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Structural reliability estimation based on quasi ideal importance sampling simulation

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Abstract. A quasi ideal importance sampling simulation method combined in the conditional expectation is proposed for the structural reliability estimation. The quasi ideal importance sampling joint probability density function (*p.d.f.*) is so composed on the basis of the ideal importance sampling concept as to be proportional to the conditional failure probability multiplied by the *p.d.f.* of the sampling variables. The respective marginal *p.d.f.*s of the ideal importance sampling joint *p.d.f.* are determined numerically by the simulations and partly by the piecewise integrations. The quasi ideal importance sampling simulations combined in the conditional expectation are executed to estimate the failure probabilities of structures with multiple failure surfaces and it is shown that the proposed method gives accurate estimations efficiently.

Keywords: structural failure probability; simulation-based reliability method; conditional expectation; ideal importance sampling.

1. Introduction

This paper describes a quasi ideal importance sampling simulation combined in the conditional expectation for the simulation-based structural reliability estimation. There have been various simulation-based methods studied for the estimation of the structural reliability. It has been a very important issue in the simulation-based structural reliability estimation to enhance the simulation efficiency by making use of various variance reduction techniques (VRT).

The conditional expectation is one method of the powerful VRT (Rubinstein 1981), which is

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utilized by randomly generating all the basic random variables except one variable referred to as a control variable and randomly generated variables referred to as sampling variables are selected as the ones of least variability. The conditional expectation VRT reduces the variance of the estimate of the structural failure probability under consideration by conditioning on the sampling variables and resulting conditional failure probabilities can be evaluated by the cumulative distribution function of the control variable (Ayyub *et al*. 1984, Karamchandani 1991).

The importance sampling method is also known as one of the powerful VRT tools. In the recent years, a number of methodologies have been studied and developed in the importance sampling techniques not only to compose the importance sampling density function (*p.d.f.*) but to enhance the simulation efficiency of the high dimensional problems (Schuëller *et al*. 2004, Katafygiotis *et al*. 2006, Hurtado 2007, Au 2008).

In this study, a quasi ideal importance sampling joint *p.d.f.* is defined on the basis of the ideal importance sampling concept (Bucher 1988, Ang *et al*. 1989). And the respective marginal *p.d.f.*s of the quasi ideal importance sampling joint *p.d.f.* are so composed as to be proportional to the expectation of the conditional failure probabilities multiplied by the *p.d.f.* of the sampling variables. These marginal *p.d.f.*s are determined numerically by the simulations based on the conditional expectation and partly by the piecewise integrations (Yonezawa *et al*. 2007).

The quasi ideal importance sampling simulations combined in the conditional expectation are executed to estimate the failure probabilities of structures with multiple failure surfaces. The samples of the basic random variables are generated by applying the inverse transformation to the cumulative distribution functions corresponding to the respective marginal *p.d.f.*s determined in the proposed method. It is shown that the proposed method gives accurate estimations with smaller sample size and frequency of function calls.

2. Simulation based on the conditional expectation

2.1 Basic definition of structural failure probability

For the analysis of time invariant structural reliability problems, the probability of failure is given by

$$P_f = \mathrm{prod}\left[\bigcup_{j=1}^{m}(g_{j_U}(\boldsymbol{u}) \le 0)\right] = \int_{D_f} f_U(\boldsymbol{u})d\boldsymbol{u} \tag{1}$$

where $g_{j_U}(\boldsymbol{u})$, (j = 1, 2,…, m) are the limit state functions and $f_U(\boldsymbol{u})$ is the joint *p.d.f.* of k-dimensional basic random variables $\boldsymbol{U} = (U_1, U_2,\ldots, U_k)^T$, which are assumed to be independent standardized normal variables. D_f is the failure domain defined by

$$D_f = \left\{\boldsymbol{u}\,\middle|\,\bigcup_{j=1}^{m}(g_{j_U}(\boldsymbol{u}) \le 0)\right\} \tag{2}$$

2.2 Conditional failure probability

The basic random variables are divided into two groups, one is a control variable U_l and the other are the sampling variables denoted as $\boldsymbol{U}_S = (U_1, U_2, \ldots, U_{l-1}, U_{l+1},\ldots, U_k)^T$. By using U_l and $\boldsymbol{U}_S$, the failure probability given by Eq. (1) is expressed as follows:

$$
\begin{aligned}
P_f &= \int_{D_f} f_U(\boldsymbol{u})d\boldsymbol{u} \\
&= \int_{all\ \boldsymbol{u}_S} prod\left[\bigcup_{j=1}^{m}(g_{j_U}(u_l, \boldsymbol{u}_S) \le 0) \middle| given\ \boldsymbol{U}_S = \boldsymbol{u}_S\right] f_{U_S}(\boldsymbol{u}_S)d\boldsymbol{u}_S \\
&= \int_{all\ \boldsymbol{u}_S} P_{fC}(u_l|\boldsymbol{u}_S)\, f_{U_S}(\boldsymbol{u}_S)(d\boldsymbol{u}_S) = E_{f_{U_S}}[P_{fC}(u_l|\boldsymbol{u}_S)]
\end{aligned}
\tag{3}
$$

where $f_{US}(\boldsymbol{u}_S)$ is the joint *p.d.f.* of the sampling variables. The notation $P_{fC}(u_l \mid \boldsymbol{u}_S)$ is the conditional failure probability of the event of $g_{jU}(u_l, \boldsymbol{u}_S) \le 0$, $(j = 1, 2, \ldots, m)$ for a given $\boldsymbol{U}_S = \boldsymbol{u}_S$. $E_{fUS}[\ ]$ is the expectation with respect to $f_{US}(\boldsymbol{u}_S)$.

The conditional failure probability $P_{fC}(u_2|u_1^{(i)})$ for a given $\boldsymbol{U}_S = u_1^{(i)}$ in a two-dimensional reliability problem with two limit state surfaces, as an example illustrated in Fig. 1, is evaluated as follows:

$$
P_{fC}(u_2|u_1^{(i)}) = \begin{cases} \displaystyle\int_{-\infty}^{u_{2,1}} f_{U_2}(u_2)du_2 + \int_{u_{2,2}}^{\infty} f_{U_2}(u_2)du_2 & \text{for} \quad u_{2,1} \le u_{2,2} \\ \displaystyle\int_{-\infty}^{\infty} f_{U_2}(u_2)du_2 = 1 & \text{for} \quad u_{2,1} > u_{2,2} \end{cases}
\tag{4}
$$

An unbiased estimate of Eq. (3) and its variance are given by

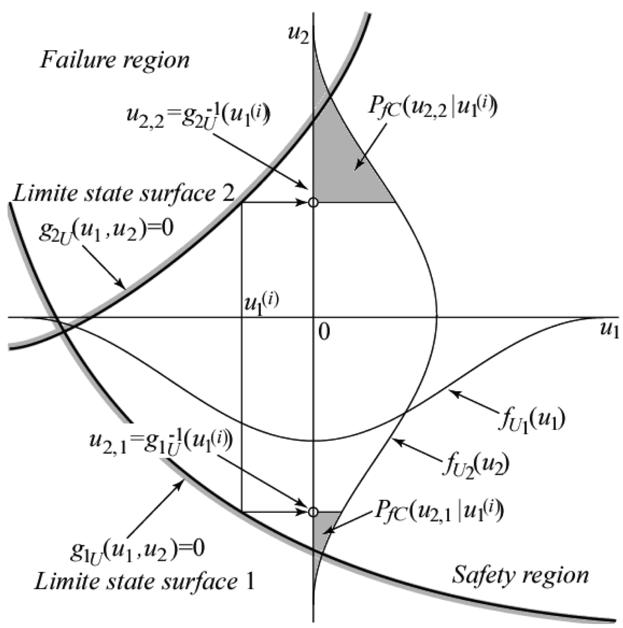

Fig. 1A conditional failure probability for a sample $u_1^{(i)}$ in a two-dimensional reliability problem with two limit state surfaces

$$\hat{P}_f = \frac{1}{N}\sum_{i=1}^{N} P_{fC}(u_l|\boldsymbol{u}_S^{(i)}) \tag{5}$$

$$\hat{\mathrm{Var}}[\hat{P}_f] = \frac{1}{N(N-1)}\sum_{i=1}^{N}\{P_{fC}(u_l|\boldsymbol{u}_S^{(i)}) - \hat{P}_f\}^2 \tag{6}$$

where N is the sample size of the simulation to estimate P_f.

In the conditional expectation VRT, most of samples are made use of yielding the conditional failure probabilities, that is to say, they contribute more or less to estimate the structural failure probability. Furthermore in this study, in order to enhance the simulation efficiency it is proposed to combine the importance sampling VRT in the conditional expectation VRT as described below.

3. Importance sampling simulation

3.1 Importance sampling probability density function

An importance sampling joint *p.d.f.* $h_{US}(\boldsymbol{u}_S)$ is introduced into Eq. (3), which is rewritten as follows:

$$\begin{aligned} P_f &= \int_{all\ \boldsymbol{u}_S} P_{fC}(u_l|\boldsymbol{u}_S)\, f_{U_S}(\boldsymbol{u}_S) d\boldsymbol{u}_S \\ &= \int_{all\ \boldsymbol{u}_S} P_{fC}(u_l|\boldsymbol{u}_S) \frac{f_{U_S}(\boldsymbol{u}_S)}{h_{U_S}(\boldsymbol{u}_S)} f_{U_S}(\boldsymbol{u}_S) d\boldsymbol{u}_S \\ &= E_{h_{U_S}}\left[P_{fC}(u_l|\boldsymbol{u}_S)\frac{f_{U_S}(\boldsymbol{u}_S)}{h_{U_S}(\boldsymbol{u}_S)}\right] \end{aligned} \tag{7}$$

where $E_{hUS}[\ ]$ is the expectation with respect to $h_{US}(\boldsymbol{u}_S)$.

An unbiased estimate of Eq. (7) and its variance are given by

$$\hat{P}_f = \frac{1}{N}\sum_{i=1}^{N}\left\{P_{fC}(u_l|\boldsymbol{u}_S^{(i)})\frac{f_{U_S}(\boldsymbol{u}_S^{(i)})}{h_{U_S}(\boldsymbol{u}_S^{(i)})}\right\} \tag{8}$$

$$\hat{\mathrm{Var}}[\hat{P}_f] = \frac{1}{N(N-1)}\sum_{i=1}^{N}\left\{P_{fC}(u_l|\boldsymbol{u}_S^{(i)})\frac{f_{U_S}(\boldsymbol{u}_S^{(i)})}{h_{U_S}(\boldsymbol{u}_S^{(i)})} - \hat{P}_f\right\}^2 \tag{9}$$

A conditional failure probability for a given sample $u_1^{(i)}$ generated from an importance sampling *p.d.f.* centered at the β-point based on the concept of ISPUD (Shuëller *et al.* 1987) is illustrated in Fig. 2 for a structural reliability problem with one failure surface of two basic random variables. This is one approach to combine the importance sampling VRT in the conditional expectation VRT.

For the structures with multiple failure surfaces, various multiple checking point (β-point) methods have been proposed (Shuëller *et al.* 1987, Murotsu *et al.* 1990, Yonezawa *et al.* 1998), where samples are allocated to the respective failure surfaces according to the weight related to the respective β-values. If the multiple checking point method is applied to cope with the structures with multiple failure surfaces, it is inevitable to identify the β-points of the respective failure surfaces.

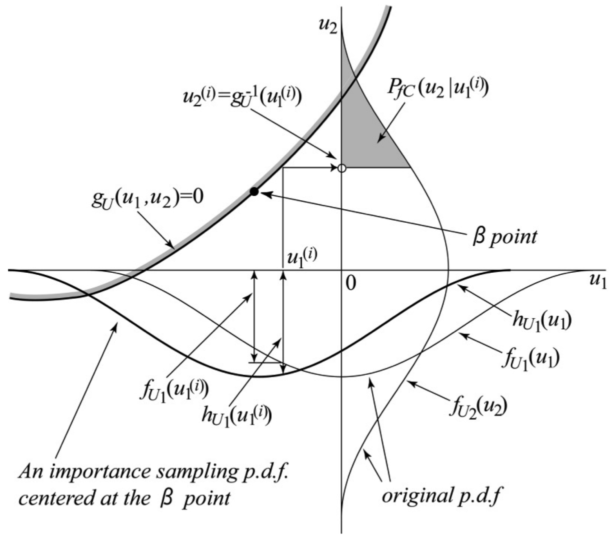

Fig. 2 A conditional failure probability for a sample $u_1^{(i)}$ taken from the importance sampling *p.d.f.* $h_{U1}(u_1)$ centered at the β point

It should be noted that the simulation based on the conditional expectation can treat the multiple failure surfaces without using the multiple β-points as illustrated in Fig. 1. Then it is proposed in this paper to combine the importance sampling in the conditional expectation for the estimation of the structural failure probability without using the β-points.

3.2 Ideal importance sampling

In the importance sampling, how to compose the importance sampling joint *p.d.f.* is the most important matter of concern. Suppose that the importance sampling *p.d.f.* in Eq. (7) is given by

$$f_{U_S}(\boldsymbol{u}_S) = \frac{P_{fC}(u_l|\boldsymbol{u}_S)\, f_{U_S}(\boldsymbol{u}_S)}{P_f} \tag{10}$$

Then the structural failure probability is expressed as the expectation of P_f itself with respect to $h_{\mathbf{U}S}(\mathbf{u}_S)$ as follows:

$$\begin{aligned} P_f &= \int_{all\ \boldsymbol{u}_S} P_{fC}(u_l|\boldsymbol{u}_S)\frac{f_{U_S}(\boldsymbol{u}_S)}{h_{U_S}(\boldsymbol{u}_S)} f_{U_S}(\boldsymbol{u}_S) d\boldsymbol{u}_S \\ &= E_{h_{U_S}}\left[P_{fC}(u_l|\boldsymbol{u}_S)\frac{f_{U_S}(\boldsymbol{u}_S)}{h_{U_S}(\boldsymbol{u}_S)}\right] = E_{h_{U_S}}[P_f] \end{aligned} \tag{11}$$

Therefore Eq. (10) gives an ideal importance sampling *p.d.f.* (Bucher 1988, Ang *et al*. 1989), however it is impossible to compose such a sampling *p.d.f.*, because it contains the P_f to be estimated and it is also difficult to determine analytically the conditional failure probability $P_{fC}(u_l|\boldsymbol{u}_S)$.

Then instead of treating the ideal importance sampling *p.d.f.* directly, it is proposed to compose approximately a quasi ideal importance sampling *p.d.f.* And the respective marginal *p.d.f.*s of the quasi ideal importance sampling joint are so composed as to be proportional to the expectation of the conditional failure probabilities multiplied by the *p.d.f.* of the sampling variables. These marginal *p.d.f.*s are determined numerically by the simulations based on the conditional expectation and partly by the piecewise integrations. In the following section, the procedure to compose the quasi ideal importance sampling marginal *p.d.f.*s is described.

4. Composition of the quasi ideal importance sampling marginal probability density functions

4.1 Marginal probability density functions in a three-dimensional reliability problem

A reliability problem with three basic random variables is considered for convenience of explanation, where u_3 is specified to be the control variable and u_1 and u_2 are specified to be the sampling variables. The two-dimensional ideal importance sampling joint *p.d.f.* denoted as $h_I(u_1, u_2)$ is expressed according to Eq. (10) as follows:

$$h_I(u_1, u_2) = \frac{P_{fC}\{u_3|(u_1, u_2)\}f_{U_S}(u_1, u_2)}{P_f} \tag{12}$$

Since u_1 and u_2 are assumed to be independent, $f_{US}(u_1, u_2)$ is rewritten as a product of the respective marginal *p.d.f.*s, $f_{U1}(u_1)$ and $f_{U2}(u_2)$, as follows:

$$f_{U_s}(u_1, u_2) = f_{U_1}(u_1) \cdot f_{U_2}(u_2) \tag{13}$$

The marginal *p.d.f.*s of $h_I(u_1, u_2)$ denoted as $h_{I1}(u_1)$ and $h_{I2}(u_2)$ are determined by integrating Eq. (12) over u_2 and u_1 respectively by taking into account of Eq. (13) as follows:

$$\begin{aligned} h_{I_1}(u_1) &= \int_{all\ u_2} h_I(u_1, u_2)du_2 \\ &= \frac{\int_{all\ u_2} P_{fC}\{u_3|(u_1, u_2)\}f_{U_1}(u_1) \cdot f_{U_2}(u_2)du_2}{P_f} \\ &= \frac{E_{f_{U_2}}[P_{fC}\{u_3|(u_1, u_2)\}] \cdot f_{U_1}(u_1)}{P_f} \end{aligned} \tag{14}$$

$$\begin{aligned} h_{I_2}(u_2) &= \int_{all\ u_1} h_I(u_1, u_2)du_1 \\ &= \frac{\int_{all\ u_1} P_{fC}\{u_3|(u_1, u_2)\}f_{U_1}(u_1) \cdot f_{U_2}(u_2)du_1}{P_f} \\ &= \frac{E_{f_{U_1}}[P_{fC}\{u_3|(u_1, u_2)\}] \cdot f_{U_2}(u_2)}{P_f} \end{aligned} \tag{15}$$

where $E_{fU2}[\ ]$ and $E_{fU1}[\ ]$ are the expectation with respect to $f_{U2}(u_2)$ and $f_{U1}(u_1)$ respectively.

It is seen from Eqs. (14) and (15) that $h_{I1}(u_1)$ and $h_{I2}(u_2)$ are proportional to the expected conditional failure probability multiplied by the respective *p.d.f.* The proportionality constant is $1/P_f$. By taking into account of Eqs. (7) and (13), P_f, the denominator of Eq. (14), is expressed so as to be related with the numerator of Eq. (14), as follows:

$$
\begin{aligned}
P_f &= \int_{all\ \boldsymbol{u}_S} P_{fC}(u_l|\boldsymbol{u}_S)\, f_{U_S}(\boldsymbol{u}_S) d\boldsymbol{u}_S \\
&= \int_{all\ u_1} \int_{all\ u_2} P_{fC}\{u_3|(u_1,u_2)\} f_{U_1}(u_1)\cdot f_{U_2}(u_2) du_1 du_2 \\
&= \int_{all\ u_1} \left[\int_{all\ u_2} P_{fC}\{u_3|(u_1,u_2)\} f_{U_2}(u_2) du_2 \right] f_{U_1}(u_1) du_1 \\
&= \int_{all\ u_1} E_{f_{U_2}}[P_{fC}\{u_3|(u_1,u_2)\}]\cdot f_{U_1}(u_1) du_1
\end{aligned} \tag{16}
$$

For the denominator of Eq. (15) in turn

$$
\begin{aligned}
P_f &= \int_{all\ \boldsymbol{u}_S} P_{fC}(u_l|\boldsymbol{u}_S)\, f_{U_S}(\boldsymbol{u}_S) d\boldsymbol{u}_S \\
&= \int_{all\ u_2} \left[\int_{all\ u_1} P_{fC}\{u_3|(u_1,u_2)\} f_{U_1}(u_1) du_1 \right] f_{U_2}(u_2) du_2 \\
&= \int_{all\ u_2} E_{f_{U_1}}[P_{fC}\{u_3|(u_1,u_2)\}]\cdot f_{U_2}(u_2) du_2
\end{aligned} \tag{17}
$$

4.2 Determination of the quasi ideal importance sampling marginal probability density functions

In this study the ideal importance sampling marginal *p.d.f.*s, $h_{I1}(u_1)$ and $h_{I2}(u_2)$ are determined numerically as the quasi ideal importance sampling marginal *p.d.f.*s denoted as $h_{Q1}(u_1)$ and $h_{Q2}(u_2)$ respectively.

First, it is proposed that the expected conditional failure probabilities contained in the integration of Eqs. (16) and (17) are evaluated by the piecewise integration. The specified interval $[u_a, u_b]$ of the respective sampling variables is split into n segments and the respective quasi ideal importance sampling marginal *p.d.f.*s at $u_1 = u_1^{(p1)}$, ($p_1 = 1, 2, ..., n$) and $u_2 = u_2^{(p2)}$, ($p_2 = 1, 2, ..., n$) are written as follows:

$$
\begin{aligned}
h_{Q_1}(u_1^{(p_1)}) &= \frac{E_{f_{U_2}}[P_{fC}\{u_3|(u_1^{(p_1)},u_2)\}] f_{U_1}(u_1^{(p_1)})}{\sum_{p_1=1}^{n} E_{f_{U_2}}[P_{fC}\{u_3|(u_1^{(p_1)},u_2)\}] f_{U_1}(u_1^{(p_1)})\Delta u} \\
&= \frac{E_{f_{U_2}}[P_{fC}\{u_3|(u_1^{(p_1)},u_2)\}] f_{U_1}(u_1^{(p_1)})}{A_1} \qquad (p_1 = 1,2,\ldots,n)
\end{aligned} \tag{18}
$$

$$h_{Q_2}(u_2^{(p_2)}) = \frac{E_{f_{U_1}}[P_{fC}\{u_3|(u_1,u_1^{(p_2)})\}]f_{U_2}(u_2^{(p_2)})}{\sum_{p_2=1}^{n} E_{f_{U_1}}[P_{fC}\{u_3|(u_1,u_2^{(p_2)})\}]f_{U_2}(u_2^{(p_2)})\Delta u}$$

$$= \frac{E_{f_{U_1}}[P_{fC}\{u_3|(u_1,u_2^{(p_2)})\}]f_{U_2}(u_2^{(p_2)})}{A_2} \qquad (p_2 = 1, 2, \ldots, n) \tag{19}$$

where n is the number of the segments of divided interval of the sampling variables, Δu is the constant width of each segment and A_1, A_2 are the results evaluated by the piecewise integrations of the respective denominators.

Next, it is proposed that the expected conditional failure probabilities both in the numerators and the denominators of Eqs. (18) and (19) are estimated by the simulations based on the conditional expectation as follows:

$$E_{f_{U_2}}[P_{fC}\{u_3|(u_1^{(p_1)},u_2)\}] \approx \frac{1}{N_Q}\sum_{j=1}^{N_Q} P_{fC}\{u_3|(u_1^{(p_1)},u_2^{(j)})\}, \quad (p_1 = 1, 2, \ldots, n) \tag{20}$$

$$E_{f_{U_1}}[P_{fC}\{u_3|(u_1,u_2^{(p_2)})\}] \approx \frac{1}{N_Q}\sum_{j=1}^{N_Q} P_{fC}\{u_3|(u_1^{(i)},u_2^{(p_2)})\}, \quad (p_1 = 1, 2, \ldots, n) \tag{21}$$

where N_Q is the sample size of the simulation to estimate the expected conditional failure probability.

The denominators A_1 and A_2 in Eqs. (18) and (19) should result in the same value, since they give the same value of P_f by the piecewise integrations of the expected conditional failure probabilities given by Eqs. (20) and (21). Some differences between them, however, may arise due to the numerical error, so A_1 and A_2 should be evaluated separately to normalize the respective marginal *p.d.f.*s.

The histograms of the respective quasi ideal importance sampling marginal probability densities thus determined are illustrated in Fig. 3.

For a reliability problem with k basic random variables, where u_l is specified as the control variable and the others as the sampling variables, the respective quasi ideal importance sampling marginal *p.d.f.*s at p_j-th segment (p_j = 1, 2, …, n) are determined as follows:

$$h_{Q_j}(u_j^{(p_j)}) = \frac{E_{f_{U_{S-j}}}[P_{fC}\{u_l|(u_1,u_2,\ldots,u_j^{(p_j)},\ldots,u_{l-1},u_{l+1},\ldots,u_k)\}]\cdot f_{U_j}(u_j^{(p_j)})}{A_j} \tag{22}$$

$$A_j = \sum_{p_j=1}^{n} E_{f_{U_{S-j}}}[P_{fC}\{u_l|(u_1,u_2,\ldots,u_j^{(p_j)},\ldots,u_{l-1},u_{l+1},\ldots,u_k)\}]\cdot f_{U_j}(u_j^{(p_j)})\Delta u$$

$$(j = 1, 2, \ldots, l-1, l+1, \ldots, k) \tag{23}$$

where $E_{fUS\text{-}j}[\ ]$ is the expectation with respect to a joint *p.d.f.* $f_{US\text{-}j}(\boldsymbol{u}_S)$, in which the j-th $f_{Uj}(u_j)$, $(j = 1, 2, \ldots, l-1, l+1, \ldots, k)$ is excluded from $f_{US}(\boldsymbol{u}_S)$, and the expected conditional failure probability is estimated by the simulation based on the conditional expectation as follows:

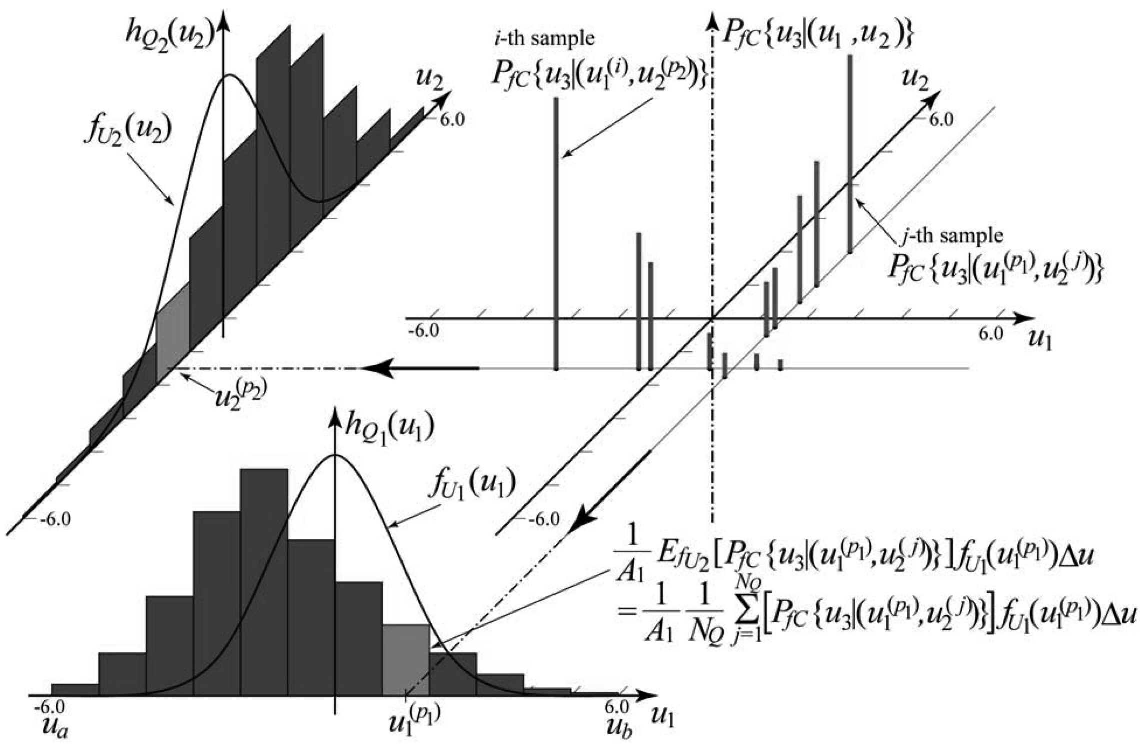

Fig. 3 The sampled distribution of the conditional failure probabilities and the histograms of the resulting quasi ideal importance sampling marginal probability densities

$$E_{f_{U_{S-j}}}[P_{fC}\{u_l|(u_1, u_2, \ldots, u_j^{(p_j)}, \ldots, u_{l-1}, u_{l+1}, \ldots, u_k)\}]$$

$$\approx \frac{1}{N_Q}\sum_{q=1}^{N_Q} P_{fC}\{u_l|(u_1^{(q)}, u_2^{(q)}, \ldots, u_j^{(p_j)}, \ldots, u_{l-1}^{(q)}, u_{l+1}^{(q)}, \ldots, u_k^{(q)})\} \quad (p_j = 1, 2, \ldots, n) \tag{24}$$

where the samples are generated from $f_{US\text{-}j}(\boldsymbol{u}_S)$.

The above procedure to construct the marginal *p.d.f.*s of the quasi ideal importance sampling *p.d.f.* is referred to as a preliminary stage, where $(k\text{-}1) \times n \times N_Q$ samples are totally required. Finally, the quasi ideal importance sampling joint *p.d.f.* $h_Q(\boldsymbol{u}_S)$ is composed of the product of the respective marginal *p.d.f.*s above determined as follows:

$$h_Q(\boldsymbol{u}_S) = \prod_{j=1, j \neq l}^{k} h_{Q_j}(u_j) \tag{25}$$

In the main stage of reliability estimation, the importance sampling simulations combined in the conditional expectation are executed to estimate the failure probabilities of structures with multiple failure surfaces by using N samples. The random outcomes of $k-1$ sampling variables are generated by applying the inverse transformation to the cumulative distribution functions corresponding to the respective marginal *p.d.f.*s given by Eq. (22). This simulation procedure for the estimation of the structural failure probability is referred to as "the quasi ideal importance sampling simulation combined in the conditional expectation."

5. Numerical examples

Numerical examples to estimate the structural failure probabilities are presented to demonstrate the feature of the proposed method. The estimated results of P_f by the proposed "Quasi ideal importance sampling simulation combined in the Conditional Expectation" (denoted as "QCE") are compared in Tables with those estimated by the conventional "Conditional Expectation" (denoted as "CE") and those by the crude Monte Carlo Simulation (denoted as "MCS"). The exact solution of P_f (denoted as "Exact") is substituted for the estimation by MCS with $N = 10^9$ samples. The value of the coefficient of variation of the estimated P_f (denoted as "cov") of 0.01 is specified as a required accuracy level in the main simulation procedure.

In the Preliminary stage to determine the quasi ideal importance sampling marginal *p.d.f.*s in "QCE" method, the specified interval $[u_a, u_b] = [-6.0, 6.0]$ of the respective sampling variables is divided into $n = 12$ segments, that is, $\Delta u = 1$ and Eq. (23) is evaluated by the piecewise integration. The expected conditional failure probabilities given in Eq. (24) are estimated through simulations based on the conditional expectation with sample size of N_Q. Then $(k-1) \cdot n \cdot N_Q$ samples are used to determine all the marginal *p.d.f.*s in the preliminary stage and N samples are used to estimate P_f in the main simulation.

The frequency of limit state functions calls is compared as an index of the calculation efficiency for both in the preliminary stage and the main simulation procedure. In the case of the reliability problem with m failure modes, the frequency of function calls is obtained by $\{(k-1) \cdot n \cdot N_Q + N\} \cdot m$, where k is the number of the dimension of the problem, n that of segments of the piecewise integration in Eq. (23) and m that of the failure modes. The effect of various N_Q on the results of the estimated P_f and the required total frequency of function calls is also compared.

In all simulation methods, the algorithm of "*Mersenne Twister*" for generating uniform pseudo-random numbers is adopted, which provides uniform random variables with a super astronomical period 2^{19937}-1 and 623-dimensional equi-distribution up to 32 bits accuracy (Matsumoto *et al.* 1998). Numerical examples are compiled in C++ on a Windows XP personal computer.

5.1 Case 1

First consider the structural system with limit state functions given in Eq. (26). The basic random variables are assumed to be standardized normal U_i: $N(0, 1^2)$, $(i = 1, 2, 3, 4)$. The variable U_4 is specified as the control variable for applying the conditional expectation. The results of the estimated failure probabilities are given in Table 1.

$$\left.\begin{aligned} g_1(\boldsymbol{U}) &= U_1^2 - 0.05U_2 - U_3U_4 + 7.55 \\ g_2(\boldsymbol{U}) &= 0.03U_1U_4 - U_2U_3 + 7.2 \\ g_3(\boldsymbol{U}) &= -U_1 - U_2 - U_3 - U_4 + 7.0 \end{aligned}\right\} \tag{26}$$

Table 1 Estimation results for *Case* 1

Method (N_Q)	Preliminary sample size $(k–1) \cdot n \cdot N_Q$	Main simulation sample size N	Estimated P_f	Frequency of function calls $\{(k–1) \cdot n \cdot N_Q+N\} \cdot m$
MCS	0	2.737×10^7	3.6524×10^{-4}	8.211×10^7
CE	0	1.030×10^7	3.5943×10^{-4}	3.091×10^7
QCE(10^2)	3.60×10^3	6.469×10^7	3.6224×10^{-4}	1.941×10^8
QCE(10^3)	3.60×10^4	4.758×10^5	3.5732×10^{-4}	1.535×10^6
QCE(10^4)	3.60×10^5	7.406×10^4	3.5903×10^{-4}	*1.302×10^6
QCE(10^5)	3.60×10^6	6.862×10^4	3.6141×10^{-4}	1.100×10^7
QCE(10^6)	3.60×10^7	7.649×10^4	3.5920×10^{-4}	1.823×10^7
Exact (by MCS)	0	1.0×10^9 (cov = 0.0044)	3.6165×10^{-4}	3.0×10^9

Note: Required accuracy level of the estimated P_f in the main simulation: cov ≤ 0.01
N_Q is the sample size of the simulation used in Eq. (24).
Parameters: $k = 4$, $n = 12$ and $m = 3$

5.2 Case 2

Next consider a ten-member truss structure shown in Fig. 4. The limit state functions of eight significant failure modes considered are given in Table 2 (Ono *et al.* 1990), in which T_i, (i = 1, 2, …, 10) are the strength of the members and F_i, (i = 1, 2) are the external forces. The standardized variable of F_1 is specified as the control variable for applying the conditional expectation. The statistical data of the basic random variables are given in Table 3 and the results of the estimated failure probabilities are given in Table 4.

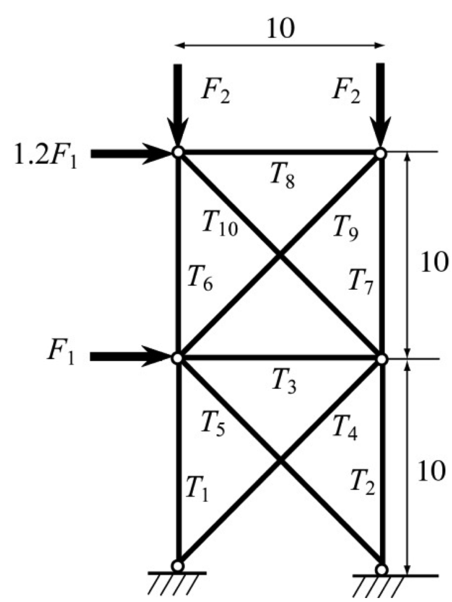

Fig. 4 A ten-member truss structure for *Case* 2

Table 2 List of the limit state functions for *Case* 2

Mode no.	Limit state function
1	$0.7071T_4 + 0.7071T_5 - 2.2F_1$
2	$T_6 + 0.7071T_{10} - 1.2F_1 - F_2$
3	$T_3 + 0.7071T_5 + 0.7071T_{10} - 2.2F_1$
4	$T_8 + 0.7071T_{10} - 1.2F_1$
5	$T_6 + T_7 - 1.2F_1$
6	$T_3 + 0.7071T_5 + T_6 - 1.2F_1 - F_2$
7	$0.7071T_9 + 0.7071T_{10} - 1.2F_1$
8	$T_1 + 0.7071T_5 - 3.4F_1 - F_2$

Table 3 Statistical data of basic random variables for *Case* 2

Variable	Mean value [kN]	Standard deviation [kN]	Distribution type
T_1, T_2	90.0	13.5	Normal
T_3	9.0	1.35	Normal
T_4, T_5	48.0	7.20	Normal
T_6, T_7	21.0	3.15	Normal
T_8	15.0	2.25	Normal
T_9, T_{10}	30.0	4.50	Normal
F_1	11.0	3.30	Normal
F_2	3.6	0.72	Normal

Table 4 Estimation results for *Case* 2

Method (N_Q)	Preliminary sample size $(k-1) \cdot n \cdot N_Q$	Main simulation sample size N	Estimated P_f	Frequency of function calls $\{(k-1) \cdot n \cdot N_Q+N\} \cdot m$
MCS	0	1.953×10^9	5.1189×10^{-5}	1.562×10^9
CE	0	2.119×10^6	5.0740×10^{-5}	1.696×10^7
QCE(10^1)	1.32×10^3	6.663×10^6	5.0609×10^{-5}	5.332×10^7
QCE(10^2)	1.32×10^4	1.045×10^5	5.0978×10^{-5}	*9.421×10^5
QCE(10^3)	1.32×10^5	7.019×10^4	5.0664×10^{-5}	1.617×10^6
QCE(10^4)	1.32×10^6	7.447×10^4	5.0941×10^{-5}	1.116×10^7
Exact (by MCS)	0	1.0×10^9 (cov = 0.0044)	5.0844×10^{-5}	8.0×10^9

Note: Required accuracy level of the estimated P_f in the main simulation: cov ≤ 0.01
N_Q is the sample size of the simulation used in Eq. (24).
Parameters: $k = 12$, $n = 12$ and $m = 8$

5.3 Results and discussions

Generally speaking, sampling variables are selected as the ones of least variability and one of larger variability should be selected as a control variable. Furthermore variables involved in the nonlinear terms among the limit state functions should be carefully treated and possibly avoided as a control variable. In the case of the reliability problems with multiple failure modes, the variables commonly appearing among the limit state functions should be preferably selected. In the present numerical examples, U_1 involved in the square term is avoided and any one of U_2, U_3, U_4 may be applicable, then U_4 is selected as the control variable for *Case* 1 and F_1 appearing most commonly among eight limit state functions is selected as the control variable for *Case* 2.

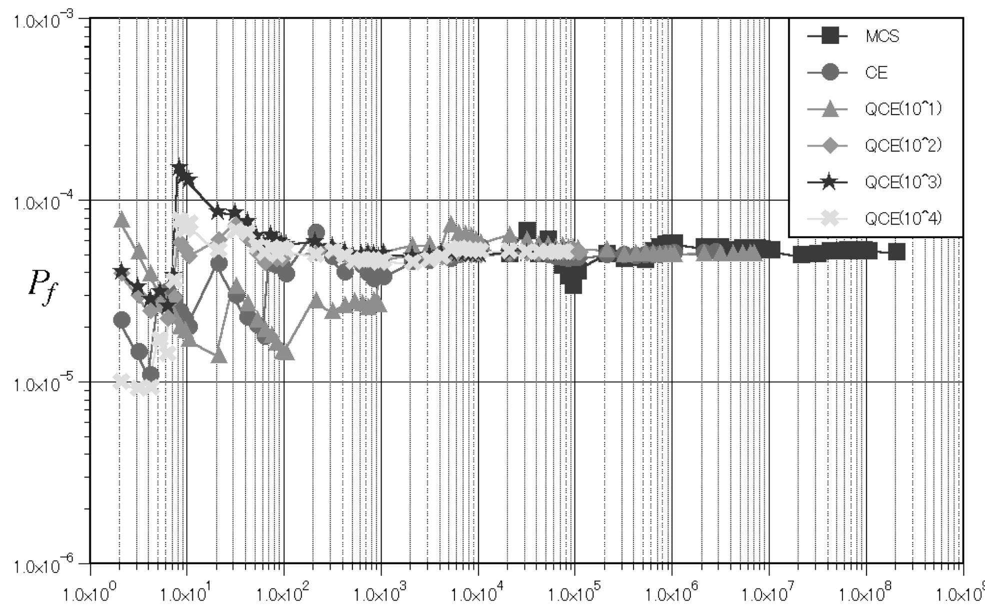

Fig. 5 Simulation results of the estimated P_f *vs*. sample size for *Case* 2

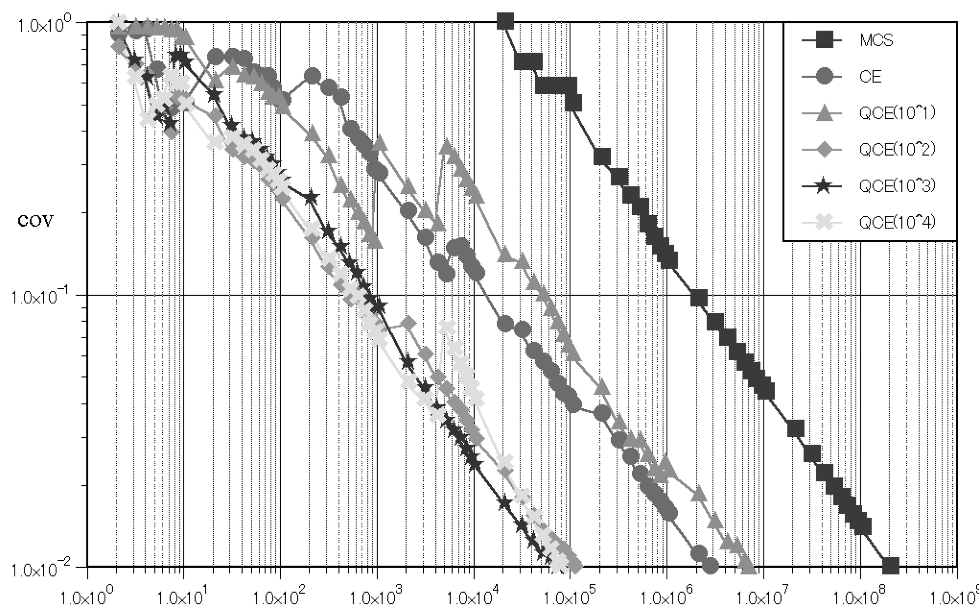

Fig. 6 Simulation results of the cov of the estimated P_f *vs*. sample size for *Case* 2

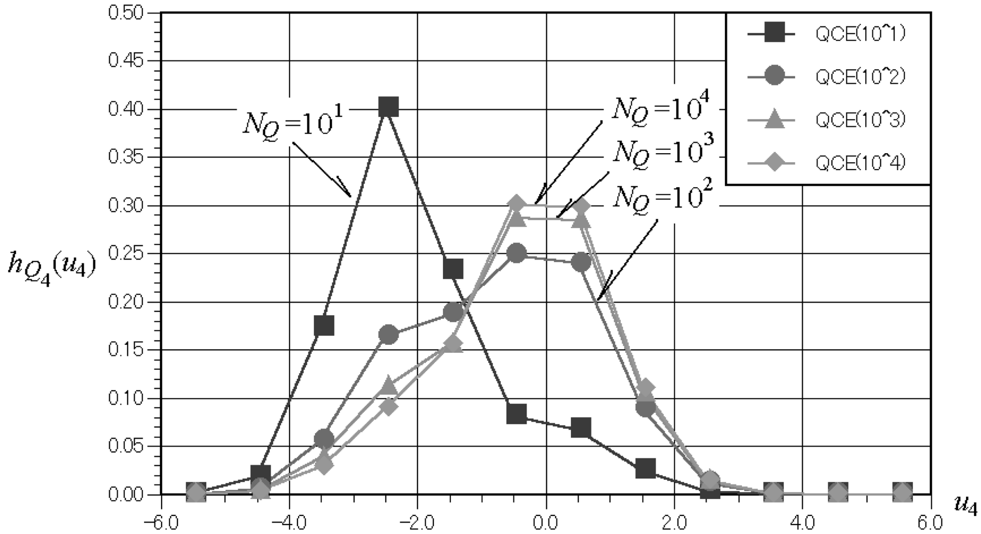

Fig. 7 The effect of N_Q on the shape of the resulting quasi ideal importance sampling marginal *p.d.f.* of u_4 (u_4: the standardized variable of T_4 in *Case* 2)

It is observed from Table 1 for *Case* 1 that the proposed QCE method with $N_Q = 10^4$ gives the best estimation at the smallest frequency of function calls compared with other methods. It is seen from Table 4 for *Case* 2 that the proposed QCE method with $N_Q = 10^2$ gives the best estimation at the smallest frequency of function calls compared with other methods. Generally the smaller N_Q is the better, but it can be said that too small or too large size of N_Q is inappropriate for enhancing the simulation efficiency in terms of the frequency of function calls.

For *Case* 2, the estimated failure probabilities and their cov *vs.* sample size are shown in Figs. 5 and 6. The effects of N_Q on the shape of the resulting quasi ideal importance sampling marginal *p.d.f.* of u_4, the standardized variable of T_4, are compared in Fig. 7 with various N_Q. It is seen from Fig. 7 that the shape of the resulting marginal *p.d.f.* at $N_Q = 10^1$ seems to converge to a certain stable shape as the value of N_Q increases. The quasi ideal importance sampling marginal *p.d.f.*s of other sampling variables also tend to reflect the same tendency. In this case, $N_Q = 10^2$ seems to be appropriate for the determining the quasi ideal importance sampling marginal *p.d.f.*s.

Since the determination of the quasi ideal importance sampling marginal *p.d.f.*s usually requires (k−1) times (k−2)-th multiple integral, then the calculation amount of the piecewise integrations will increase drastically and confront the lack of computer memories as the number of k increases. Therefore it can be said that instead of executing the piecewise integrations in the entire process of the preliminary stage, the simulation approach is effective when constructing the quasi ideal importance sampling marginal *p.d.f.*s. for the reliability problems whose dimension is not too high.

6. Conclusions

A quasi ideal importance sampling simulation combined in the conditional expectation is proposed for the estimation of the structural failure probability. A quasi ideal importance sampling joint *p.d.f.* is defined on the basis of the ideal importance sampling concept. For all sampling variables, the respective marginal *p.d.f.*s. of the quasi ideal importance sampling *p.d.f.* are determined numerically by the simulation based on the conditional expectation and partly by the piecewise integration.

It is shown by the numerical examples that the proposed method gives accurate estimations with smaller sample size and frequency of function calls and the proposed method can be applied for the estimations of the failure probabilities of the structures with multiple failure surfaces.

References

Ang, G.L., Ang, H.-S. and Tang, W.H. (1989), "Kernel method in importance sampling density estimation", *Proc. of ICOSSAR'89*, 1193-1200.

Au, S.K. (2008), "First passage probability of elasto-plastic systems by importance sampling with adaptive process", *Probabilist. Eng. Mech.*, **23**, 114-124.

Ayyub, B.M. and Haldar, A. (1984), "Practical structural reliability techniques", *J. Struct. Eng.*, ASCE, **110**(8), 1707-1724.

Bucher, C.G. (1988), "Adaptive sampling - An iterative fast Monte Carlo procedure", *Struct. Safety*, **5**, 119-126.

Hurtado, J.E. (2007), "Filtered importance sampling with support vector margin: A powerful method for structural reliability analysis", *Struct. Safety*, **29**, 2-15.

Karamchandani, A. and Cornell, C.A. (1991), "Adaptive hybrid conditional expectation approaches for reliability estimation", *Struct. Safety,* **11**, 59-74.

Katafygiotis, L.S. and Cheung, S.H. (2006), "Domain decomposition method for calculating the failure probability of linear dynamic systems subjected to Gaussian stochastic loads", *J. Eng. Mech.*, **132**, 475-486.

Matsumoto, M. and Nishimura, T. (1998), "Mersenne Twister, A 623-dimendionally equi-distributed uniform pseudorandom number generator", *ACM*, *Transactions on Modeling and Computer Simulations*, 8(1), 3-30. Retrieved from the World Wide Web: http://www. math. hiroshima-u.ac.jp/~m-mat/MT/mt. html.

Murotsu, Y., Shao, S., Murayama, N. and Yonezawa, M. (1990), "Importance sampling method for reliability assessment of structures with multi-modal limit states", *Proc.of 1st ISUMA,* Maryland, 49-54.

Ono, T., Idota, H. and Totsuka, H. (1990), "Reliability evaluation of structural systems using higher-order moment standardization technique", *J. Struct. Constr. Eng. T. AIJ* (Architectural Institute of Japan), **448**, 71-79 (in Japanese).

Rubinstein, R.Y. (1981), *Simulation and the Monte Carlo Method*, John Wiley & Sons, 114-157.

Schuëller, G.I. and Stix, R. (1987), "A critical appraisal of methods to determine failure probabilities", *Struct. Safety*, **4**, 293-309.

Schuëller, G.I., Pradlwarter, H.J. and Koutsourelakis, P.S. (2004), "A critical appraisal of reliability estimation procedures for high dimensions", *Probabilist. Eng. Mech.*, **19**, 463-474.

Yonezawa, M., Okuda, S. and Park, Y.T. (1998), "Adaptive sampling for reliability estimation of structure with multimode failure", In A. S. Nowak (ed.), *Reliability and Optimization of Structural Systems*, Proc. of 8th IFIP WG7.5, 337-344, Michigan, Book Crafters.

Yonezawa, M., Okuda, S. and Kobayashi, K. (2007), "Structural Failure Probability Estimation Based on Quasi Ideal Importance Sampling Simulation", *J. School Sci. Eng.,* Kinki University, **43**, 9 (in Japanese).