Structural Engineering and Mechanics, Vol. 31, No. 4 (2009) 453-476

DOI: http://dx.doi.org/10.12989/sem.2009.31.4.453

Free vibration of axially loaded Reddy-Bickford beam on elastic soil using the differential transform method

Yusuf Yesilce† and Seval Catal‡

Dokuz Eylul University, Civil Engineering Department, Engineering Faculty, 35160, Buca, Izmir, Turkey

(Received December 9, 2008, Accepted January 19, 2009)

Abstract. The literature regarding the free vibration analysis of Bernoulli-Euler and Timoshenko beams on elastic soil is plenty, but the free vibration analysis of Reddy-Bickford beams on elastic soil with/without axial force effect using the Differential Transform Method (DTM) has not been investigated by any of the studies in open literature so far. In this study, the free vibration analysis of axially loaded Reddy-Bickford beam on elastic soil is carried out by using DTM. The model has six degrees of freedom at the two ends, one transverse displacement and two rotations, and the end forces are a shear force and two end moments in this study. The governing differential equations of motion of the rectangular beam in free vibration are derived using Hamilton's principle and considering rotatory inertia. Parameters for the relative stiffness, stiffness ratio and nondimensionalized multiplication factor for the axial compressive force are incorporated into the equations of motion in order to investigate their effects on the natural frequencies. At first, the terms are found directly from the analytical solutions of the differential equations that describe the deformations of the cross-section according to the high-order theory. After the analytical solution, an efficient and easy mathematical technique called DTM is used to solve the governing differential equations of the motion. The calculated natural frequencies of one end fixed and the other end simply supported Reddy-Bickford beam on elastic soil using DTM are tabulated in several tables and figures and are compared with the results of the analytical solution where a very good agreement is observed and the mode shapes are presented in graphs.

Keywords: differential transform method; elastic soil; free vibration; partial differential equation; Reddy-Bickford beam.

1. Introduction

The analysis of beams has been performed over the years mostly using Bernoulli-Euler beam theory. The classical Bernoulli-Euler beam is well studied for slender beams, where the transverse shear deformation can be safely disregarded. This theory is based on the assumption that plane sections of the cross-section remain plane and perpendicular to the beam axis. The cross-sectional displacements are shown in (Fig. 1(a)), and expresses as

$$u(x,z,t) = -z \cdot \frac{\partial w_0(x,t)}{\partial x} \tag{1}$$

† Ph.D. Student, Corresponding author, E-mail: yusuf.yesilce@deu.edu.tr
‡ Assistant Professor, Ph.D., E-mail: seval.catal@deu.edu.tr

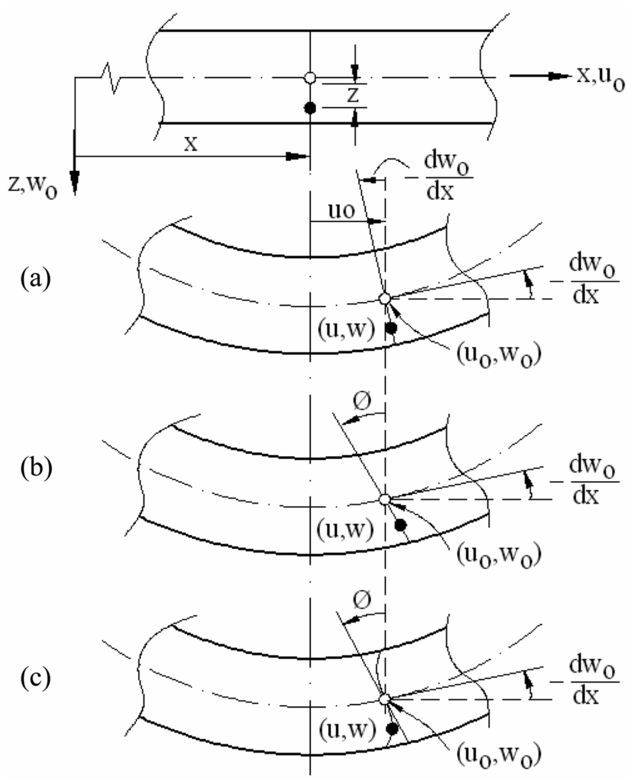

Fig. 1 Cross-section displacements in different beam theories (Wang *et al.* 2000). (a) Bernoulli-Euler Beam Theory, (b) Timoshenko Beam Theory, (c) Reddy-Bickford Beam Theory

$$w(x, z, t) = w_0(x, t) \tag{2}$$

where $w_0(x, t)$ is the lateral displacement of the beam neutral axis, z is the distance from the beam neutral axis (Timoshenko 1921).

For moderately thick beams Bernoulli-Euler beam theory can be modified in order to take into account the transverse shear effect in a simplified way. For example, the well-known Timoshenko beam theory predicts a uniform shear distribution, so necessitating the use of a so-called shear factor (Cowper 1966, Murthy 1970, Gruttmann and Wagner 2001). The cross-sectional displacements of Timoshenko beam theory are shown in (Fig. 1(b)) and the equations for Timoshenko beam theory which relaxes the restriction on the angle of shearing deformations are

$$u(x, z, t) = z \cdot \phi(x, t) \tag{3}$$

$$w(x, z, t) = w_0(x, t) \tag{4}$$

where $\phi(x, t)$ represents the rotation of a normal to the axis of the beam. Han *et al.* presented a comprehensive study of Bernoulli-Euler, Rayleigh, Shear and Timoshenko beam theories (Han *et al.* 1999).

The real shear deformation distribution is not uniform along the depth of the beam, so that Timoshenko beam theory is not recommended for composite beams, where the accurate determination of the shear stresses is required. Especially, it was found that the Timoshenko shear deformation theory has some major numerical problems such as locking in the numerical analysis for composite materials. The other problem was the need to supply an artificially derived shear correction factor. Although some remedies were devised, as a result, several higher-order theories have emerged. These theories, with small variations, are due to Bickford, Levinson, Heyliger and Reddy, Wang *et al*. and others all relax the restriction on the warping of the cross-section and allow variation in the longitudinal direction of the beam which is cubic (Bickford 1982, Levinson 1981, Heyliger and Reddy 1988, Wang *et al.* 2000).

In this paper, Reddy-Bickford beam theory is used, which seems a good compromise between accuracy and simplicity (Bickford 1982, Wang *et al.* 2000). The cross-sectional displacements of Reddy-Bickford beam theory are shown in (Fig. 1(c)) and according to Reddy-Bickford beam theory, the displacements of the rectangular beam can be written as Wang *et al.* 2000, Reddy 2002, Reddy 2007)

$$u(x,z,t) = z \cdot \phi(x,t) - \gamma \cdot z^3 \cdot \left[\phi(x,t) + \frac{\partial w(x,t)}{\partial x}\right] \tag{5}$$

$$w(x,z,t) = w_0(x,t) \tag{6}$$

where $\gamma = 4/(3 \cdot h^2)$; h is the height of the beam. Yesilce and Catal compared the free vibration analysis of Reddy-Bickford pile with the results of Timoshenko pile by using analytical method (Yesilce and Catal 2008).

Bernoulli-Euler beam theory does not consider the shear stress in the cross-section and the associated strains. Thus, the shear angle is taken as zero through the height of the cross-section. Timoshenko beam theory assumes constant shear stress and shear strain in the cross-section. On the top and bottom edges of the beam the free surface condition is thus violated. The use of a shear correction factor, in various forms including the effect of Poisson's ratio, does not correct this fault of the theory, but rather artificially adjusts the solutions to match the static or dynamic behavior of the beam. Reddy-Bickford beam theory and the other high-order theories remedy this physical mismatch at the free edges by assuming variable shear strain and shear stress along the height of the cross-section. Then there is no need for the shear correction factor. The high-order theory is more exact and represents much better the physics of the problem. It results in a sixth-order theory compared to the fourth order of the other less-accurate theories. This yields a six-degree-of-freedom element with six end forces, a shear force, bending moment and a high-order moment, at the two ends of the beam element.

Previously, numerous researchers studied the behavior of beams supported by elastic foundations (Hetenyi 1955). Doyle and Pavlovic solved the partial differential equation for free vibration of beams partially attached to elastic foundation using variable separating method and neglecting axial force and shear effects (Doyle and Pavlovic 1982). West and Mafi solved the partial differential equation for free vibration of an elastic beam on elastic foundation that is subjected to axial force by using initial value method (West and Mafi 1984). Yokoyama studied the free vibration motion of Timoshenko beam on two-parameters elastic foundation (Yokoyama 1991). Esmailzadeh and Ohadi investigated vibration and stability analysis of non-uniform Timoshenko beams under axial and distributed tangential loads (Esmailzadeh and Ohadi 2000).

DTM was applied to solve linear and non-linear initial value problems and partial differential equations by many researches. The concept of DTM was first introduced by Zhou and he used DTM to solve both linear and non-linear initial value problems in electric circuit analysis (Zhou 1986). Chen and Ho solved eigenvalue problems for the free and transverse vibration problems of a rotating twisted Timoshenko beam under axial loading by using DTM (Chen and Ho 1996, 1999). DTM was applied to solve a second order non-linear differential equation that describes the under damped and over damped motion of a system subject to external excitation by Jang and Chen (1997). Chen and Liu considered first order both the linear and non-linear two-point boundary value problems by using DTM (Chen and Liu 1998). In the other study, Jang *et al*. investigated the linear and non-linear initial value problems by using DTM (Jang *et al*. 2000). Malih and Dang applied DTM to the free vibration of Bernoulli-Euler beams (Malih and Dang 1998). Hassan studied the solution of Sturm-Lioville eigenvalue problem and solved partial differential equations by using DTM (Hassan 2002a, 2002b). Ayaz obtained numerical solution of linear differential equations by using DTM (Ayaz 2004). Bert and Zeng used DTM to investigate analysis of axial vibration of compound bars (Bert and Zeng 2004). Kurnaz *et al*. studied n-dimensional DTM to solve partial differential equations (Kurnaz *et al*. 2005). Özdemir and Kaya investigated flapwise bending vibration analysis of a rotating tapered cantilever Bernoulli-Euler beam by DTM (Özdemir and Kaya 2006). In the other study, the out-of-plane free vibration analysis of a double tapered Bernoulli-Euler beam, mounted on the periphery of a rotating rigid hub is performed using DTM by Ozgumus and Kaya (2006). Catal suggested DTM for the free vibration analysis of both ends simply supported and one end fixed, the other end simply supported Timoshenko beams resting on elastic soil foundation (Catal 2006, 2008). Catal and Catal calculated the critical buckling loads of partially embedded Timoshenko pile in elastic soil by DTM (Catal and Catal 2006). Ho and Chen investigated the vibration problems of an axially loaded non-uniform spinning twisted Timoshenko beam by using DTM (Ho and Chen 2006). Bildik *et al*. expressed the definitions and operations of DTM and Adomian's decomposition method on different partial differential equations (Bildik *et al*. 2006). Free vibration analysis of a rotating, double tapered Timoshenko beam featuring coupling between flapwise bending and torsional vibrations is performed using DTM by Ozgumus and Kaya (2007). In the other study, Kaya and Ozgumus introduced DTM to analyze the free vibration response of an axially loaded, closed-section composite Timoshenko beam which features material coupling between flapwise bending and torsional vibrations due to ply orientation (Kaya and Ozgumus 2007). Ertürk and Momani presented a numerical comparison between DTM and Adomian's decomposition method for solving fourth-order boundary value problems (Ertürk and Momani 2007). DTM was applied to construct semi numerical-analytic solutions of linear sixth-order boundary value problems with two-point boundary value conditions by Ertürk (2007). Numerical solution to buckling analysis of Bernoulli-Euler beams and columns were obtained using DTM and harmonic differential quadrature for various support conditions considering the variation of flexural rigidity by Rajasekaran (2008). In this study, solution technique is applied to find the buckling load of fully or partially embedded columns such as piles. Since previous studies have shown DTM to be an efficient tool and it has been applied to solve boundary value problems for many linear, non-linear integro-differential and differential-difference equations that are very important in fluid mechanics, viscoelasticity, control theory, acoustics, etc. Besides the variety of the problems to that DTM may be applied, its accuracy and simplicity in calculating the natural frequencies and plotting the mode shapes makes this method outstanding among many other methods.

In this study, the free vibration analysis of a rectangular and one end fixed, the other end simply supported Reddy-Bickford beam resting on elastic soil is performed. At the beginning of the study, the governing equations of motion are obtained applying Hamilton's principle and Winkler hypothesis and considering rotatory inertia. In the solution part, the equations of motion, including the parameters for the relative stiffness, stiffness ratio and nondimensionalized multiplication factor for the axial compressive force, are solved using analytical method and an efficient mathematical technique, called DTM. Finally, the natural frequencies of Reddy-Bickford beam are calculated, the mode shapes are plotted and effects of the parameters, mentioned above, are investigated by using the computer package, Matlab. Unfortunately, a suitable example that studies the free vibration analysis of Reddy-Bickford beams on elastic soil with/without axial force effect using DTM has not been investigated by any of the studies in open literature so far.

2. The mathematical model and formulation

A Reddy-Bickford beam resting on elastic soil is presented in (Fig. 2). It is assumed that the elastic soil that the beam is on behaves due to Winkler hypothesis.

The relation between displacement function $w(x, t)$ of the beam on elastic soil and the distributed force $q(x, t)$ existing at the elastic soil under the beam can be written as

$$q(x,t) = C_S \cdot w(x,t) \tag{7}$$

where $C_S = C_0 \cdot b$, C_0 is the modulus of subgrade reaction, b is the width of the beam.

Using Hamilton's principle and Eqs. (5) and (6) and considering rotatory inertia, the equations of motion for a rectangular Reddy-Bickford beam on elastic soil can be written as

$$\begin{aligned} -\frac{68}{105}\cdot EI_x \cdot \frac{\partial^2 \phi(x,t)}{\partial x^2} + \frac{16}{105}\cdot EI_x \cdot \frac{\partial^3 w(x,t)}{\partial x^3} + \frac{8}{15}\cdot AG \cdot \left[\phi(x,t) + \frac{\partial w(x,t)}{\partial x}\right] = \\ -\frac{68}{105}\cdot \frac{m \cdot I_x}{A} \cdot \frac{\partial^2 \phi(x,t)}{\partial t^2} + \frac{16}{105}\cdot \frac{m\cdot I_x}{A}\cdot \frac{\partial^3 w(x,t)}{\partial x \cdot \partial t^2} \end{aligned} \tag{8}$$

$$\begin{aligned} -m \cdot \frac{\partial^2 w(x,t)}{\partial t^2} + \frac{8}{15}\cdot AG \cdot \left[\frac{\partial \phi(x,t)}{\partial x} + \frac{\partial^2 w(x,t)}{\partial x^2}\right] + \frac{16}{105}\cdot EI_x \cdot \frac{\partial^3 \phi(x,t)}{\partial x^3} - \frac{1}{21}\cdot EI_x \cdot \frac{\partial^4 w(x,t)}{\partial x^4} \\ -C_S \cdot w(x,t) - N \cdot \frac{\partial^2 w(x,t)}{\partial x^2} = -\frac{1}{21}\cdot \frac{m\cdot I_x}{A}\cdot \frac{\partial^4 w(x,t)}{\partial x^2 \cdot \partial t^2} + \frac{16}{105}\cdot \frac{m \cdot I_x}{A}\cdot \frac{\partial^3 \phi(x,t)}{\partial x \cdot \partial t^2} \end{aligned} \tag{9}$$

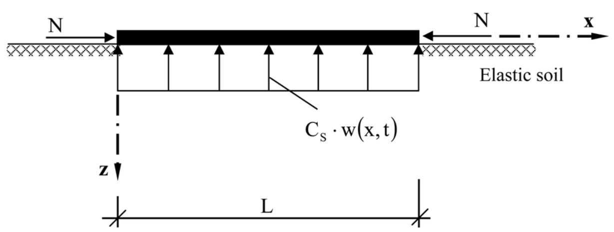

Fig. 2 A beam on elastic soil

where $\phi(x,t)$ represents the rotation of a normal to the axis of the beam, m is mass per unit length of the beam, L is length of the beam, N is the axial compressive force, A is the cross-section area, I_x is moment of inertia, E, G are Young's modulus and shear modulus of the beam, respectively, x is the beam position, t is time variable.

Assuming that the motion is harmonic we substitute for $w(z,t)$ and $\phi(z,t)$ the following

$$w(z,t) = w(z)\cdot\sin(\omega\cdot t) \tag{10}$$

$$\phi(z,t) = \phi(z)\cdot\sin(\omega\cdot t) \tag{11}$$

and obtain a system of two coupled ordinary equation as

$$\begin{gathered}-\frac{68}{105}\cdot\frac{EI_x}{L^2}\cdot\frac{d^2\phi(z)}{dz^2}+\frac{16}{105}\cdot\frac{EI_x}{L^3}\cdot\frac{d^3w(z)}{dz^3}+\frac{8}{15}\cdot AG\cdot\left[\phi(z)+\frac{1}{L}\cdot\frac{dw(z)}{dz}\right]= \\ \frac{68}{105}\cdot\frac{m\cdot I_x\cdot\omega^2}{A}\cdot\phi(z)-\frac{16}{105}\cdot\frac{m\cdot I_x\cdot\omega^2}{A\cdot L}\cdot\frac{dw(z)}{dz}\end{gathered} \tag{12}$$

$$\begin{gathered}m\cdot\omega^2\cdot w(z)+\frac{8}{15}\cdot\frac{AG}{L}\cdot\left[\frac{d\phi(z)}{dz}+\frac{1}{L}\cdot\frac{d^2w(z)}{dz^2}\right]+\frac{16}{105}\cdot\frac{EI_x}{L^3}\cdot\frac{d^3\phi(z)}{dz^3}-\frac{1}{21}\cdot\frac{EI_x}{L^4}\cdot\frac{d^4w(z)}{dz^4} \\ -C_S\cdot w(z)-\frac{N}{L^2}\cdot\frac{d^2w(z)}{dz^2}=\frac{1}{21}\cdot\frac{m\cdot I_x\cdot\omega^2}{A\cdot L^2}\cdot\frac{d^2w(z)}{dz^2}-\frac{16}{105}\cdot\frac{m\cdot I_x\cdot\omega^2}{A\cdot L}\cdot\frac{d\phi(z)}{dz}\end{gathered} \tag{13}$$

where $z = x/L$.

It is assumed that the solution is

$$w(z) = C\cdot e^{isz} \tag{14}$$

$$\phi(z) = P\cdot e^{isz} \tag{15}$$

and substituting Eqs. (14) and (15) into Eqs. (12) and (13) results in

$$\begin{gathered}\left(\frac{8}{15}\cdot AG-\frac{68}{105}\cdot\frac{m\cdot I_x\cdot\omega^2}{A}+\frac{68}{105}\cdot\frac{EI_x}{L^2}\cdot s^2\right)\cdot P \\ +\left(\frac{8}{15}\cdot\frac{AG}{L}\cdot s\cdot i+\frac{16}{105}\cdot\frac{m\cdot I_x\cdot\omega^2}{A\cdot L}\cdot s\cdot i-\frac{16}{105}\cdot\frac{EI_x}{L^3}\cdot s^3\cdot i\right)\cdot C = 0\end{gathered} \tag{16}$$

$$\begin{gathered}\left(\frac{8}{15}\cdot\frac{AG}{L}\cdot s\cdot i+\frac{16}{105}\cdot\frac{m\cdot I_x\cdot\omega^2}{A\cdot L}\cdot s\cdot i-\frac{16}{105}\cdot\frac{EI_x}{L^3}\cdot s^3\cdot i\right)\cdot P \\ +\left(m\cdot\omega^2-\frac{8}{15}\cdot\frac{AG}{L^2}\cdot s^2-\frac{1}{21}\cdot\frac{EI_x}{L^4}\cdot s^4-C_S+\frac{N}{L^2}\cdot s^2+\frac{1}{21}\cdot\frac{m\cdot I_x\cdot\omega^2}{A\cdot L^2}\cdot s^2\right)\cdot C = 0\end{gathered} \tag{17}$$

Eqs. (16) and (17) can be written in matrix form for the two unknowns P and C as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{Bmatrix} P \\ C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{18}$$

where

$$A_{11} = \frac{8}{15} \cdot AG - \frac{68}{105} \cdot \frac{m \cdot I_x \cdot \omega^2}{A} + \frac{68}{105} \cdot \frac{EI_x}{L^2} \cdot s^2 \tag{19.a}$$

$$A_{12} = A_{21} = \frac{8}{15} \cdot \frac{AG}{L} \cdot s \cdot i + \frac{16}{105} \cdot \frac{m \cdot I_x \cdot \omega^2}{A \cdot L} \cdot s \cdot i - \frac{16}{105} \cdot \frac{EI_x}{L^3} \cdot s^3 \cdot i \tag{19.b}$$

$$A_{22} = m \cdot \omega^2 - \frac{8}{15} \cdot \frac{AG}{L^2} \cdot s^2 - \frac{1}{21} \cdot \frac{EI_x}{L^4} \cdot s^4 - C_S + \frac{N}{L^2} \cdot s^2 + \frac{1}{21} \cdot \frac{m \cdot I_x \cdot \omega^2}{A \cdot L^2} \cdot s^2 \tag{19.c}$$

and the non-trivial solution will be when the determinant of the coefficient matrix will be zero. Thus, we have a sixth-order equation with the unknowns, resulting in six values and the general solution functions can be written as

$$w(z,t) = [C_1 \cdot e^{is_1 z} + C_2 \cdot e^{is_2 z} + C_3 \cdot e^{is_3 z} + C_4 \cdot e^{is_4 z} + C_5 \cdot e^{is_5 z} + C_6 \cdot e^{is_6 z}] \cdot \sin(\omega \cdot t) \tag{20}$$

$$\phi(z,t) = [P_1 \cdot e^{is_1 z} + P_2 \cdot e^{is_2 z} + P_3 \cdot e^{is_3 z} + P_4 \cdot e^{is_4 z} + P_5 \cdot e^{is_5 z} + P_6 \cdot e^{is_6 z}] \cdot \sin(\omega \cdot t) \tag{21}$$

The twelve constants, C_1, ..., C_6 and P_1, ..., P_6 will be found from Eqs. (16), (17) and boundary conditions.

The expression for bending rotation $w'(z,t)$ is given by

$$w'(z,t) = \frac{1}{L} \cdot \frac{dw(z)}{dz} \cdot \sin(\omega \cdot t) \tag{22}$$

The shear force function $Q(z, t)$ can be obtained by using Eqs. (20) and (21) as

$$\begin{aligned} Q(z,t) = & \left[\frac{EI_x}{21 \cdot L^3} \cdot \frac{d^3 w(z)}{dz^3} + \left(-\frac{8 \cdot AG}{15 \cdot L} + \frac{N}{L} + \frac{m \cdot I_x \cdot \omega^2}{21 \cdot A \cdot L} \right) \cdot \frac{dw(z)}{dz} \right] \cdot \sin(\omega \cdot t) \\ & - \left[\frac{16 \cdot EI_x}{105 \cdot L^2} \cdot \frac{d^2 \phi(z)}{dz^2} + \left(\frac{8 \cdot AG}{15} + \frac{16 \cdot m \cdot I_x \cdot \omega^2}{105 \cdot A} \right) \cdot \phi(z) \right] \cdot \sin(\omega \cdot t) \end{aligned} \tag{23}$$

Similarly, the bending moment function $M(z, t)$ can be obtained by using Eqs. (20) and (21) as

$$M(z,t) = \left(-\frac{EI_x}{21 \cdot L^2} \cdot \frac{d^2 w(z)}{dz} - N \cdot w(z) + \frac{16 \cdot EI_x}{105 \cdot L} \cdot \frac{d\phi(z)}{dz} \right) \cdot \sin(\omega \cdot t) \tag{24}$$

The higher-order moment function $M_h(z, t)$ can be obtained as

$$M_h(z,t) = \left(\frac{16 \cdot EI_x}{105 \cdot L^2} \cdot \frac{d^2 w(z)}{dz^2} - \frac{68 \cdot EI_x}{105 \cdot L} \cdot \frac{d\phi(z)}{dz} \right) \cdot \sin(\omega \cdot t) \tag{25}$$

3. The differential transform method (DTM)

Partial differential equations are often used to describe engineering problems whose closed form solutions are very difficult to establish in many cases. Therefore, approximate numerical methods are often preferred. However, in spite of the advantages of these on hand methods and the computer codes that are based on them, closed form solutions are more attractive due to their implementation of the physics of the problem and their convenience for parametric studies. Moreover, closed form solutions have the capability and facility to solve inverse problem of determining and designing the geometry and characteristics of an engineering system and to achieve a prescribed behavior of the system. Considering the advantages of the closed form solutions mentioned above, DTM is introduced in this study as the solution method. DTM is a semi-analytic transformation technique based on Taylor series expansion and is a useful tool to obtain analytical solutions of the differential equations. Certain transformation rules are applied and the governing differential equations and the boundary conditions of the system are transformed into a set of algebraic equations in terms of the differential transforms of the original functions in DTM. The solution of these algebraic equations gives the desired solution of the problem. The different from high-order Taylor series method is; Taylor series method requires symbolic computation of the necessary derivatives of the data functions and is expensive for large orders. DTM is an iterative procedure to obtain analytic Taylor series solutions of differential equations (Ozgumus and Kaya 2007).

A function $w(z)$, which is analytic in a domain D, can be represented by a power series with a center at $z = z_0$, any point in D. The differential transform of the function $w(z)$ is given by

$$W(k) = \frac{1}{k!} \cdot \left(\frac{d^k w(k)}{dz^k}\right)_{z = z_0} \tag{26}$$

where $w(z)$ is the original function and $W(k)$ is the transformed function. The inverse transformation is defined as

$$w(z) = \sum_{k=0}^{\infty} (z-z_0)^k \cdot W(k) \tag{27}$$

From Eqs. (26) and (27) we get

$$w(z) = \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{k!} \cdot \left(\frac{d^k w(k)}{dz^k}\right)_{z = z_0} \tag{28}$$

Eq. (28) implies that the concept of the differential transformation is derived from Taylor's series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivative are calculated by iterative procedure that are described by the transformed equations of the original functions. In real applications, the function $w(z)$ in Eq. (27) is expressed by a finite series and can be written as

$$w(z) = \sum_{k=0}^{\bar{N}} (z-z_0)^k \cdot W(k) \tag{29}$$

Eq. (29) implies that $\sum_{k=\bar{N}+1}^{\infty} (z-z_0)^k W(k)$ is negligibly small. Where $\bar{N}$ is series size and the value of $\bar{N}$ depends on the convergence of the eigenvalues.

Table 1 DTM theorems used for equations of motion

Original function	Transformed function
$w(z) = u(z) \pm v(z)$	$W(k) = U(k) \pm V(k)$
$w(z) = a \cdot u(z)$	$W(k) = a \cdot U(k)$
$w(z) = \dfrac{d^m u(z)}{dz^m}$	$W(k) = \dfrac{(k+m)!}{k!} \cdot U(k+m)$
$w(z) = u(z) \cdot v(z)$	$W(k) = \sum_{r=0}^{k} U(r) \cdot V(k-r)$
$w(z) = z^m$	$W(k) = \delta(k-m) = \begin{cases} 0 \text{ if } k \neq m \\ 1 \text{ if } k = m \end{cases}$

Table 2 DTM theorems used for boundary conditions

$z = 0$		$z = 1$	
Original Boundary Conditions	Transformed Boundary Conditions	Original Boundary Conditions	Transformed Boundary Conditions
$w(0) = 0$	$W(0) = 0$	$w(1) = 0$	$\sum_{k=0}^{\infty} W(k) = 0$
$\dfrac{dw}{dz}(0) = 0$	$W(1) = 0$	$\dfrac{dw}{dz}(1) = 0$	$\sum_{k=0}^{\infty} k \cdot W(k) = 0$
$\dfrac{d^2 w}{dz^2}(0) = 0$	$W(2) = 0$	$\dfrac{d^2 w}{dz^2}(1) = 0$	$\sum_{k=0}^{\infty} k \cdot (k-1) \cdot W(k) = 0$
$\dfrac{d^3 w}{dz^3}(0) = 0$	$W(3) = 0$	$\dfrac{d^3 w}{dz^3}(1) = 0$	$\sum_{k=0}^{\infty} k \cdot (k-1) \cdot (k-2) \cdot W(k) = 0$

Theorems that are frequently used in differential transformation of the differential equations and the boundary conditions are introduced in (Table 1) and (Table 2), respectively.

3.1 Using differential transformation to solve motion equations

Eqs. (12) and (13) can be rewritten as follows

$$\frac{d^3 w(z)}{dz^3} = \left(\frac{17 \cdot L}{4}\right) \cdot \frac{d^2 \phi(z)}{dz^2} - \left(\frac{7}{2} \cdot \beta + \frac{\lambda^4 \cdot I_x}{A \cdot L^2}\right) \cdot \frac{dw(z)}{dz} + \left(\frac{17 \cdot \lambda^4 \cdot I_x}{4 \cdot A \cdot L} - \frac{7}{2} \cdot \beta \cdot L\right) \cdot \phi(z) \tag{30}$$

$$\begin{aligned} \frac{d^4 w(z)}{dz^4} &= \left(\frac{16 \cdot L}{5}\right) \cdot \frac{d^3 \phi(z)}{dz^3} + \left(\frac{56}{5} \cdot \beta - 21 \cdot N_r \cdot \pi^2 - \frac{\lambda^4 \cdot I_x}{A \cdot L^2}\right) \cdot \frac{d^2 w(z)}{dz^2} \\ &+ \left(\frac{56}{5} \cdot \beta \cdot L + \frac{16 \cdot \lambda^4 \cdot I_x}{5 \cdot A \cdot L}\right) \cdot \frac{d\phi(z)}{dz} + 21 \cdot (\lambda^4 - \alpha) \cdot w(z) \end{aligned} \tag{31}$$

where

$$\lambda = \sqrt[4]{\frac{m \cdot \omega^2 \cdot L^4}{EI_x}} \quad \text{(Frequency factor)} \tag{32.a}$$

$$\alpha = \frac{C_S \cdot L^4}{EI_x} \quad \text{(Relative stiffness)} \tag{32.b}$$

$$\beta = \frac{AG \cdot L^2}{EI_x} \quad \text{(Stiffness ratio)} \tag{32.c}$$

$$N_r = \frac{N \cdot L^2}{\pi^2 \cdot EI_x} \quad \text{(Nondimensionalized multiplication factor for the axial force)} \tag{32.d}$$

The differential transform method is applied to Eqs. (30) and (31) by using the theorems introduced in (Table 1) and the following expression are obtained

$$W(k+3) = \left(\frac{17 \cdot L}{4}\right) \cdot \frac{\Phi(k+2)}{(k+3)} - \left(\frac{7}{2} \cdot \beta + \frac{\lambda^4 \cdot I_x}{A \cdot L^2}\right) \cdot \frac{W(k+1)}{(k+2) \cdot (k+3)} + \left(\frac{17 \cdot \lambda^4 \cdot I_x}{4 \cdot A \cdot L} - \frac{7}{2} \cdot \beta \cdot L\right) \cdot \frac{\Phi(k)}{(k+1) \cdot (k+2) \cdot (k+3)} \tag{33}$$

$$W(k+4) = \left(\frac{16 \cdot L}{5}\right) \cdot \frac{\Phi(k+3)}{(k+4)} + \left(\frac{56}{5} \cdot \beta - 21 \cdot N_r \cdot \pi^2 - \frac{\lambda^4 \cdot I_x}{A \cdot L^2}\right) \cdot \frac{W(k+2)}{(k+3) \cdot (k+4)} + \left(\frac{56}{5} \cdot \beta \cdot L + \frac{16 \cdot \lambda^4 \cdot I_x}{5 \cdot A \cdot L}\right) \cdot \frac{\Phi(k+1)}{(k+2) \cdot (k+3) \cdot (k+4)} + 21 \cdot (\lambda^4 - \alpha) \cdot \frac{W(k)}{(k+1) \cdot (k+2) \cdot (k+3) \cdot (k+4)} \tag{34}$$

where $W(k)$ and $\Phi(k)$ are the transformed functions of $w(z)$ and $\phi(z)$, respectively.

The boundary conditions of Reddy-Bickford beam resting on elastic foundation and one end fixed, the other end simply supported shown in (Fig. 3) are given below (Wang *et al.* 2000)

$$w(z=0) = 0 \tag{35.a}$$

$$w'(z=0) = 0 \tag{35.b}$$

$$\phi(z=0) = 0 \tag{35.c}$$

$$w(z=1) = 0 \tag{35.d}$$

$$M(z=1) = 0 \tag{35.e}$$

$$M_h(z=1) = 0 \tag{35.f}$$

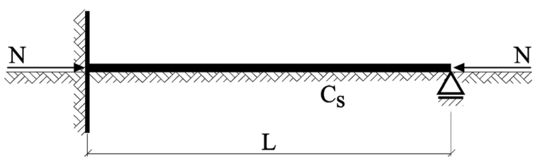

Fig. 3 One end fixed and the other end simply supported beam on elastic soil

Applying the differential transform method to Eqs. (35.a)-(35.f) and using the theorems introduced in (Table 2), the transformed boundary conditions are obtained as

for $z = 0$; $$W(0) = W(1) = \Phi(0) = 0 \tag{36.a}$$

for $z = 1$; $$\sum_{k=0}^{N} W(k) = \sum_{k=0}^{N} \overline{M}(k) = \sum_{k=0}^{N} \overline{M}_h(k) = 0 \tag{36.b}$$

where $\overline{M}(k)$ and $\overline{M}_h(k)$ are the transformed functions of $M(z)$ and $M_h(z)$, respectively.

Substituting the boundary conditions expressed in Eqs. (36.a) and (36.b) into Eqs. (33) and (34) and taking $W(2) = c_1$, $\Phi(1) = c_2$ and $\Phi(2) = c_3$, the following matrix expression is obtained

$$\begin{bmatrix} A_{11}^{(\overline{N})}(\omega) & A_{12}^{(\overline{N})}(\omega) & A_{13}^{(\overline{N})}(\omega) \\ A_{21}^{(\overline{N})}(\omega) & A_{22}^{(\overline{N})}(\omega) & A_{23}^{(\overline{N})}(\omega) \\ A_{31}^{(\overline{N})}(\omega) & A_{32}^{(\overline{N})}(\omega) & A_{33}^{(\overline{N})}(\omega) \end{bmatrix} \cdot \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{37}$$

where c_1, c_2 and c_3 are constants and $A_{j1}^{(\overline{N})}(\omega), A_{j2}^{(\overline{N})}(\omega)$ and $A_{j3}^{(\overline{N})}(\omega)$ ($j = 1, 2, 3$) are polynomials of ω corresponding $\overline{N}$.

In the last step, for non-trivial solution, equating the coefficient matrix that is given in Eq. (37) to zero one determines the natural frequencies of the vibrating system as is given in Eq. (38)

$$\begin{vmatrix} A_{11}^{(\overline{N})}(\omega) & A_{12}^{(\overline{N})}(\omega) & A_{13}^{(\overline{N})}(\omega) \\ A_{21}^{(\overline{N})}(\omega) & A_{22}^{(\overline{N})}(\omega) & A_{23}^{(\overline{N})}(\omega) \\ A_{31}^{(\overline{N})}(\omega) & A_{32}^{(\overline{N})}(\omega) & A_{33}^{(\overline{N})}(\omega) \end{vmatrix} = 0 \tag{38}$$

The j^{th} estimated eigenvalue, $\omega_j^{(\overline{N})}$ corresponds to $\overline{N}$ and the value of $\overline{N}$ is determined as

$$\left|\omega_j^{(\overline{N})} - \omega_j^{(\overline{N}-1)}\right| \le \varepsilon \tag{39}$$

where $\omega_j^{(\overline{N}-1)}$ is the j^{th} estimated eigenvalue corresponding to $(\overline{N}-1)$ and ε is the small tolerance parameter. If Eq. (39) is satisfied, the j^{th} estimated eigenvalue, $\omega_j^{(\overline{N})}$ is obtained.

The procedure that is explained below can be used to plot the mode shapes of Reddy-Bickford beam. The following equalities can be written by using Eq. (37)

$$A_{11}(\omega) \cdot c_1 + A_{12}(\omega) \cdot c_2 + A_{13}(\omega) \cdot c_3 = 0 \tag{40.a}$$

$$A_{21}(\omega) \cdot c_1 + A_{22}(\omega) \cdot c_2 + A_{23}(\omega) \cdot c_3 = 0 \tag{40.b}$$

Using Eqs. (40.a) and (40.b) the constants c_2 and c_3 can be obtained in terms of c_1 as follows

$$c_2 = -\frac{\begin{vmatrix} A_{11}(\omega) & A_{13}(\omega) \\ A_{22}(\omega) & A_{23}(\omega) \end{vmatrix}}{\begin{vmatrix} A_{12}(\omega) & A_{13}(\omega) \\ A_{22}(\omega) & A_{23}(\omega) \end{vmatrix}} \cdot c_1 \tag{41.a}$$

$$c_3 = -\frac{\begin{vmatrix} A_{12}(\omega) & A_{11}(\omega) \\ A_{22}(\omega) & A_{21}(\omega) \end{vmatrix}}{\begin{vmatrix} A_{12}(\omega) & A_{13}(\omega) \\ A_{22}(\omega) & A_{23}(\omega) \end{vmatrix}} \cdot c_1 \tag{41.b}$$

All transformed functions can be expressed in terms of ω, c_1, c_2 and c_3. Since c_2 and c_3 have been written in terms of c_1 above, $W(k), \Phi(k), \overline{M}(k)$ and $\overline{M}_h(k)$ can be expressed in terms c_1 as follows

$$W(k) = W(\omega, c_1) \tag{42.a}$$

$$\Phi(k) = \Phi(\omega, c_1) \tag{42.b}$$

$$\overline{M}(k) = \overline{M}(\omega, c_1) \tag{42.c}$$

$$\overline{M}_h(k) = \overline{M}_h(\omega, c_1) \tag{42.d}$$

The mode shapes can be plotted for several values of ω by using Eq. (42.a)

4. Numerical analysis and discussions

For numerical analysis, one end fixed and the other end simply supported Reddy-Bickford beam shown in (Fig. 3) is considered in the paper. Natural frequencies of the beam, ω_i (i = 1, 2, 3) are calculated by using computer program prepared in Matlab by the authors. Natural frequencies are found by determining values for which the determinant of the coefficient matrix is equal to zero. There are various methods for calculating the roots of the frequency equation. One common used and simple technique is the secant method in which a linear interpolation is employed. The eigenvalues, the natural frequencies, are determined by a trial and error method based on interpolation and the bisection approach. One such procedure consists of evaluating the determinant for a range of frequency values, ω_i. When there is a change of sign between successive evaluations, there must be a root lying in this interval. The iterative computations are determined when the value of the determinant changed sign due to a change of 10^{-4} in the value of ω_i.

The numerical results of this paper are obtained based on a uniform, rectangular Reddy-Bickford beam with the following data as:

m = 0.50968 kN·sec^2/m; EI_x = 1.900×10^4 kN.m^2 ; L = 3.0 m; β = 10, 11 and 12; N_r = 0.25 and 0.50; α = 1, 10, 100, 1000 and 100000.

The values of C_S are calculated due to relative stiffness values (α) and are presented in (Table 3).

Table 3 The values of C_S due to relative stiffness values (α)

$\alpha = \frac{C_S \cdot L^4}{EI_x}$	C_S (kN/m^2)
1	2.345679 × 10^2
10	2.345679 × 10^3
100	2.345679 × 10^4
1000	2.345679 × 10^5
10000	2.345679 × 10^6

Table 4 The first three natural frequencies of one end fixed and the other end simply supported Reddy-Bickford beam on elastic soil for $\beta = 10$ and $N_r = 0.25$

Method	$\overline{N}$	$\beta = 10$ and $N_r = 0.25$								
		$\alpha = 1$			$\alpha = 10$			$\alpha = 100$		
		ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)
DTM	64	128.8873	316.2505	525.0289	144.0624	322.7327	528.9589	249.3478	381.5452	566.7606
	66	128.8861	316.2635	524.6847	144.0613	322.7455	528.6172	249.3472	381.5560	566.4421
	68	128.8860	316.2652	524.6387	144.0612	322.7472	528.5715	249.3471	381.5573	566.3995
	70	128.8860	316.2654	524.6334	144.0612	322.7473	528.5661	249.3471	381.5574	566.3942
	72	128.8860	316.2654	524.6323	144.0612	322.7473	528.5650	249.3471	381.5575	566.3934
	74	128.8860	316.2654	524.6319	144.0612	322.7473	528.5649	249.3471	381.5575	566.3931
Analytic Method		128.8860	316.2654	524.6319	144.0612	322.7473	528.5649	249.3471	381.5575	566.3931

Method	$\overline{N}$	$\alpha = 1000$			$\alpha = 10000$		
		ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)
DTM	64	690.2011	748.1845	857.5669	2149.0490	2168.3664	2208.4958
	66	690.2010	748.1900	857.3563	2149.0487	2168.3680	2208.4140
	68	690.2009	748.1907	857.3282	2149.0486	2168.3682	2208.4031
	70	690.2009	748.1908	857.3245	2149.0486	2168.3685	2208.4018
	72	690.2009	748.1908	857.3242	2149.0486	2168.3685	2208.4017
	74	690.2009	748.1908	857.3241	2149.0486	2168.3685	2208.4015
Analytic Method		690.2009	748.1908	857.3241	2149.0486	2168.3685	2208.4015

Table 5 The first three natural frequencies of one end fixed and the other end simply supported Reddy-Bickford beam on elastic soil for $\beta = 10$ and $N_r = 0.50$

Method	$\overline{N}$	$\beta = 10$ and $N_r = 0.50$								
		$\alpha = 1$			$\alpha = 10$			$\alpha = 100$		
		ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)
DTM	64	66.7847	230.6031	412.7813	92.7483	239.4155	417.7684	223.6573	314.2294	464.7050
	66	66.7846	230.6039	412.7252	92.7482	239.4163	417.7130	223.6572	314.2300	464.6552
	68	66.7846	230.6040	412.7185	92.7482	239.4164	417.7063	223.6572	314.2301	464.6492
	70	66.7846	230.6040	412.7177	92.7482	239.4164	417.7056	223.6572	314.2301	464.6486
	72	66.7846	230.6040	412.7176	92.7482	239.4164	417.7055	223.6572	314.2301	464.6485
	74	66.7846	230.6040	412.7176	92.7482	239.4164	417.7055	223.6572	314.2301	464.6485
Analytic Method		66.7846	230.6040	412.7176	92.7482	239.4164	417.7055	223.6572	314.2301	464.6485

Method	$\overline{N}$	$\alpha = 1000$			$\alpha = 10000$		
		ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	$_1$ (rad/sec)	$_2$ (rad/sec)	$_3$ (rad/sec)
DTM	64	681.3413	716.2007	793.8224	2146.2197	2157.5393	2184.5335
	66	681.3412	716.2009	793.7933	2146.2196	2157.5394	2184.5229
	68	681.3412	716.2010	793.7898	2146.2196	2157.5394	2184.5220
	70	681.3412	716.2010	793.7894	2146.2196	2157.5394	2184.5216
	72	681.3412	716.2010	793.7893	2146.2196	2157.5394	2184.5215
	74	681.3412	716.2010	793.7893	2146.2196	2157.5394	2184.5215
Analytic Method		681.3412	716.2010	793.7893	2146.2196	2157.5394	2184.5215

Table 6 The first three natural frequencies of one end fixed and the other end simply supported Reddy-Bickford beam on elastic soil for $\beta = 11$ and $N_r = 0.25$

Method	$\overline{N}$	$\beta = 11$ and $N_r = 0.25$								
		$\alpha = 1$			$\alpha = 10$			$\alpha = 100$		
		ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)
DTM	64	136.0371	332.7289	568.8002	150.4931	338.8960	572.4312	253.1176	395.3109	607.5345
	66	135.9678	333.3298	553.8136	150.4302	339.4859	557.5408	253.0802	395.8169	593.5239
	68	135.9585	333.4140	552.0528	150.4220	339.5684	555.7917	253.0753	395.8877	591.8821
	70	135.9575	333.4248	551.8105	150.4209	339.5792	555.5512	253.0747	395.8969	591.6548
	72	135.9574	333.4256	551.7777	150.4208	339.5803	555.5169	253.0746	395.8977	591.6241
	74	135.9574	333.4258	551.7728	150.4208	339.5804	555.5129	253.0746	395.8978	591.6203
Analytic Method		135.9574	333.4258	551.7728	150.4208	339.5804	555.5129	253.0746	395.8978	591.6203

Method	$\overline{N}$	$\alpha = 1000$			$\alpha = 10000$		
		ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)
DTM	64	691.5720	755.2973	885.0422	2149.4893	2170.8309	2219.3087
	66	691.5583	755.5622	875.4856	2149.4849	2170.9230	2215.5151
	68	691.5570	755.5992	874.3726	2149.4847	2170.9359	2215.0757
	70	691.5565	755.6041	874.2198	2149.4844	2170.9360	2215.0155
	72	691.5563	755.6046	874.1983	2149.4843	2170.9378	2215.0072
	74	691.5563	755.6047	874.1954	2149.4843	2170.9379	2215.0059
Analytic Method		691.5563	755.6047	874.1954	2149.4843	2170.9379	2215.0059

Table 7 The first three natural frequencies of one end fixed and the other end simply supported Reddy-Bickford beam on elastic soil for $\beta = 11$ and $N_r = 0.50$

Method	$\overline{N}$	$\beta = 11$ and $N_r = 0.50$								
		$\alpha = 1$			$\alpha = 10$			$\alpha = 100$		
		ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)
DTM	64	79.8521	253.7198	453.0899	102.5592	261.7552	457.6380	227.9007	331.5661	500.8521
	66	79.8402	253.8541	447.7198	102.5500	261.8853	452.3218	227.8965	331.6689	495.9993
	68	79.8388	253.8710	447.0389	102.5489	261.9017	451.6479	227.8961	331.6818	495.3848
	70	79.8387	253.8729	446.9390	102.5487	261.9036	451.5498	227.8960	331.6834	495.3046
	72	79.8387	253.8732	446.9377	102.5487	261.9039	451.5490	227.8960	331.6835	495.2939
	74	79.8387	253.8732	446.9376	102.5487	261.9039	451.5476	227.8960	331.6835	495.2934
Analytic Method		79.8387	253.8732	446.9376	102.5487	261.9039	451.5476	227.8960	331.6835	495.2934

Method	$\overline{N}$	$\alpha = 1000$			$\alpha = 10000$		
		ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)
DTM	64	682.7459	723.9747	815.5097	2146.6661	2160.1324	2192.5074
	66	682.7449	724.0218	815.5384	2146.6657	2160.1482	2191.4039
	68	682.7445	724.0277	812.1634	2146.6655	2160.1500	2191.2649
	70	682.7444	724.0284	812.1144	2146.6654	2160.1503	2191.2468
	72	682.7444	724.0285	812.1084	2146.6654	2160.1504	2191.2446
	74	682.7444	724.0285	812.1076	2146.6654	2160.1504	2191.2443
Analytic Method		682.7444	724.0285	812.1076	2146.6654	2160.1504	2191.2443

Table 8 The first three natural frequencies of one end fixed and the other end simply supported Reddy-Bickford beam on elastic soil for $\beta = 12$ and $N_r = 0.25$

Method	$\overline{N}$	$\beta = 12$ and $N_r = 0.25$								
		$\alpha = 1$			$\alpha = 10$			$\alpha = 100$		
		ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)
DTM	64	145.3485	331.1567	---	158.9595	337.3542	---	258.2411	393.9884	---
	66	142.7904	346.0904	---	156.6240	352.0244	---	256.8098	406.6221	---
	68	142.4192	348.8300	586.3335	156.2855	354.7178	589.8537	256.6037	408.9566	623.9780
	70	142.3673	349.2433	578.4373	156.2385	355.1238	582.0098	256.5752	409.3071	616.5659
	72	142.3608	349.3006	577.4016	156.2325	355.1807	580.9763	256.5713	409.3561	615.5944
	74	142.3597	349.3064	577.2500	153.2318	355.1865	580.8255	256.5711	409.3625	615.4516
	76	142.3597	349.3064	577.2495	156.2318	355.1865	580.8253	256.5711	409.3625	615.4513
Analytic Method		142.3597	349.3064	577.2495	156.2318	355.1865	580.8253	256.5711	409.3625	615.4513

Method	$\overline{N}$	$\alpha = 1000$			$\alpha = 10000$		
		ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)
DTM	64	693.4636	754.6064	---	2150.0987	2170.5905	---
	66	692.9319	761.2780	---	2149.9273	2172.9190	---
	68	692.8556	762.5272	896.4090	2149.9027	2173.3570	2223.8651
	70	692.8449	762.7167	891.2674	2149.8993	2173.4233	2221.7974
	72	692.8436	762.7421	890.5940	2149.8988	2173.4327	2221.5270
	74	692.8434	762.7461	890.4950	2149.8987	2173.4335	2221.4903
	76	692.8434	762.7461	890.4947	2149.8987	2173.4335	2221.4902
Analytic Method		692.8434	762.7461	890.4947	2149.8987	2173.4335	2221.4902

Table 9 The first three natural frequencies of one end fixed and the other end simply supported Reddy-Bickford beam on elastic soil for $\beta = 12$ and $N_r = 0.50$

Method	$\overline{N}$	$\beta = 12$ and $N_r = 0.50$								
		$\alpha = 1$			$\alpha = 10$			$\alpha = 100$		
		ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)
DTM	64	91.7482	264.8089	---	112.0703	272.5175	---	232.3362	340.1266	---
	66	90.6419	272.9897	---	111.1665	280.4736	---	231.9015	346.5339	---
	68	90.4885	274.3650	485.0932	111.0415	281.8124	489.3439	231.8416	347.6184	529.9791
	70	90.4686	274.5593	479.0814	111.0252	282.0015	483.3850	231.8338	347.7717	524.4820
	72	90.4661	274.5844	478.2003	111.0232	282.0260	482.5117	231.8329	347.7916	523.6773
	74	90.4658	274.5875	478.1989	111.0230	282.0289	482.5103	231.8328	347.7940	523.6760
	76	90.4658	274.5879	478.1987	111.0230	282.0293	482.5101	231.8328	347.7943	523.6757
Analytic Method		90.4658	274.5879	478.1987	111.0230	282.0293	482.5101	231.8328	347.7943	523.6757

Method	$\overline{N}$	$\alpha = 1000$			$\alpha = 10000$		
		ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)
DTM	64	684.2393	727.9350	---	2147.1414	2161.4629	---
	66	684.0913	730.9508	---	2147.0944	2162.4804	---
	68	684.0715	731.4655	833.7152	2147.0879	2162.6545	2199.3439
	70	684.0689	731.5384	829.7893	2147.0871	2162.6792	2198.0258
	72	684.0686	731.5486	829.7308	2147.0870	2162.6823	2197.8339
	74	684.0685	731.5490	829.7236	2147.0869	2162.6827	2197.8336
	76	684.0685	731.5491	829.7226	2147.0869	2162.6828	2197.8335
Analytic Method		684.0685	731.5491	829.7226	2147.0869	2162.6828	2197.8335

The frequency values of one end fixed and the other end simply supported Reddy-Bickford beam on elastic soil obtained for the first three modes by using DTM are presented in (Tables 4-9) being compared with the frequency values obtained by using analytical method for the different values of stiffness ration (β), relative stiffness (α) and nondimensionalized multiplication factor for the axial compressive force (N_r).

For the different values of β and N_r, the variations of frequency factors (λ) due to relative stiffness for the first three modes are presented in (Figs. 4-6) and for β = 12, N_r = 0.50 and α = 100, the mode shapes of one end fixed and the other end simply supported Reddy-Bickford beam on elastic soil are presented in (Fig. 7).

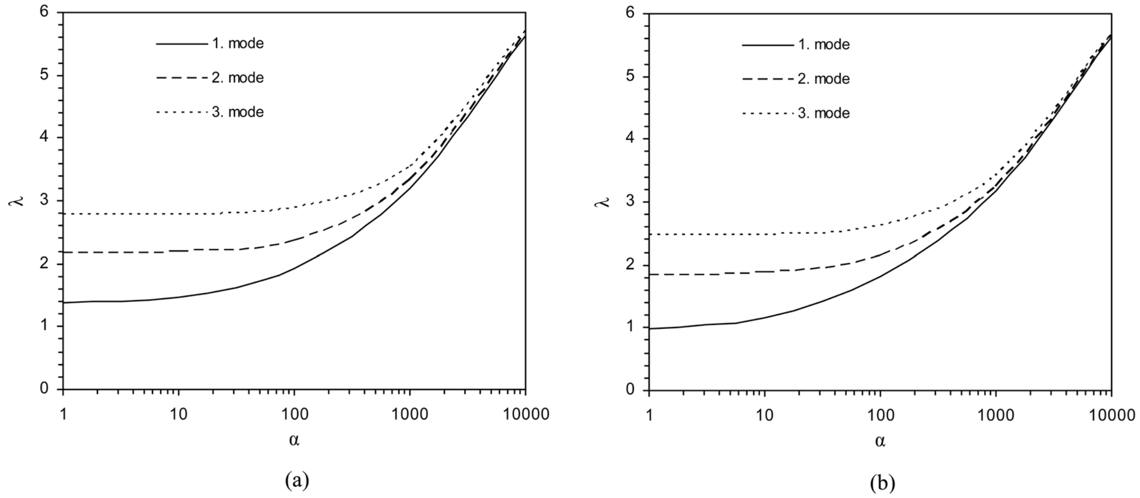

Fig. 4 Variation of frequency factors due to relative stiffness for the first three modes. (a) For β = 10 and N_r = 0.25, (b) For β = 10 and N_r = 0.50

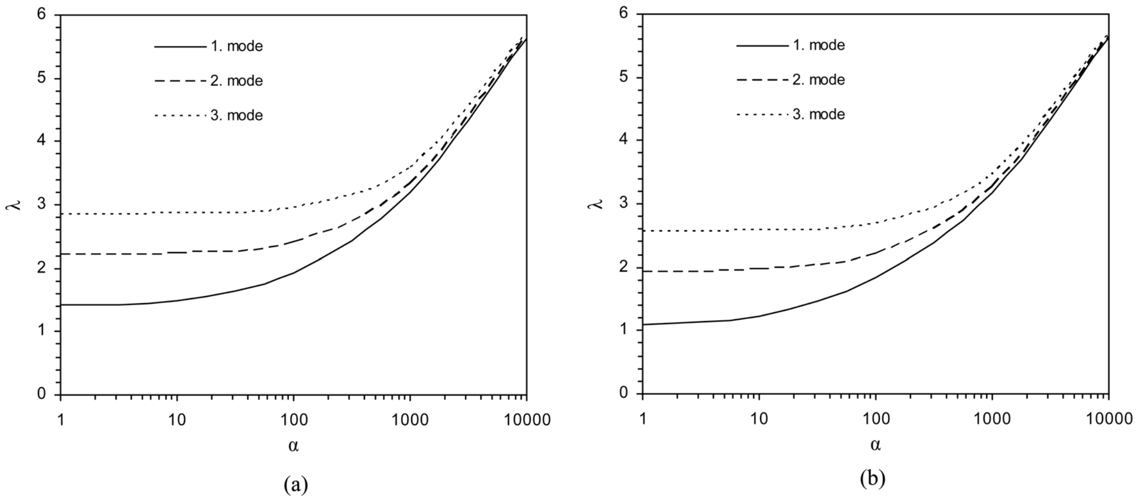

Fig. 5 Variation of frequency factors due to relative stiffness for the first three modes. (a) For β = 11 and N_r = 0.25, (b) For β = 11 and N_r = 0.50

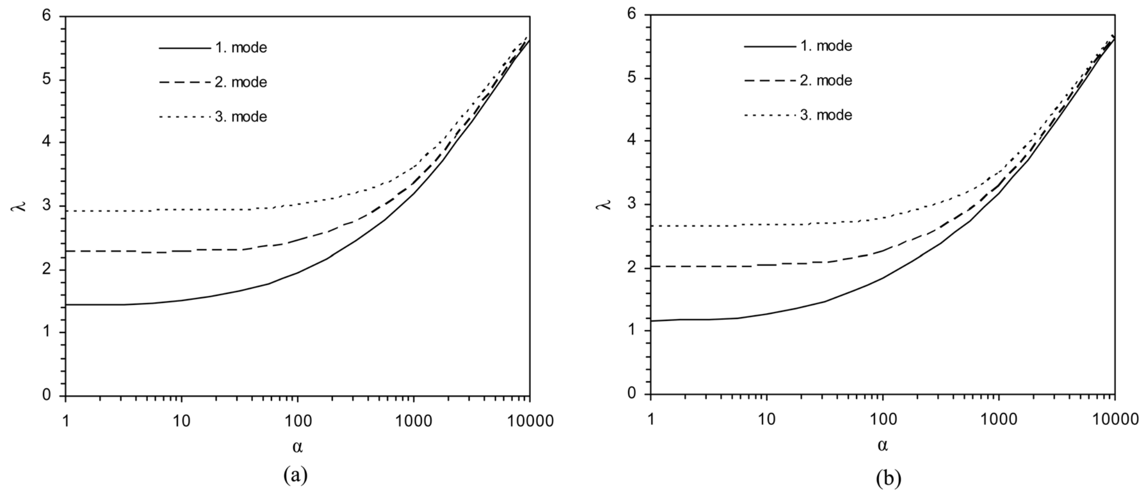

Fig. 6 Variation of frequency factors due to relative stiffness for the first three modes. (a) For $\beta = 12$ and $N_r = 0.25$, (b) For $\beta = 12$ and $N_r = 0.50$

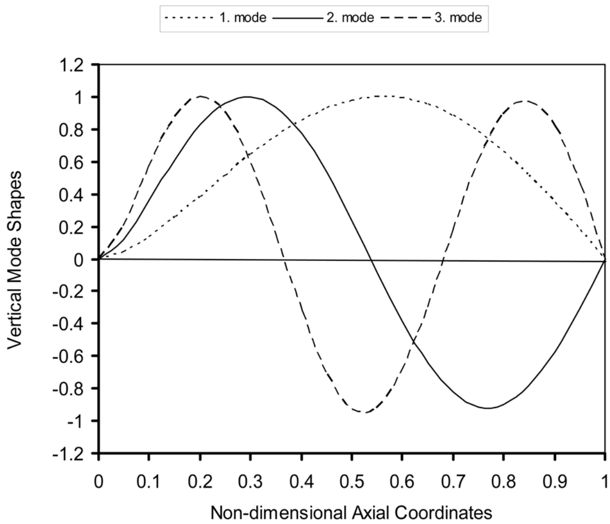

Fig. 7 The first three mode shapes of one end fixed and the other and simply supported Reddy-Bickford beam on elastic soil, for $\beta = 12$, $N_r = 0.50$ and $\alpha = 100$

As the axial compressive force acting to the beam is increased for the other variables (β and α) are constant, the natural frequency values of one end fixed and the other simply supported Reddy-Bickford beam resting on elastic soil are decreased. This result indicates that, the increasing for the axial compressive force leads to reduction for Reddy-Bickford beam theory. This result is very important for the effect of axial force.

An increase is observed in natural frequency values of the first three modes of Reddy-Bickford beam for the conditions of β and N_r ratio being constant and the values of the relative stiffness are increased. This result indicates that, the increasing for the relative stiffness leads to augmentation in natural frequency values for Reddy-Bickford beam theory.

For the other variables (N_r and α) are constant, as the stiffness ratio is increased, an increase is observed in natural frequency values of the first three modes of Reddy-Bickford beam. The increasing for the stiffness ratio leads to augmentation in natural frequency values for Reddy-Bickford beam theory.

In application of DTM, the natural frequency values of one end fixed and the other end simply supported Reddy-Bickford beam are calculated by increasing series size $\overline{N}$. In (Tables 4-9), convergences of the first three natural frequencies are introduced. Here, it is seen that, when the series size is taken 76, the natural frequency values of the third mode can be appeared. Additionally, here it is seen that higher modes appear when more terms are taken into account in DTM applications. Thus, depending on the order of the required mode, one must try a few values for the term number at the beginning of the calculations in order to find the adequate number of terms.

5. Conclusions

In this study, starting from the governing differential equations of motion in free vibration, analytical solution and DTM algorithm are developed by using Reddy-Bickford beam theory and the iterative-based computer programs are developed for solution of linear-homogeneous frequency equation set relating to free vibration of one end fixed and the other end simply supported beam resting on elastic soil. Variation in free vibration natural frequencies for the first three modes of the beam is investigated for the different values of the relative stiffness, stiffness ratio and nondimensionalized multiplication factor for the axial compressive force. The calculated natural frequencies of Reddy-Bickford beam on elastic soil by using DTM are compared with the results of the analytical solution. The essential steps of the DTM application includes transforming the governing equations of motion into algebraic equations, solving the transformed equations and then applying a process of inverse transformation to obtain any desired natural frequency. All the steps of the DTM are very straightforward and the application of the DTM to both the equations of motion and the boundary conditions seem to be very involved computationally. However, all the algebraic calculations are finished quickly using symbolic computational software. Besides all these, the analysis of the convergence of the results show that DTM solutions converge fast. When the results of the DTM are compared with the results of analytical method, very good agreement is observed.

References

Ayaz, F. (2004), “Application of differential transforms method to differential-algebraic equations”, *Appl. Math. Comput.*, **152**, 648-657.

Bert, C.W. and Zeng, H. (2004), “Analysis of axial vibration of compound bars by differential transformation method”, *J. Sound Vib.*, **275**, 641-647.

Bickford, W.B. (1982), “A consistent higher order beam theory”, *Development Theor. Appl. Mech.*, **11**, 137-150.

Bildik, N., Konuralp, A., Bek, F.O. and Küçükarslan, S. (2006), “Solution of different type of the partial differential equation by differential transform method and Adomian’s decomposition method”, *Appl. Math. Comput.*, **172**, 551-567.

Çatal, S. (2006), “Analysis of free vibration of beam on elastic soil using differential transform method”, *Struct. Eng. Mech.*, **24**(1), 51-62.

Çatal, S. (2008), "Solution of free vibration equations of beam on elastic soil by using differential transform method", *Appl. Math. Model.*, **32**, 1744-1757.

Çatal, S. and Çatal, H.H. (2006), "Buckling analysis of partially embedded pile in elastic soil using differential transform method", *Struct. Eng. Mech.*, **24**(2), 247-268.

Chen, C.K. and Ho, S.H. (1996), "Application of differential transformation to eigenvalue problem", *J. Appl. Math. Comput.*, **79**, 173-188.

Chen, C.K. and Ho, S.H. (1999), "Transverse vibration of a rotating twisted Timoshenko beams under axial loading using differential transform", *Int. J. Mech. Sci.*, **41**, 1339-1356.

Chen, C.L. and Liu, Y.C. (1998), "Solution of two-point boundary-value problems using the differential transformation method", *J. Optimiz. Theory App.*, **99**, 23-35.

Cowper, G.R. (1966), "The shear coefficient in Timoshenko's beam theory", *J. Appl. Mech.*, **33**(2), 335-340.

Doyle, P.F. and Pavlovic, M.N. (1982), "Vibration of beams on partial elastic foundations", *J. Earthq. Eng. Struct. Dyn.*, **10**, 663-674.

Ertürk, V.S. (2007), "Application of differential transformation method to linear sixth-order boundary value problems", *Appl. Math. Sci.*, **1**, 51-58.

Ertürk, V.S. and Momani, S. (2007), "Comparing numerical methods for solving fourth-order boundary value problems", *Appl. Math. Comput.*, **188**, 1963-1968.

Esmailzadeh, E. and Ohadi, A.R. (2000), "Vibration and stability analysis of non-uniform Timoshenko beams under axial and distributed tangential loads", *J. Sound Vib.*, **236**, 443-456.

Gruttmann, F. and Wagner, W. (2001), "Shear coefficient factors in Timoshenko's beam theory for arbitrary shaped cross-section", *Comput. Mech.,* **27**, 199-207.

Han, S.M., Benaroya, H. and Wei, T. (1999), "Dynamics of transversely vibrating beams using four engineering theories", *J. Sound Vib.*, **225**, 935-988.

Hassan, I.H.A.H (2002*a*), "On solving some eigenvalue problems by using differential transformation", *Appl. Math. Comput.*, **127**, 1-22.

Hassan, I.H.A.H (2002*b*), "Different applications for the differential transformation in the differential equations", *Appl. Math. Comput.*, **129**, 183-201.

Hetenyi, M. (1955), *Beams on Elastic Foundations*, 7th edn., The University of Michigan Press, Michigan.

Heyliger, P.R. and Reddy, J.N. (1988), "A higher-order beam finite element for bending and vibration problems", *J. Sound Vib.*, **126**(2), 309-326.

Ho, S.H. and Chen, C.K. (2006), "Free transverse vibration of an axially loaded non-uniform sinning twisted Timoshenko beam using differential transform", *Int. J. Mech. Sci.*, **48**, 1323-1331.

Jang, M.J. and Chen, C.L. (1997), "Analysis of the response of a strongly non-linear damped system using a differential transformation technique", *Appl. Math. Comput.*, **88**, 137-151.

Jang, M.J., Chen, C.L. and Liu, Y.C. (2000), "On solving the initial-value problems using differential transformation method", *Appl. Math. Comput.*, **115**, 145-160.

Kaya, M.O. and Ozgumus, O.O. (2007), "Flexural-torsional-coupled vibration analysis of axially loaded closed-section composite Timoshenko beam by using DTM", *J. Sound Vib.*, **306**, 495-506.

Kurnaz, A., Oturanç, G. and Kiris, M.E. (2005), "n-Dimensional differential transformation method for solving PDEs", *Int. J. Comput. Math.*, **82**(3), 369-380.

Levinson, M. (1981), "A new rectangular beam theory", *J. Sound Vib*., **74**, 81-87.

Malik, M. and Dang, H.H. (1998), "Vibration analysis of continuous systems by differential transformation", *Appl. Math. Comput.*, **96**, 17-26.

Murthy, A.V. (1970), "Vibration of short beams", *AIAA*, **8**, 34-38.

Özdemir, Ö. and Kaya, M.O. (2006), "Flapwise bending vibration analysis of a rotating tapered cantilever Bernoulli-Euler beam by differential transform method", *J. Sound Vib.*, **289**, 413-420.

Ozgumus, O.O. and Kaya, M.O. (2006), "Flapwise bending vibration analysis of double tapered rotating Euler-Bernoulli beam by using the differential transform method", *Meccanica*, **41**, 661-670.

Ozgumus, O.O. and Kaya, M.O. (2007), "Energy expressions and free vibration analysis of a rotating double tapered Timoshenko beam featuring bending-torsion coupling", *Int. J. Eng. Sci.*, **45**, 562-586.

Rajasekaran, S. (2008), "Buckling of fully and partially embedded non-prismatic columns using differential quadrature and differential transformation methods", *Struct. Eng. Mech.*, **28**(2), 221-238.

Reddy, J.N. (2002), *Energy Principles and Variational Methods in Applied Mechanics*, Second Edition, John Wiley, NY.

Reddy, J.N. (2007), *Theory and Analysis of Elastic Plates and Shells*, Second Edition, Taylor & Francis; Philadelphia, PA.

Timoshenko, S.P. (1921), "On the correction for shear of the differential equation for transverse vibrations of prismatic bars", *Philos. Mag.*, **41**, 744-746.

Wang, C.M., Reddy, J.N. and Lee, K.H. (2000), *Shear Deformable Beams and Plates: Relationships with Classical Solutions*, Elsevier Science Ltd., The Netherlands.

West, H.H. and Mafi, M. (1984), "Eigenvalues for beam columns on elastic supports", *J. Struct. Eng.*, **110**(6), 1305-1320.

Yesilce, Y. and Catal, H.H. (2008), "Free vibration of semi-rigid connected Reddy-Bickford piles embedded in elastic soil", *Sadhana-Academy Proceedings in Engineering Science*, **33**(6), 781-801.

Yokoyama, T. (1991), "Vibrations of Timoshenko beam-columns on two parameter elastic foundations", *Earthq. Eng. Struct. Dyn.*, **20**, 355-370.

Zhou, J.K. (1986), *Differential Transformation and Its Applications for Electrical Circuits*, Huazhong University Press, Wuhan China.