j^*) = P^*_{F,j} \tag{21}$$

where L_j^θ is the L^θ function for the *j*-th performance index. The *j*-th reliability constraint can then be converted into the safety factor constraint $L_j^{\theta^{-1}}[Q_{L(G),j}(1-P^*_{F,j})]\cdot\bar{R}_j(\theta)\le 1$.

5.1 Solving the quantiles with Monte Carlo simulation

Although it seems nontrivial, the solution of the $(1-P^*_{F,j})$ quantile $Q_{L(G),j}(1-P^*_{F,j})$ can be quite simple. Observe that

$$P(L_j^\theta[G_j(Z,\theta)]>\eta_j^*) = P(L_j^\theta[G_j(Z,\theta)]>\eta_j^*\,|\,G_j(Z,\theta)>\mu_j)P(G_j(Z,\theta)>\mu_j) \tag{22}$$

where μ_j is the adaptive GPD threshold for the *j*-th normalized performance index. Clearly $P(G_j(Z,\theta)>\mu_j)\approx 0.2$ since μ_j is adaptively chosen to ensure there are 20% exceeding samples. If the exceeding samples are denoted by $\{G^{(i)}_{j,\mu}:i=1,\dots,0.2\cdot N\}$, the corresponding $P^*_{F,j}$ value for a chosen η_j^* value can be estimated by Law of Large Number

$$\begin{aligned} P^*_{F,j} &= P(L_j^\theta[G_j(Z,\theta)]>\eta_j^*)\approx 0.2\cdot P(L_j^\theta[G_j(Z,\theta)]>\eta_j^*\,|\,G_j(Z,\theta)>\mu_j) \\ &\approx \frac{1}{N}\sum_{i=1}^{0.2N} I(L_j^\theta[G^*_{j,\mu}]>\eta_j^*)\equiv \hat{P}^*_{F,j} \end{aligned} \tag{23}$$

By changing the η_j^* value, one can find the corresponding $\hat{P}^*_{F,j}$ values by repetitively applying (23), i.e., the entire functional relationship between η_j^* and $\hat{P}^*_{F,j}$ can be obtained. Note that the same N MCS samples can be repetitively used to estimate the entire functional relationships for all performance indices; therefore, the η_j^*-$P^*_{F,j}$ relationships for all performance indices can be estimated with N structural analyses. Once these η_j^*-$P^*_{F,j}$ relationships are obtained, the $(1-P^*_{F,j})$ quantile $Q_{L(G),j}(1-P^*_{F,j})$ can be easily found.

5.2 Summary of the procedure

The following algorithm summarizes the procedure of the proposed approach.

1. Obtain MCS samples $\{(Z^{(i)},\theta^{(i)}):i=1,\dots,N\}$. Also compute the corresponding $\{G_j^{(i)}:i=1,\dots,N;\ j=1,\dots,M\}$ samples with N structural analyses. Find the adaptive thresholds $\{\mu_j:j=1,\dots,M\}$ for all performance indices, and denote the exceeding samples by $\{G^{(i)}_{\mu,j}:i=1,\dots,0.2N;\ j=1,\dots,M\}$.

2. For each performance index, find the maximum likelihood estimate for the L_j^θ mapping. The η_j^* - $P_{F,j}^*$ relationship for the j-th performance index can be estimated based on the following equation:

$$P_{F,j}^* \approx \frac{1}{N}\sum_{i=1}^{0.2N} I(L_j^\theta[G_{j,\mu}^{(i)}] > \eta_j^*) .$$

3. Based on the prescribed target failure probabilities $\{P_{F,j}^*: j = 1, \ldots, M\}$, find the corresponding quantiles $\{Q_{L(G),j}(1 - P_{F,j}^*): j = 1, \ldots, M\}$ from the estimated η_j^* - $P_{F,j}^*$ relationships.
4. Solve the following ordinary optimization problem to get an approximate solution of the original RBO problem

$$\min_\theta c_0(\theta) \quad s.t. \quad L_j^{\theta^{-1}}[Q_{L(G),j}(1 - P_{F,j}^*)] \cdot \overline{R}_j(\theta) \le 1 , \;\; j = 1, \ldots, M \quad c_l(\theta) \le 0 , \; l = 1, \ldots, L \qquad (24)$$

6. Numerical examples

We present two examples in this section: (a) a consolidation problem; and (b) a retaining wall. The goal of the first example is to demonstrate the validity of the new theorem, while the second example is to demonstrate the use of the novel RBO approach. The failure event F, design parameters θ and allowable design region D will be defined differently in the examples.

For both examples, the following brute-force approach is adopted to verify the results from the proposed approach: Let the region Ω be the intersection of design region D and the feasible set formed by the deterministic constraints $\{c_l(\theta) \le 0: l = 1, \ldots, L\}$. The region Ω is then filled with dense grid points, and at each grid point, the failure probability estimates are found with large-sample Monte Carlo simulation. Then all the grid points where the failure probability estimates are less than the target failure probabilities $P_{F,j}^*$ are found. The set containing all these grid points is the actual feasible set satisfying the j-th reliability constraint, named the jth reliability feasible set, denoted by Σ_j^R. This set will be compared with the set satisfying $L_j^{\theta^{-1}}[Q_{L(G),j}(1 - P_{F,j}^*)] \cdot \overline{R}_j(\theta) \le 1$, i.e.: the approximation made by the proposed novel approach, named the jth safety-factor feasible set, denoted by Σ_j^S. If the safety-factor feasible set of a performance index is close to the reliability feasible set, the novel approach is then verified to be effective for that performance index.

For the second example, a further verification for the RBO is taken: compute the objective functions at the grid points that are in the intersection of all reliability feasible sets $\cap_{j=1}^M \Sigma_j^R$. Among these grid points, the one minimizing the objective function represents the brute-force solution of the RBO problem. This brute-force solution will be compared with the approximate solution obtained with the proposed approach.

For all examples, θ is treated as uncertain and its prior PDF is taken to be uniform over the design region D, the sample number for both examples is taken to be N = 10,000.

6.1 Consolidation

Consider a strip shallow foundation underlain by two soil layers: a sandy soil layer near the ground surface and a clayey soil layer underneath (see Fig. 2). The thickness H of the clay layer is uncertain. Besides H, the uncertainties include the saturated unit weight γ_{sat}^{clay}, compression index C_c, re-compression index C_r, initial void ratio e_0, and over-consolidation ratio OCR of the clay and the

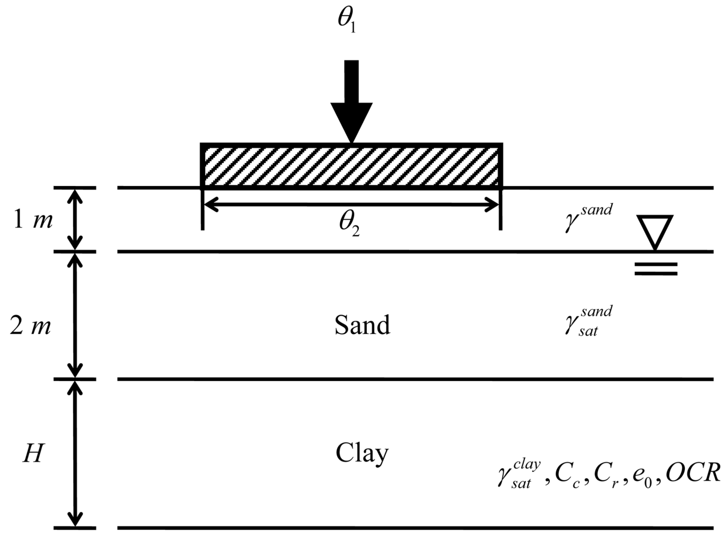


Fig. 2 The cross section of the consolidation example. θ_1 and θ_2 are the bearing pressure and foundation width, respectively

saturated unit weight of the sand γ_{sat}^{sand} and the unsaturated unit weight of the sand γ^{sand}. The uncertain variables are modeled as follows: H is Gaussian with mean value = 5 m and standard deviation = 0.5 m; [γ_{sat}^{clay}, γ_{sat}^{sand}, γ^{sand}] are Gaussian random variables with means equal to [18, 20, 18] kN/m^3 and standard deviations equal to [1 1.5 1] kN/m^3; [C_c, e_0] are log-normal random variables with means equal to [0.4, 0.8] and coefficients of variation (c.o.v.) both equal to [0.2, 0.1]; C_r is equal to C_c multiplied by a coefficient uniformly distributed over the interval [0.1, 0.3]; OCR is uniformly distributed over the interval [1.0, 1.5]. The design parameters include the bearing pressure of the shallow foundation $q(\theta_1)$ and the width of the foundation $B(\theta_2)$ of the wall. The allowable design region is chosen to be the rectangle formed by the following two constraints: $\theta_1 \in [75\ 150]$ kN/m^2 and $\theta_2 \in [2\ 4]$ m.

The long-term consolidation settlement of the foundation can be calculated as follows

$$S = \begin{cases} \frac{C_r H}{1+e_0}\log_{10}\left(\frac{OCR\cdot\sigma_0}{\sigma_0}\right) + \frac{C_c H}{1+e_0}\log_{10}\left(\frac{\sigma_0+\Delta\sigma}{OCR\cdot\sigma_0}\right) & \text{if} \quad \sigma_0+\Delta\sigma \geq OCR\cdot\sigma_0 \\ \frac{C_r H}{1+e_0}\log_{10}\left(\frac{\sigma_0+\Delta\sigma}{\sigma_0}\right) & \text{if} \quad \sigma_0+\Delta\sigma < OCR\cdot\sigma_0 \end{cases} \tag{25}$$

where $\sigma_0 = \gamma^{sand} + 2\cdot(\gamma_{sat}^{sand} - 9.81) + (H/2)\cdot(\gamma_{sat}^{clay} - 9.81)$ is the vertical effective stress at the middle of the clayey layer before the foundation is constructed; $\Delta\sigma$ is the vertical stress increment due to the construction of the foundation. According to the 2:1 method

$$\Delta\sigma \approx \frac{q\cdot B}{3+B+H/2} \tag{26}$$

The considered performance index is the exceedance of the consolidation settlement over 0.3 m

$$R(Z, \theta) = S/0.3 \tag{27}$$

For this example, the nominal performance index is taken to be of the following form

$$\bar{R}(\theta) = \exp[a_0 + a_1\theta_1 + a_2\theta_2 + a_{11}\theta_1^2 + a_{22}\theta_2^2 + a_{12}\theta_1\theta_2] \tag{28}$$

that is, the exponential of a second-order polynomial, where the coefficients $a_0, a_1, a_2, a_{11}, a_{22}, a_{12}$ are estimated according to the following procedure: given the N MCS samples $\{\theta^{(i)}: i = 1, \ldots, N\}$ and $\{R^{(i)}: i = 1, \ldots, N\}$, use least-square method to fit a second-order polynomial to the sample pairs $\{(\theta^{(i)}, \log[R^{(i)}]): i = 1, \ldots, N\}$ to estimate $a_0, a_1, a_2, a_{11}, a_{22}, a_{12}$. One can verify that such $\bar{R}(\theta)$ is similar to $E_Z[R(Z, \theta)]$.

With a sample size N = 10,000, the estimated η_j^*-P_F^* relationship is shown in Fig. 3. The corresponding quantiles for the target failure probabilities of 0.01, 0.001 and 0.0001 are tabulated in the figure. In Fig. 4, the resulting safety-factor feasible sets are shown as shaded regions for various target failure probabilities. The results obtained from the old theorem developed by Ching and Hsu (2008) are also shown in the figure for comparison. The borders of the feasible sets include several break points: this is due to the coarse discretization in plotting the sets. Similar features will be seen in the next numerical example.

The aforementioned brute-force approach is taken to examine the consistency of the novel approach in converting reliability constraints into safety-factor constraints. The reliability feasible sets obtained by the brute-force analysis are shown in Fig. 4, where the regions with label O indicate the feasible region, while the label × regions are infeasible. The comparison shows that the safety-factor feasible sets obtained by the new theorem are very close to the actual reliability feasible sets. The safety-factor feasible sets obtained by the old theorem are also close to the actual reliability feasible sets, but it is clear that the results from the new theorem are superior to those from the old one.

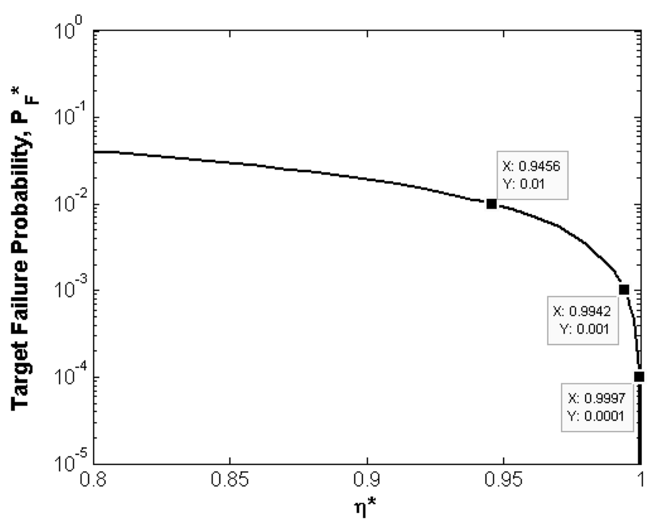


Fig. 3 The estimated η^*-P_F^* relationships with N = 10,000 samples for Example 1

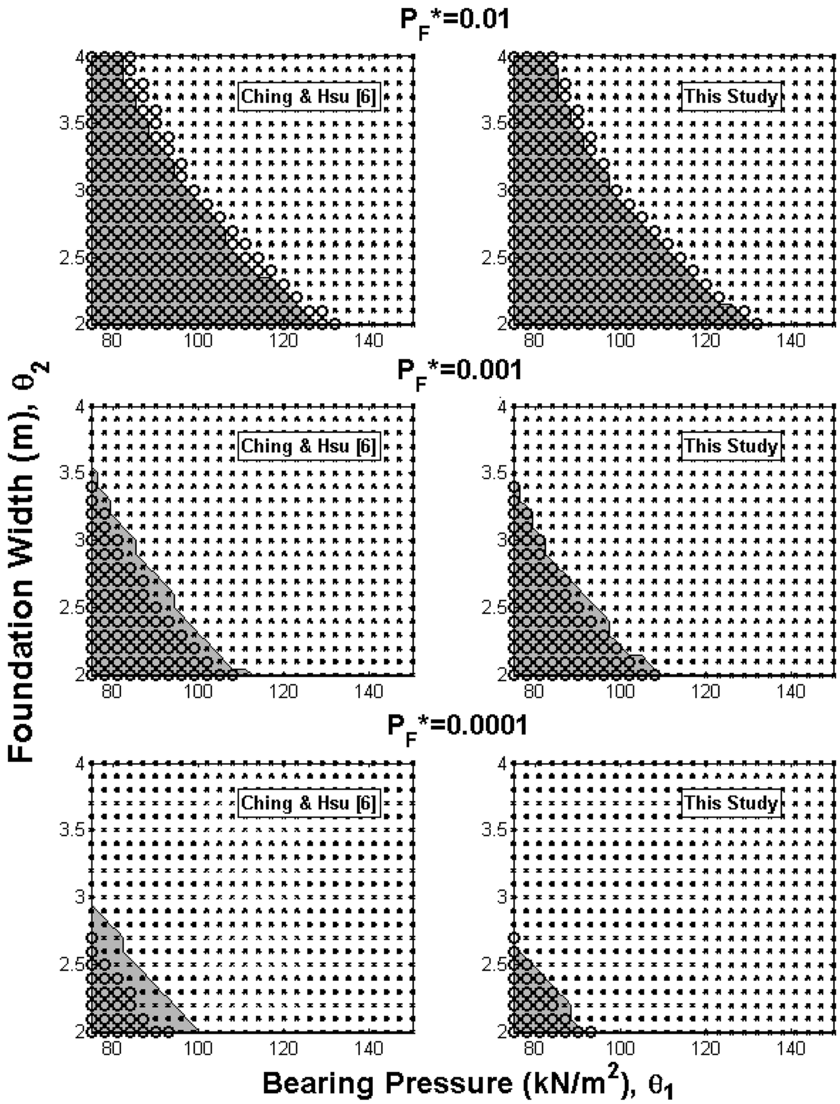


Fig. 4 The safety-factor feasible sets (shaded regions) for Example 1 for various target failure probabilities and their comparison with the actual reliability feasible sets (the region with label O). The plots in the left column are based on the old theorem, while those in the right column are based on the new theorem

6.2 Retaining wall

Consider the retaining wall in Fig. 5 that is subject to self weight and earthquake excitation. The backfill angle α is 20°. The uncertainties include the density γ and friction angle ϕ of the backfill cohesiveless soil, friction angle between the wall and soil δ and peak horizontal and vertical earthquake acceleration k_h and k_v. The uncertain variables are modeled as follows: γ and ϕ are Gaussian random variables with means equal to $[\mu_\gamma\ \mu_\phi]$ = [17.5 KN/m^3 33°] and standard deviations

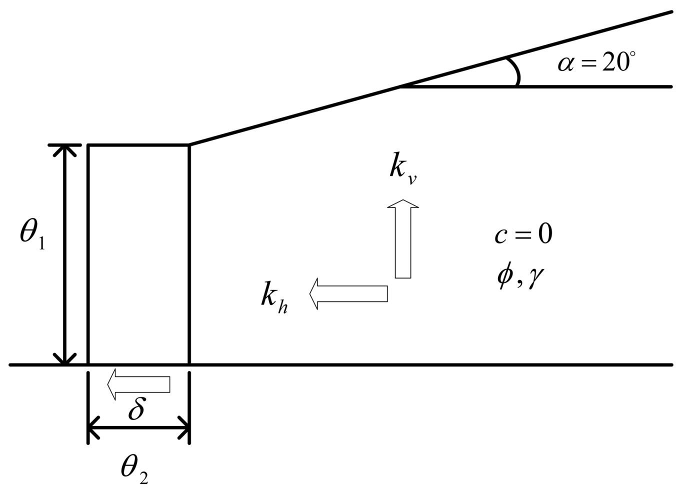


Fig. 5 The cross section of the retaining wall. θ_1 and θ_2 are the height and width of the retaining wall, respectively

equal to $[\sigma_\gamma\ \sigma_\phi]$ = [1.5 KN/m^3 2^o] (the Gaussian PDF for γ is truncated at zero, and the one for ϕ is truncated at zero and 90^o); k_h and k_v are log-normal random variables with means equal to $[\mu_{kh}\ \mu_{kv}]$ = [0.2 g 0.02 g] and coefficients of variation both equal to 0.2; δ is equal to ϕ multiplied by a coefficient uniformly distributed over the interval [1/2, 3/4]. The design parameters include the height $H(\theta_1)$ and the width $W(\theta_2)$ of the wall. Eight performance indices are considered: (a) sliding of the wall under static loading; (b) overturning of the wall under static loading; (c) failure of the foundation soil beneath the wall under static loading; (d) exceedance of the maximum moment within the wall body over a appointed threshold under static loading; (e)-(h) same as (a)-(d) but under earthquake loading. Furthermore, the designated moment thresholds of (d) and (h) are 120 kN·m and 160 kN·m respectively.

The following summarizes the eight performance indices that are based on the mechanical model developed by Coulomb, which is based on simple force equilibrium of a triangular block behind the retaining wall (1990)

$$R_a = \frac{P_a\cos(\delta)}{[P_a\sin(\delta)+W_W]\tan(\delta)} \quad R_b = \frac{P_a\cos(\delta)\cdot z}{\theta_2\left[\frac{W_W}{2}+P_a\sin(\delta)\right]} \quad R_c = \frac{\frac{K_0\gamma\theta_1^3}{6}}{120} \quad R_d = \frac{W_W+P_a\sin(\delta)}{q_u}$$

$$R_e = \frac{P_{ae}\cos(\delta)}{[P_{ae}\sin(\delta)+W_W]\tan(\delta)} \quad R_f = \frac{P_{ae}\cos(\delta)\cdot \bar{z}}{\theta_2\left[\frac{W_W}{2}+P_{ae}\sin(\delta)\right]} \quad R_g = \frac{\frac{(1.8k_h\times 9.8)\theta_1^2}{2}+\frac{K_0\gamma\theta_1^3}{6}}{160} \tag{29}$$

$$R_h = \frac{W_W+P_{ae}\sin(\delta)}{q_{ue}}$$

where

$$P_a = \frac{1}{2}K_a\gamma\theta_1^2 \quad z = \frac{\theta_1}{3} \quad W_W = 23.58\,\theta_1\theta_2 \quad K_a = \frac{\cos^2(\phi)}{\cos(\delta)\left[1+\sqrt{\frac{\sin(\delta+\phi)\sin(\phi-\alpha)}{\cos(\delta)\cos(-\alpha)}}\right]^2}$$

$$q_u = \frac{1}{2}\gamma N_\gamma\left(\theta_2 - 2\left|\frac{W_W\frac{\theta_2}{2}+P_a\sin(\delta)\theta_2 - P_a\cos(\delta)z}{W_W+P_a\sin(\delta)} - \frac{\theta_2}{2}\right|\right) \tag{30}$$

$$N_\gamma = 2(N_q+1)\tan\phi \quad N_q = e^{\pi\tan\phi}\tan^2\left(45+\frac{\phi}{2}\right)$$

and

$$P_{ae} = \left[\frac{1}{2}\gamma\theta_1^2(1-k_v)\right]K_{ae}\cos\rho \quad \alpha' = \alpha+\rho \quad \rho = \tan^{-1}\left(\frac{k_h}{1-k_v}\right) \quad \bar{z} = \frac{P_a\left(\frac{\theta_1}{3}\right)+\Delta P_{ae}(0.6\,\theta_1)}{P_{ae}}$$

$$\Delta P_{ae} = P_{ae}-P_a \quad K_{ae} = \frac{\cos^2(\phi-\rho)}{\cos^2(\rho)\cos(\delta+\rho)\left[1+\sqrt{\frac{\sin(\delta+\phi)\sin(\phi-\alpha')}{\cos(\delta+\rho)\cos(\rho-\alpha')}}\right]^2} \tag{31}$$

$$q_{ue} = \frac{1}{2}\gamma N_\gamma\left(\theta_2 - 2\left|\frac{W_W\frac{\theta_2}{2}+P_{ae}\sin(\delta)\theta_2 - P_{ae}\cos(\delta)\bar{z}}{W_W+P_{ae}\sin(\delta)} - \frac{\theta_2}{2}\right|\right)$$

The objective of this example is to minimize the volume of the concrete used to construct the wall subject to the eight reliability constraints that the probabilities of failure of the first four performance indices should not exceed 10^{-3} and that the probabilities of failure of the last four performance indices should not exceed 10^{-2}. The allowable design region is chosen to be the rectangle formed by the following two constraints: $\theta_1 \in [3\ 7]\text{m}$ and $\theta_2 \in [2\ 9]\text{m}$. Therefore, the RBO problem is as the following

$$\min_\theta c_0(\theta) \quad s.t. \begin{cases} P_{F_a}(\theta)\le 10^{-3}, P_{F_b}(\theta)\le 10^{-3}, P_{F_c}(\theta)\le 10^{-3}, P_{F_d}(\theta)\le 10^{-3} \\ P_{F_e}(\theta)\le 10^{-2}, P_{F_f}(\theta)\le 10^{-2}, P_{F_g}(\theta)\le 10^{-2}, P_{F_h}(\theta)\le 10^{-2} \end{cases} \tag{32}$$
$$3\le\theta_1\le 7 \qquad 2\le\theta_2\le 9$$

where $c_0(\theta) = \theta_1\cdot\theta_2$ is the volume of the concrete used to construct the wall.

As a first attempt, the nominal performance indices are taken to be $\bar{R}_j(\theta) = R_j(E(Z),\theta)$ for all performance indices. Compared to the first example, a different nominal performance index is taken herein to demonstrate the effect of various choices of the nominal performance indices to the analysis results. Although not shown here, the choice of the exponential of a second-order polynomial in the first example works equally well for the current example. With a sample size N = 10,000, the estimated η_j^*-$P_{F,j}^*$ relationships are shown in Fig. 6. The corresponding quantiles for the target failure probabilities are tabulated in the figure. In Fig. 7, the resulting safety-factor

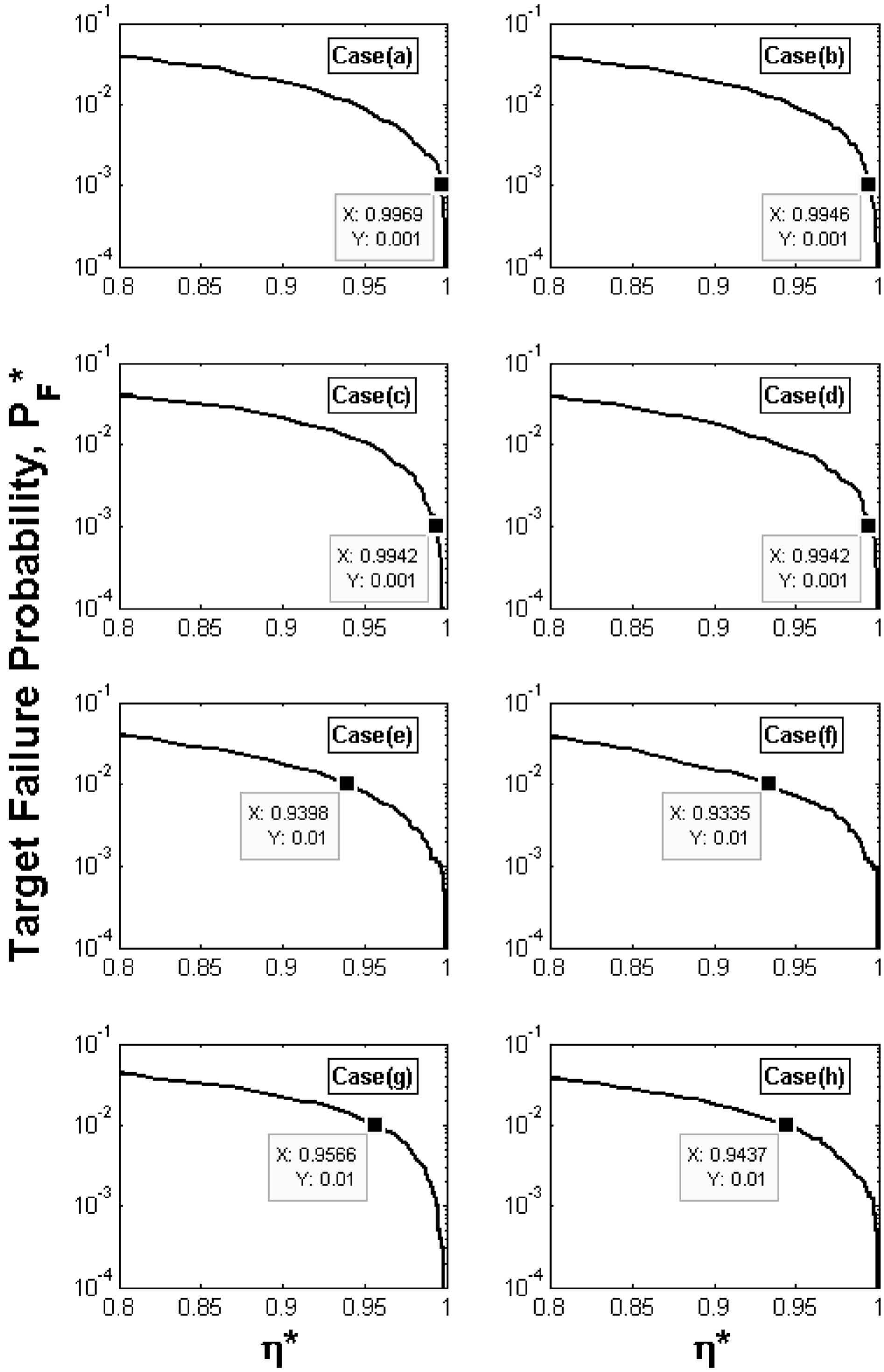


Fig. 6 The estimated η_j^*-$P_{F,j}^*$ relationships with N = 10,000 samples for Example 2

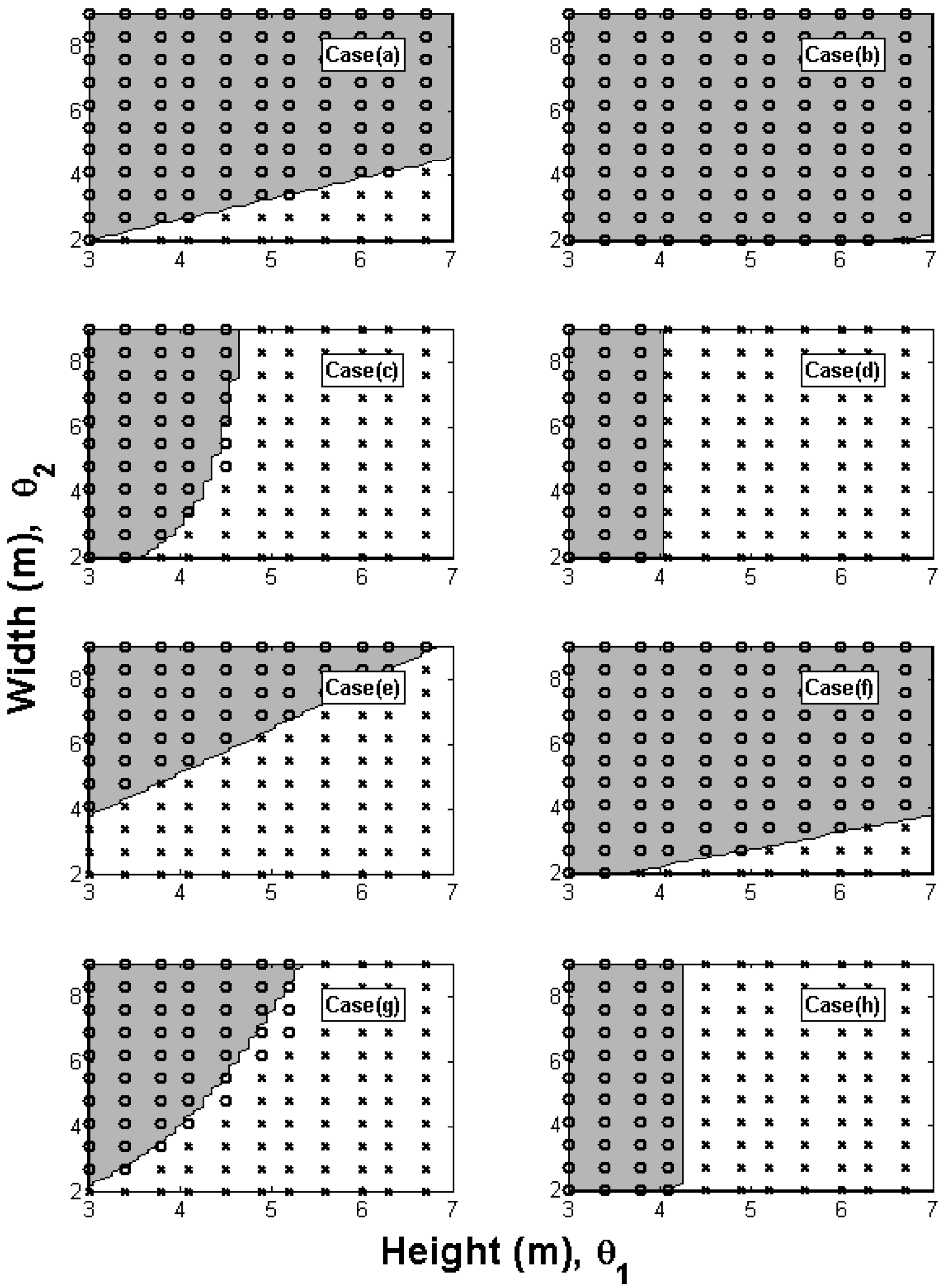


Fig. 7 The safety-factor feasible sets for Example 2 and their comparison with the actual reliability feasible sets (the region with label O)

feasible sets are shown as shaded regions. Fig. 8 shows the intersection of the eight safety-factor feasible sets. The genetic algorithm (GA) is taken to find the solution of the optimization problem subject to the safety-factor constraints to determine the approximate solution of the original RBO problem, which is found to be $(H^*, W^*) = (3.0091\text{m}, 3.9048\text{m})$.

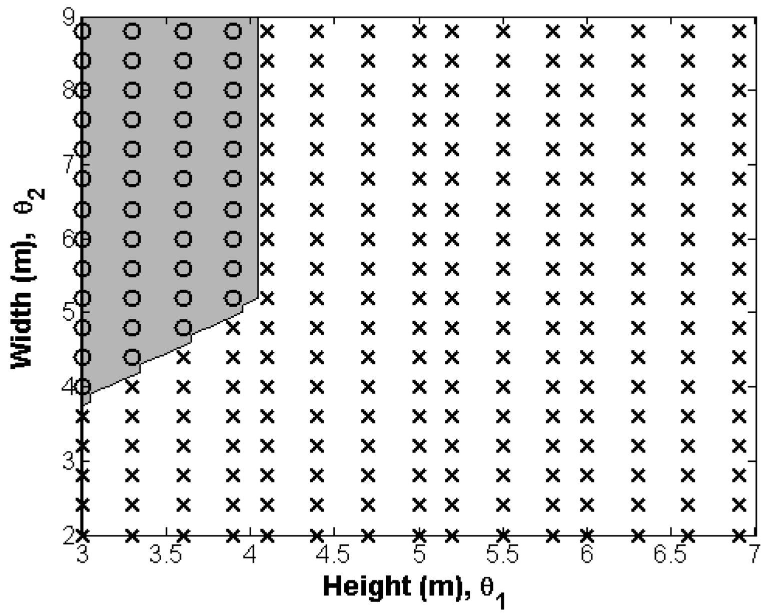


Fig. 8 The intersection of the eight safety-factor feasible sets for Example 2

Table 1 The results from the approximate method and brute-force method for Example 2

		Brute-force method	Approximate method
Objective function		11.70	11.76
Failure probability	$P_{F_{(a)}}(\theta)$	0	0
	$P_{F_{(b)}}(\theta)$	0	0
	$P_{F_{(c)}}(\theta)$	0	0
	$P_{F_{(d)}}(\theta)$	4.00×10^{-6}	4.00×10^{-6}
	$P_{F_{(e)}}(\theta)$	9.99×10^{-3}	1.00×10^{-2}
	$P_{F_{(f)}}(\theta)$	0	0
	$P_{F_{(g)}}(\theta)$	0	0
	$P_{F_{(h)}}(\theta)$	1.40×10^{-5}	2.10×10^{-5}

The aforementioned brute-force approach is taken to examine the consistency of the novel approach in converting reliability constraints into safety-factor constraints. The reliability feasible sets obtained by the brute-force analysis for the eight performance indices are shown in Fig. 7, where the regions with label *O* indicate the feasible region, while the label× regions are infeasible. The comparison shows that the safety-factor feasible sets are close to the actual reliability feasible sets. The solution of the RBO problem obtained by the brute-force approach is $(H^*, W^*) = (3\text{m}, 3.8988\text{m})$.

More efforts are spent to examine the approximate solutions obtained with the proposed approach: MCS with large samples is taken to evaluate the failure probabilities of all performance indices by holding the design parameters at their approximate solution values to get the "attained" failure probabilities. Table 1 lists the attained failure probabilities of all performance indices as well as the attained value of the objective function for the brute-force solution and the approximate solution. It is evident that all attained failure probabilities are satisfactorily bounded by the target failure probabilities, and the performance index (e) dominates. The approximate solution is very close to the brute-force result.

7. Conclusions

1. The final solution obtained from the proposed RBO approach is an approximation of the exact solution. The accuracy of the proposed method mainly depends on two factors: (a) the validity of the premise that the distribution of $L_\theta[G(Z,\theta)]$ is invariant over D; and (b) the accuracy of the Monte Carlo simulation. The aspect (b) can be easily improved by increasing the sample size N or may be improved by using a more efficient simulation technique, e.g., importance sampling and subset simulation.
2. The current method is originated from a method proposed by Ching and Hsu (2008) that requires the distribution of $G(Z,\theta)$ to be invariant over θ. In some cases this requirement cannot be easily achieved. The contribution of the current method is to introduce the L_θ mapping to alleviate this restriction: now we only require the distribution of $L_\theta[G(Z,\theta)]$ to be invariant over θ. From our experience, the introduction of the L_θ mapping is helpful: for the cases where the distribution of $G(Z,\theta)$ significantly varies over θ, the approximate feasible region may deviate from the actual reliability feasible region. For these cases, the introduction of the L_θ mapping, i.e., the current method, can help to reduce the deviation. However, when the chosen $\overline{R}(\theta)$ function is poor, it is found that the deviation usually cannot be completely removed though reduced.
3. In essence, the purpose of the proposed method is to transform reliability constraints into approximate ordinary constraints. Notice that even if the approximate constraints are close to the actual reliability constraints, it is not for sure that the resulting approximate optimal solution will be close to the actual solution. However, if the approximate ordinary constraints are close to the actual reliability constraints, the following consequences can be guaranteed: (a) the approximate optimal solution should be approximately feasible with respect to the actual reliability constraints; and (b) the approximate optimal solution is the one that minimizes the objective function among all the approximately feasible designs. The latter implies that even if the resulting approximate optimal solution is not close to the actual solution, the corresponding objective function value should be close to the actual optimal value.
4. Since the proposed method is based on Monte Carlo simulation, it inherits most advantages of MCS: it is robust against uncertainty dimension and complexity of the target system. However, it may also inherit the disadvantages of MCS: when there are rare failure modes, the proposed method may require many samples to be able to capture those modes, therefore may require much computational effort when finite element analyses are involved. To resolve this issue may require more advance simulation methods, e.g. subset simulation: this topic is still an ongoing research.
5. From our experience, the proposed method works well for many examples, besides the two examples, although most of them are relatively simple examples. For complicated examples, as

long as satisfactory $G(Z, \theta)$ function and L_θ mapping can be chosen so that the distribution of $L_\theta[G(Z, \theta)]$ is roughly invariant over θ, the proposed method will work well. However, the method has not yet thoroughly tested over many complicated examples (e.g., those with switching failure modes: failure modes depend on load level). For those examples, finding suitable G function and L mapping may be challenging.

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