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Technical Note

A stochastic optimal time-delay control for nonlinear structural systems

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Abstract. The time delay in active and semi-active controls is an important research subject. Many researches on the time-delay control for deterministic systems have been made (Hu and Wang 2002, Yang *et al.* 1990, Abdel-Mooty and Roorda 1991, Pu 1998, Cai and Huang 2002), while the study on that for stochastic systems is very limited. The effects of the time delay on the control of nonlinear systems under Gaussian white noise excitations have been studied by Bilello *et al.* (2002). The controlled linear systems with deterministic and random time delay subjected to Gaussian white noise excitations have been treated by Grigoriu (1997). Recently, a stochastic averaging method for quasi-integrable Hamiltonian systems with time delay has been proposed (Liu and Zhu 2007). In the present paper, a stochastic optimal time-delay control method for stochastically excited nonlinear structural systems is proposed based on the stochastic averaging method for quasi Hamiltonian systems with time delay and the stochastic dynamical programming principle. An example of stochastically excited and controlled hysteretic column is given to illustrate the proposed control method.

1. Optimal time-delay control problem and its transformation

The stochastic optimal time-delay control problem of a nonlinear structural system can be expressed as

$$\mathbf{M}\ddot{\mathbf{X}}+\mathbf{C}\dot{\mathbf{X}}+\frac{\partial V_s(\mathbf{X})}{\partial \mathbf{X}} = \mathbf{F}\mathbf{W}(t)+\mathbf{B}\mathbf{U}(\mathbf{X}_\tau,\dot{\mathbf{X}}_\tau) \tag{1}$$

$$J = \lim_{t_f\to\infty}\frac{1}{t_f}\int_0^{t_f} L(\mathbf{X}(t),\dot{\mathbf{X}}(t),\mathbf{U}(\mathbf{X}_\tau,\dot{\mathbf{X}}_\tau))\mathrm{d}t \tag{2}$$

where $\mathbf{X}$ is the n-dimensional structural displacement vector, $\mathbf{M}$ and $\mathbf{C}$ are respectively symmetric positive-definite structural mass and damping matrices, $V_s(\mathbf{X}) \geq 0$ is the structural potential energy with $V_s(-\mathbf{X}) = V_s(\mathbf{X})$, $\mathbf{W}(t)$ is the m-dimensional stochastic process vector assumed as Gaussian

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white noise with intensity $2\mathbf{D}$, $\mathbf{U}(\mathbf{X}_\tau, \dot{\mathbf{X}}_\tau)$ is the s-dimensional control force vector dependent on the past states $\mathbf{X}_\tau = \mathbf{X}(t-\tau)$ and $\dot{\mathbf{X}}_\tau = \dot{\mathbf{X}}(t-\tau)$, τ is the delayed time in control, $\mathbf{F}$ and $\mathbf{B}$ are constant matrices, J is the performance index of the infinite time-interval ergodic control, t_f is the terminal time and $L(\mathbf{X}, \dot{\mathbf{X}}_\tau, \mathbf{U})$ is a continuous differential convex function. Eq. (1) is rewritten in the form of quasi Hamiltonian equations

$$\dot{\mathbf{Q}} = \frac{\partial H}{\partial \mathbf{P}},\ \dot{\mathbf{P}} = -\frac{\partial H}{\partial \mathbf{Q}} - \mathbf{C}\frac{\partial H}{\partial \mathbf{P}} + \mathbf{F}\mathbf{W}(t) + \mathbf{B}\mathbf{U}(\mathbf{Q}_\tau, \mathbf{P}_\tau) \tag{3}$$

where $\mathbf{Q} = \mathbf{X}$, $\mathbf{P} = \mathbf{M}\dot{\mathbf{X}}$, $H = \mathbf{P}^T\mathbf{M}^{-1}\mathbf{P}/2 + V_s(\mathbf{Q})$ is the Hamiltonian representing total structural energy. It is assumed that the Hamiltonian system corresponding to (3) is completely integrable, as many engineering structures are modeled generally. According to the stochastic averaging method for quasi-integrable Hamiltonian systems with time-delay (Liu and Zhu 2007), the past structural displacement and momentum can be expressed approximately by the present structural displacement and momentum as

$$Q_{\tau i} = Q_i(t)\cos\omega_i\tau - P_i(t)\sin\omega_i\tau/\omega_i,\quad P_{\tau i} = Q_i(t)\,\omega_i\sin\omega_i\tau + P_i(t)\cos\omega_i\tau \tag{4}$$

where Q_i and P_i are the i-th elements of state vectors $\mathbf{Q}$ and $\mathbf{P}$, respectively, ω_i is the averaged frequency of the i-th integrated sub-system. The following averaged Itô stochastic differential equations with the corresponding performance index can be derived further

$$\mathrm{d}H_r = \left[m_r(\mathbf{H}) + \langle(\mathbf{B}\overline{\mathbf{U}}(\mathbf{Q},\mathbf{P}))_i\frac{\partial H_r}{\partial P_i}\rangle\right]\mathrm{d}t + \sigma_{rk}(\mathbf{H})\mathrm{d}B_k(t) \tag{5}$$

$$\bar{J} = \lim_{t_f\to\infty}\frac{1}{t_f}\int_0^{t_f} L(\mathbf{H}(t), <\overline{\mathbf{U}}(\mathbf{Q}(t), \mathbf{P}(t))>)\mathrm{d}t \tag{6}$$

where H_r is the independent integral of motion, $\mathbf{H} = [H_1, H_2, \ldots, H_n]^T$, $<\cdot>$ denotes the averaging operation, $\overline{\mathbf{U}}(\mathbf{Q}, \mathbf{P})$ is the transformed control force vector, $B_k(t)$ is the unit Wiener process, $m_r(\mathbf{H})$ and $\sigma_{rk}(\mathbf{H})$ are the drift and diffusion coefficients (Ying *et al.* 2003b). The optimal time-delay control problem [(1) and (2)] has been converted into another optimal control problem without time delay [(5) and (6)].

2. Optimal active and semi-active time-delay control laws

Based on the stochastic dynamical programming principle, the dynamical programming equation for (5) and (6) is established as (Ying *et al.* 2003a)

$$\min_{\overline{\mathbf{U}}}\left\{L(\mathbf{H}, <\overline{\mathbf{U}}>) + \left[m_r(\mathbf{H}) + <(\mathbf{B}\overline{\mathbf{U}})_i\frac{\partial H_r}{\partial P_i}>\right]\frac{\partial V}{\partial H_r} + \sigma_{rk}(\mathbf{H})\sigma_{sk}(\mathbf{H})\frac{\partial^2 V}{2\partial H_r\partial H_s}\right\} = \lambda \tag{7}$$

where V is called the value function and λ is a constant. The optimal control law $\overline{\mathbf{U}}^*(\mathbf{Q}, \mathbf{P})$ can be determined by minimizing the left side of Eq. (7), which depends on the present structural displacement and momentum. By using Eq. (4), the present structural states can be expressed approximately by the past structural states as

$$Q_i = Q_{\tau i}\cos\omega_i\tau + P_{\tau i}\sin\omega_i\tau/\omega_i\,,\; P_i = -Q_{\tau i}\omega_i\sin\omega_i\tau + P_{\tau i}\cos\omega_i\tau \tag{8}$$

Therefore, the optimal active time-delay control force dependent completely on the past structural displacement and velocity is derived as follows

$$\begin{aligned}\mathbf{U}^*(\mathbf{X}_\tau, \dot{\mathbf{X}}_\tau) &= \mathbf{U}^*(Q_i\cos\omega_i\tau + (-P_i\sin\omega_i\tau/\omega_i),\; Q_i\omega_i\sin\omega_i\tau + P_i\cos\omega_i\tau)\\ &= \overline{\mathbf{U}}^*(\mathbf{Q},\mathbf{P}) = \overline{\mathbf{U}}^*(Q_{\tau i}\cos\omega_i\tau + P_{\tau i}\sin\omega_i\tau/\omega_i, -Q_{\tau i}\omega_i\sin\omega_i\tau + P_{\tau i}\cos\omega_i\tau)\\ &= -\mathbf{R}^{-1}\mathbf{B}^T\left(\frac{\partial H_r}{2\partial\mathbf{P}}\frac{\partial V}{\partial H_r}\right)\left(X_{\tau i}\cos\omega_i\tau + (\mathbf{M}\dot{\mathbf{X}}_\tau)_i\frac{\sin\omega_i\tau}{\omega_i}, (\mathbf{M}\dot{\mathbf{X}}_\tau)_i\cos\omega_i\tau - X_{\tau i}\omega_i\sin\omega_i\tau\right)\end{aligned} \tag{9}$$

For the semi-active control using MR dampers, the control force $\mathbf{U}$ can be split into passive part $\mathbf{U}_{ps}$ and semi-active part $\mathbf{U}_{sa}$ (Ying *et al.* 2003b). The passive part is incorporated in the structural system. The semi-active part is determined by Eq. (9) according to Bingham model as

$$\mathbf{U}^*_{sa}(\mathbf{X}(t-\tau), \dot{\mathbf{X}}(t-\tau)) = [U^*_{sa1}(\mathbf{X}_\tau, \dot{\mathbf{X}}_\tau), U^*_{sa2}(\mathbf{X}_\tau, \dot{\mathbf{X}}_\tau), \ldots, U^*_{sas}(\mathbf{X}_\tau, \dot{\mathbf{X}}_\tau)]$$

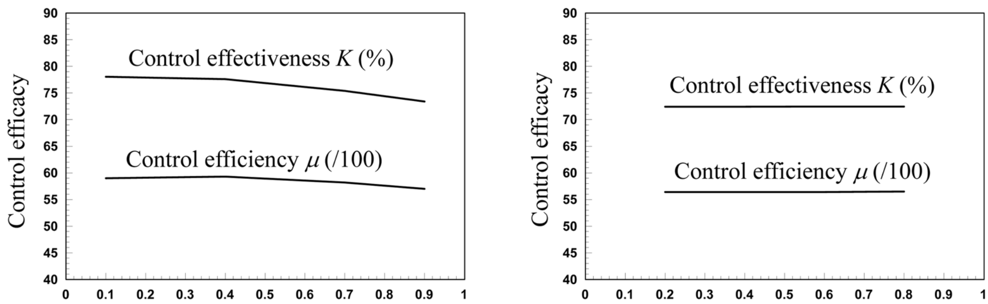
$$U^*_{saj}(\mathbf{X}_\tau, \dot{\mathbf{X}}_\tau) = \frac{F^*_{dj} + |F^*_{dj}|}{-2}\mathrm{sgn}\{(\mathbf{B}^T\mathbf{M}^{-1}\mathbf{P})_j\}\,,\quad F^*_{dj} = \left(\mathbf{R}^{-1}\mathbf{B}^T\frac{\partial H_r}{2\partial\mathbf{P}}\frac{\partial V}{\partial H_r}\right)_j\mathrm{sgn}\{(\mathbf{B}^T\mathbf{M}^{-1}\mathbf{P})_j\} \tag{10}$$

The $\mathbf{U}^*_{sa}$ dependent completely on the past structural displacement and velocity is an optimal semi-active time-delay control force. In the case of $\partial V/\partial H_1 = \partial V/\partial H_2 = \ldots = \partial V/\partial H_n \geq 0$, the optimal semi-active time-delay control force becomes an optimal active time-delay control force so that the semi-active MR dampers can implement the active time-delay control law without clipping in this case.

Consider the nonlinear hysteretic column under stochastic support excitations and a time-delay control force $u(X_\tau, \dot{X}_\tau)$ (Zhu *et al.* 2000). By using the proposed control method, the optimal active and semi-active time-delay control force can be obtained, for $L(H,\bar{u}) = g(H) + \langle r\bar{u}^2\rangle$ as

$$u^* = \frac{X_\tau\omega\sin\omega\tau - \dot{X}_\tau\cos\omega\tau}{2r}\frac{\mathrm{d}}{\mathrm{d}H}V\left(H\left(X_\tau\cos\omega\tau + \dot{X}_\tau\frac{\sin\omega\tau}{\omega}\right), \dot{X}_\tau\cos\omega\tau - X_\tau\omega\sin\omega\tau\right) \tag{11}$$

$$u^*_{sa} = -\frac{1}{2}(F^*_d + |F^*_d|)\mathrm{sgn}\{-X_\tau\omega\sin\omega\tau + \dot{X}_\tau\cos\omega\tau\}$$



(a) Excitation intensity D_1 (b) Excitation intensity D_2

Fig. 1 Control efficacy for different excitation intensity

$$F_d^* = \frac{\left|X_\tau \omega \sin\omega\tau - \dot{X}_\tau \cos\omega\tau\right|}{2r} \frac{\mathrm{d}}{\mathrm{d}H} V\left(H\left(X_\tau \cos\omega\tau + \dot{X}_\tau \frac{\sin\omega\tau}{\omega}, \dot{X}_\tau \cos\omega\tau - X_\tau \omega \sin\omega\tau\right)\right) \tag{12}$$

Taking r and $g(H)$ such that $dV/dH \geq 0$ yields $u_{sa}^* = u^*$. The mean square displacements of the uncontrolled and controlled hysteretic columns can be calculated by using the stochastic averaging method. The control efficacy can be evaluated by control effectiveness K and efficiency μ (Ying *et al.* 2003b). Figs. 1(a) and 1(b) show the high control efficacy of the hysteretic column under the proposed optimal time-delay control.

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References

Abdel-Mooty, M. and Roorda, J. (1991), "Time-delay compensation in active damping of structures", *J. Eng. Mech.*, ASCE, **117**, 2549-2570.

Bilello, C., Di Paola, M. and Pirrotta, A. (2002), "Time delay induced effects on control of non-linear systems under random excitation", *Mechanica*, **37**, 207-220.

Cai, G. and Huang, J. (2002), "Optimal control method with time delay in control", *J. Sound Vib.*, **251**, 383-394.

Grigoriu, M. (1997), "Control of time delay linear systems with Gaussian white noise", *Probabilist. Eng. Mech.*, **12**, 89-96.

Hu, H.Y. and Wang, Z.H. (2002), *Dynamics of Controlled Mechanical Systems with Delay Feedback*, Springer, Berlin.

Liu, Z.H. and Zhu, W.Q. (2007), "Stochastic averaging of quasi-integrable Hamiltonian systems with delayed feedback control", *J. Sound Vib.*, **299**, 178-195.

Pu, J.P. (1998), "Time delay compensation in active control of structures", *J. Eng. Mech.*, ASCE, **124**, 1018-1028.

Yang, J.N., Akbarpour, A. and Askar, G. (1990), "Effect of time delay on control of seismic-excited buildings", *J. Struct. Eng.*, ASCE, **116**, 2801-2814.

Ying, Z.G., Ni, Y.Q. and Ko, J.M. (2003a), "Stochastic optimal control of coupled structures", *Struct. Eng. Mech.*, **15**, 669-683.

Ying, Z.G., Zhu, W.Q. and Soong, T.T. (2003b), "A stochastic optimal semi-active control strategy for ER/MR dampers", *J. Sound Vib.*, **259**, 45-62.

Zhu, W.Q., Ying, Z.G., Ni, Y.Q. and Ko, J.M. (2000), "Optimal nonlinear stochastic control of hysteretic systems", *J. Eng. Mech.*, ASCE, **126**, 1027-1032.