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Hybrid displacement FE formulations including a hole

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Abstract. The paper deals with the problem related to the modelling of riveted assemblies for crashworthiness analysis of full-scale complete aircraft structures. Comparisons between experiments and standard FE computations on high-energy accidental situations onto aluminium riveted panels show that macroscopic plastic strains are not sufficiently localised in the FE shells connected to rivet elements. The main reason is related to the structural embrittlement caused by holes, which are currently not modelled. Consequently, standard displacement FE models do not succeed in initialising and propagating the rupture in sheet metal plates and along rivet rows as observed in the experiments. However, the literature survey show that it is possible to formulate super-elements featuring defects that both give accurate singular strain fields and are compatible with standard displacement finite elements. These super-elements can be related to the displacement model of the hybrid-Trefftz principle of the finite element method, which is a kind of domain decomposition method. A feature of hybrid-Trefftz finite elements is that they are mainly used for elastic computations. It is thus proposed to investigate the possibility of formulating a hybrid displacement finite element, including the effects of a hole, dedicated to crashworthiness analysis of full-scale aeronautic structures.

Keywords: hybrid-displacement finite element; complex variable; structural mechanics; riveted assemblies; crashworthiness.

1. Introduction

Finite element simulations of airframe High Velocity Impacts hardly succeed in representing the failure of the structure when it occurs in riveted joints areas. Computational and experimental results were compared for bird impacts onto aluminium riveted panels (Langrand *et al.* 2002). The analysis shows that the macroscopic plastic strains are not sufficiently localised within the shell finite elements (that do not model holes), to which beam type spring elements are connected, so as

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to initiate and propagate failure along rivet lines. The structural embrittlement, caused by holes, is not taken into account in the standard shell finite element formulation that is used for structural computations. Indeed, modelling the geometrical defects (holes) with a really fine mesh remains not suitable for full-scale aircraft crashworthiness, as an aircraft can feature more than hundreds of thousands of riveted assemblies.

However, the literature survey shows that there are some finite element procedures allowing to build super-elements containing defects, so that the fine meshing of the considered defect is not required anymore (Piltner 1985, Dhanasekar *et al.* 2006). All these finite elements pertain to the category of hybrid-Trefftz finite elements. However, the major drawback of such elements was shown to be their restriction to linear computations (Leconte *et al.* 2008a). Therefore, there is a need for these formulations to be extended to crashworthiness analysis, i.e. high plastic strains and strain rates. However, as reminded by Felippa, the extension of the so-called mixed and hybrid principles is still under the scope of nowadays research (Felippa 2006). Nevertheless, several authors have investigated the extension of such principles. In particular, Grimaldi et al. have given some clues on how to take into account material non-linearity (Grimaldi *et al.* 2004), Liu *et al.* focused on geometric non-linearity (Liu and To 1995) and Darilmaz *et al.* formulated a mass matrix (Darilmaz and Kumbasar 2006).

The article proposes to focus on hybrid displacement variational principles that allow the building of super-elements that are compatible with standard displacement FE, with emphasis on an existing hybrid-Trefftz displacement element featuring a hole (Piltner 1985). Firstly, the building of the element interpolation functions is reminded. Following the framework of Freitas (1999), it is then demonstrated that Piltner's hybrid-Trefftz element results from the constraining of a hybrid-mixed principle to satisfy pointwise inner domain equations. Also, the extension of the perforated element to non-linear problems is discussed. Finally, the mass matrix of the perforated element is formulated as a first step to its extension.

2. Interpolation functions formulation

The first step to formulate a perforated element is to build interpolation functions that take into account the hole's presence. Complex variables, in particular Kolosov-Muskhelishvili formalism (Muskhelishvili 1953), are generally used to this aim. Indeed, Kolosov-Muskhelischvili system of equations allows solving plane elasticity problems in a systematic way. This formalism is particularly suitable to solve the problem of an infinite elastic plate featuring a circular hole. To obtain the expressions of displacement and stress fields, one has to start with the following statement

$$\begin{cases} 2\mu(u+iv) = k\Phi(z) - z\overline{d\Phi(z)} - \overline{\Psi(z)} \\ \sigma_{xx} + i\tau_{xy} = d\Phi(z) + \overline{d\Phi(z)} - z\overline{d^2\Phi(z)} - \overline{d\Psi(z)} \\ \sigma_{yy} - i\tau_{xy} = d\Phi(z) + \overline{d\Phi(z)} + z\overline{d^2\Phi(z)} + \overline{d\Psi(z)} \end{cases} \tag{1}$$

$$\text{and } k\Phi(z) - z\overline{d\Phi(z)} - \overline{\Psi(z)} = 2\mu(\breve{u} + i\breve{v}) \text{ on } \Gamma_u$$

$$\text{and } \Phi(z) + z\overline{d\Phi(z)} + \overline{\Psi(z)} = i\int(\breve{T}_x + i\breve{T}_y)dS \text{ on } \Gamma_t$$

where $d(\,)$ means complex differentiation, the over bar represents complex conjugate, i is the imaginary number, z is a complex variable, k is Muskhelishvili's constant ($k = 3-4\nu$ for plane strain, $k = (3-\nu)/(1+\nu)$ for plane stress), μ is Lamé's coefficient, $\Psi(z)$ and $\Phi(z)$ are Kolosov-Muskhelishvili potentials, u and v are the components of the displacement vector, $\sigma_{xx}, \sigma_{yy}, \tau_{xy}$ are the components of the stress tensor, $\breve{u}, \breve{v}, \breve{T}_x, \breve{T}_y$ are prescribed values of displacement and traction vector, respectively.

In order to describe easily the free boundary condition along the hole, it is convenient to map the circular hole onto a unit circle using conformal mapping

$$z = f(\zeta) = r_0\zeta \tag{2}$$

where f is a mapping function, ζ a complex variable and r_0 the hole radius.

Then, Eq. (1) becomes

$$\begin{cases} 2\mu(u+iv) = k\Phi - f\overline{\dfrac{d\Phi}{df}} - \overline{\Psi} \\[2ex] \sigma_{xx} - i\tau_{xy} = \overline{\dfrac{d\Phi}{df}} + \dfrac{d\Phi}{df} - \bar{f}\left(\dfrac{d^2\Phi}{df^2} - d\Phi\dfrac{d^2f}{df^3}\right) - \dfrac{d\Psi}{df} \\[2ex] \sigma_{yy} + i\tau_{xy} = \overline{\dfrac{d\Phi}{df}} + \dfrac{d\Phi}{df} + \bar{f}\left(\dfrac{d^2\Phi}{df^2} - d\Phi\dfrac{d^2f}{df^3}\right) + \dfrac{d\Psi}{df} \end{cases} \tag{3}$$

$$\begin{aligned} &\text{and } k\overline{\Phi} - \bar{f}\frac{d\Phi}{df} - \Psi = 2\mu\overline{(\breve{u}+i\breve{v})} \text{ on } \Gamma'_u \\ &\text{and } \overline{\Phi} + \bar{f}\frac{d\Phi}{df} + \Psi = \overline{i\int(\breve{T}_x + i\breve{T}_y)dS} \text{ on } \Gamma'_t \end{aligned} \tag{4}$$

Then, the Kolosov-Muskhelishvili potential Φ is assumed as a Laurent series

$$\Phi(\zeta) = \sum_{j=-N}^{M} a_j\zeta^j \tag{5}$$

The second term of Eq. (4) is determined within the whole domain, thanks to Eq. (5)

$$\bar{f}\frac{d\Phi}{df} = \bar{\zeta}\sum_{j=-N}^{M} ja_j\zeta^{j-1} \tag{6}$$

Finally, the Kolosov-Muskhelishvili potential Ψ featuring the stress-free boundary condition along the hole, which is the third term of Eq. (4), is determined. This is done by describing the boundary of the unit circular hole with the following equation

$$\overline{\zeta^{\alpha}} = \zeta^{-\alpha} \quad \text{on} \quad |\zeta| = 1 \tag{7}$$

and by expressing homogenous static boundary conditions $(\breve{T}_x = \breve{T}_y = 0)$ along the boundary of the hole (Eq. (7)). Making use of these conditions into Eq. (4) provides Ψ potential such as

$$\Psi(\zeta) = -\sum_{j=-N}^{M}[\bar{a}_j\zeta^{-j} + a_j j\zeta^{j-2}] \tag{8}$$

Now that the expressions of $\Psi(\zeta)$ and $\Phi(\zeta)$ are known, expressions of u, v, σ_{xx}, σ_{yy} and τ_{xy} can be identified by introducing $\zeta^j = R^j\cos j\theta + iR^j\sin j\theta$ and $a_j = \alpha_j + i\beta_j$ (with $R = \sqrt{x^2+y^2}/r_0$ and $\theta = \arctan(y/x)$). Indeed, the expressions of displacement and stress components can be obtained by identifying the real and imaginary parts of Eq. (3).

Finally, the obtained expressions of displacement and stress components are

$$\begin{aligned}2\mu u^h = \sum_{j=-N}^{M}\, & \alpha_j[(kR^j + R^{-j})\cos j\theta - j(R^j - R^{j-2})\cos(j-2)\theta] + \\ & \beta_j[-(kR^j + R^{-j})\sin j\theta + j(R^j - R^{j-2})\sin(j-2)\theta]\end{aligned} \tag{9}$$

$$\begin{aligned}2\mu v^h = \sum_{j=-N}^{M}\, & \alpha_j[(kR^j + R^{-j})\sin j\theta + j(R^j - R^{j-2})\sin(j-2)\theta] + \\ & \beta_j[(kR^j + R^{-j})\cos j\theta + j(R^j - R^{j-2})\cos(j-2)\theta]\end{aligned} \tag{10}$$

$$\begin{aligned}\sigma_{xx} = \frac{1}{r_0}\sum_{j=-N}^{M}\, & \alpha_j j[2R^{j-1}\cos(j-1)\theta - R^{-j-1}\cos(j+1)\theta - [(j-1)R^{j-1} - (j-2)R^{j-3}]\cos(j-3)\theta] \\ & + \beta_j j[-2R^{j-1}\sin(j-1)\theta + R^{-j-1}\sin(j+1)\theta + [(j-1)R^{j-1} - (j-2)R^{j-3}]\sin(j-3)\theta]\end{aligned} \tag{11}$$

$$\begin{aligned}\sigma_{yy} = \frac{1}{r_0}\sum_{j=-N}^{M}\, & \alpha_j j[2R^{j-1}\cos(j-1)\theta + R^{-j-1}\cos(j+1)\theta + [(j-1)R^{j-1} - (j-2)R^{j-3}]\cos(j-3)\theta] \\ & + \beta_j j[-2R^{j-1}\sin(j-1)\theta - R^{-j-1}\sin(j+1)\theta - [(j-1)R^{j-1} - (j-2)R^{j-3}]\sin(j-3)\theta]\end{aligned} \tag{12}$$

$$\begin{aligned}\tau_{xy} = \frac{1}{r_0}\sum_{j=-N}^{M}\, & \alpha_j j[-R^{-j-1}\sin(j+1)\theta + [(j-1)R^{j-1} - (j-2)R^{j-3}]\sin(j-3)\theta] \\ & + \beta_j j[-R^{-j-1}\cos(j+1)\theta + [(j-1)R^{j-1} - (j-2)R^{j-3}]\cos(j-3)\theta]\end{aligned} \tag{13}$$

It can be deduced from these expressions that

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}\begin{Bmatrix} \alpha_j \\ \beta_j \end{Bmatrix} = \frac{1}{2\mu}N_{ij}c_j \tag{14}$$

$$\mathbf{T} = \begin{bmatrix} \sigma_{xx}n_x + \tau_{xy}n_y \\ \sigma_{yy}n_x + \tau_{xy}n_x \end{bmatrix} = \begin{bmatrix} \mathbf{T}_x \\ \mathbf{T}_y \end{bmatrix}\begin{Bmatrix} \alpha_j \\ \beta_j \end{Bmatrix} = \frac{1}{r_0}P_{ij}c_j \tag{15}$$

where $\mathbf{T}$ is the traction vector, $\mathbf{c}$ is the vector of parameters (α, β), N and P are matrices of special shape functions.

3. A set of hybrid displacement variational principles

The second step of the formulation consists in building a variationnal principle to be coupled with the previously obtained interpolation functions. Specific variational principles are required when one needs to build a super-element that is both able to feature the chosen interpolation functions and to ensure displacement continuity with standard displacement based finite elements. It appears that hybrid displacement variational principles are recognized as a domain decomposition method, and allow the compatibility of the built hybrid displacement super-element with standard displacement elements. Our interest is thus focused on these principles. Three kinds of hybrid displacement principles can be enumerated: the hybrid-mixed displacement, the hybrid displacement and the hybrid-Trefftz displacement principles. It is reminded that following the most common nomenclature, a hybrid element is an element where one field is interpolated in the interior domain and at least one field is interpolated on the interface (Felippa 2006), a hybrid-mixed element is an element where at least two fields are interpolated in the interior domain and at least one field is interpolated on the interface (Zienkiewicz *et al.* 2005, Freitas 1999), and a hybrid-Trefftz formulation is a hybrid principle constrained (Freitas 1999) so that interior domain equations are a priori satisfied (for further information on Trefftz method, see: Trefftz 1926, Kita and Kamiya 1995, Zienkiewicz *et al.* 1977). The link between these three principles is demonstrated hereafter following Freitas (1999), to help discuss on the extension of the formulation of an existing perforated hybrid-Trefftz displacement element (Piltner 1985) to non-linear problems.

In order to build the three different kinds of hybrid displacement variational principles, the strong form equations of elasto-statics are considered:

Kinematic equation (in V): $$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{16}$$

Constitutive equation (in V): $$\sigma_{ij} = E_{ijkl}\varepsilon_{kl} \tag{17}$$

Equilibrium equation (in V): $$\sigma_{ij,j} + b_i = 0 \tag{18}$$

Flux boundary condition (on S_t): $$\sigma_{ij}n_j = \hat{T}_i \tag{19}$$

Primary boundary condition (on S_u): $$u_i = \hat{u}_i \tag{20}$$

Where S_u and S_t are the complementary boundaries on which the displacement and traction boundary conditions apply, respectively, and V is an interior domain.

Eqs. (16), (17) can be recast so that:

$$\varepsilon_{ij}^{u} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{21}$$

$$\varepsilon_{ij}^{\sigma} = E_{ijkl}^{-1}\sigma_{kl} \tag{22}$$

where the superscript u or σ indicates the field to which the considered strain field is connected through strong form equations.

Thanks to the recast Eqs. (21), (22), the weak statement of strain compatibility can be built

$$\int_V (\varepsilon_{ij}^{u} - \varepsilon_{ij}^{\sigma})\delta\sigma_{ij}dV = 0 \tag{23}$$

Moreover, the weak statements of Eqs. (18), (19) can be built respectively as

$$\int_V (\sigma_{ij,j} + b_i)\delta u_i dV = 0 \tag{24}$$

$$\int_{S_t} (\sigma_{ij} n_j - \hat{T}_i)\, \delta u_i dS = 0 \tag{25}$$

Following Felippa (2006), it is possible to modify Eq. (24) by applying the divergence theorem to its first term

$$\int_V \sigma_{ij,j} \delta u_i dV = -\int_V \sigma_{ij,j} \delta u_{i,j} dV + \int_S \sigma_{ij} n_j \delta u_i dS \tag{26}$$

For a symmetric tensor σ_{ij}, Eq. (26) can be transformed to

$$\int_V \sigma_{ij,j} \delta u_i dV = -\int_V \sigma_{ij} \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) dV + \int_S \sigma_{ij} n_j \delta u_i dS \tag{27}$$

With the help of Eq. (16), it can be deduced that

$$\int_V \sigma_{ij,j} \delta u_i dV = -\int_V \sigma_{ij} \varepsilon_{ij}^u dV + \int_S \sigma_{ij} n_j \delta u_i dS \tag{28}$$

Replacing Eq. (28) into the opposite of Eq. (24) and using Eq. (20) leads to

$$\int_V \sigma_{ij} \delta \varepsilon_{ij}^u dV - \int_{S_t} \sigma_{ij} n_j \delta u_i dS - \int_V b_i \delta u_i dV = 0 \tag{29}$$

When summing Eqs. (25), (29) and Eq. (23), the first variation $\delta\Pi$ is obtained

$$\delta\Pi = \int_V [\sigma_{ij} \delta \varepsilon_{ij}^u + (\varepsilon_{ij}^u - \varepsilon_{ij}^\sigma) \delta \sigma_{ij} - b_i \delta u_i] dV - \int_{S_t} \hat{T}_i \delta u_i dS = 0 \tag{30}$$

Eq. (30) can be recognized as the exact variation with respect to u_i and σ_{ij} of

$$\Pi_{HR}[u_i, \sigma_{ij}] = \int_V \sigma_{ij} \varepsilon_{ij}^u dV - \frac{1}{2} \int_V \sigma_{ij} E_{ijkl}^{-1} \sigma_{kl} dV - \int_V b_i u_i dV - \int_{S_t} \hat{T}_i u_i dS \tag{31}$$

which is recognized as the (mixed) Hellinger-Reissner principle.

Following Tong (1970), a hybrid-mixed displacement principle (Π_{HM}) can be obtained by modifying the Hellinger-Reissner principle (Π_{HR}) so that the inter-element displacement continuity requirement is relaxed along the inter-element boundary S_i, thanks to the use of Lagrangian interface constraints

$$\Pi_{HM}[u_i, \sigma_{ij}, T_i, \tilde{u}_i] = \Pi_{HR}[u_i, \sigma_{ij}] + \int_{Su+Si} (\tilde{u}_i - u_i) T_i dS \tag{32}$$

$$\Pi_{HM}[u_i, \sigma_{ij}, T_i, \tilde{u}_i] = \int_V \sigma_{ij} \varepsilon_{ij}^u dV - \frac{1}{2} \int_V \sigma_{ij} E_{ijkl}^{-1} \sigma_{kl} dV - \int_V b_i u_i dV - \int_{St} \hat{T}_i u_i dS + \int_{Su+Si} (\tilde{u}_i - u_i) T_i dS \tag{33}$$

where $\tilde{u}_i = \tilde{N}_{ij} q_j$ is a vector of independently assumed boundary displacements, where q is a vector of nodal degrees of freedom. For example, $\tilde{\mathbf{u}}$ can be written for linear neighbouring elements as $\tilde{\mathbf{u}} = \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} 1 - \frac{s}{1} & \frac{s}{1} & 0 & 0 \\ 0 & 0 & 1 - \frac{s}{1} & \frac{s}{1} \end{bmatrix} \begin{Bmatrix} u_p \\ u_{p+1} \\ v_p \\ v_{p+1} \end{Bmatrix} = \tilde{N}_{ij} q_j$, where 1 is the distance between two nodes, s represents a boundary co-ordinate, which is measured from node p.

Assuming that the interpolation functions satisfy pointwise the strong forms Eqs. (16), (17), Eq. (33) become

$$\Pi_H[u_i, T_i, \tilde{u}_i] = \frac{1}{2}\int_V \varepsilon_{ij}^u E_{ijkl}\varepsilon_{ij}^u dV - \int_V b_i u_i dV - \int_{St} \hat{T}_i u_i dS + \int_{Su+Si} (\tilde{u}_i - u_i) T_i dS \tag{34}$$

which is recognized as a hybrid displacement principle.

When using the divergence theorem (Leconte *et al.* 2008a) in Eq. (34), the following expression is obtained

$$\Pi_H[u_i, T_i, \tilde{u}_i] = \frac{1}{2}\int_S T_i u_i dS - \frac{1}{2}\int_V (\sigma_{ij,j}^u + b_i) u_i dV - \frac{1}{2}\int_V b_i u_i dV - \int_{St} \hat{T}_i u_i dS + \int_{Su+Si} (\tilde{u}_i - u_i) T_i dS \tag{35}$$

Assuming that the strong form Eq. (18) is satisfied pointwise and that $b_i = 0$ (body forces are neglected) in Eq. (35) leads to

$$\Pi_{HT}[u_i, T_i, \tilde{u}_i] = \frac{1}{2}\int_S T_i u_i dS - \int_{St} \hat{T}_i u_i dS + \int_{Su+Si} (\tilde{u}_i - u_i) T_i dS \tag{36}$$

which is recognized as a hybrid-Trefftz displacement principle.

The hybrid-Trefftz principle of Eq. (36), coupled with the interpolation functions of fields u_i and T_i obtained from Kolosov-Muskhelishvili analytical solution (Eqs. (14), (15)), leads to the formulation of the stiffness matrix and load vector of Piltner's perforated elastic element by taking stationary conditions with respect to c and q (defined in Eqs. (14), (15), (33)). Indeed, the first stationary condition provides a linear relationship between c and q. Then, parameter c is eliminated thanks to this relationship, allowing the building of a system $N_{ij}q_j = f_i$. For further details on the formulation steps of hybrid-Trefftz displacement elements featuring a hole, see Leconte 2006, Piltner 1985, Chen 1994, Wang *et al.* 2004 or Dhansekhar *et al.* 2006.

It is demonstrated here, in accordance with Freitas (1999) that hybrid-mixed displacement, hybrid displacement and hybrid-Trefftz displacement principles are linked. It is shown that the hybrid-Trefftz displacement principle results in the constraining of a hybrid-mixed displacement principle to satisfy all inner domain strong form equations. This demonstration is valid in the particular case of a hybrid-Trefftz element featuring a hole, which can be defined as the constraining of a hybrid or hybrid-mixed formulation featuring a hole to satisfy all inner domain equations (by interpolation functions properties enforcement). It is thus concluded that Piltner's formulation is somehow the restriction of a more general hybrid-mixed formulation featuring a hole. Moreover, it can be seen from Eq. (36) that the hybrid-Trefftz perforated formulation of Piltner can be interpreted as the direct coupling, along the boundary S_i, of the perforated plate analytical solution (obtained thanks to Kolosov-Muskhelishvili solution) with a finite element mesh. Indeed, each term in the hybrid-Trefftz displacement functional appears to result from the integration of products of the analytical fields only (Eq. (36)), and no possibility is left to modify the material law, contrary to hybrid and hybrid-mixed principles (Eqs. (33), (34)). The solution obtained is thus restrained to the characteristics of the analytical solution, and it is thus only suitable for elastic computations. However, the choice of using a hybrid-Trefftz formulation is fully justified for elastic problems, as it leads to higher computational efficiency due to one dimension reduction of the integration (Eq. (36)).

4. Discussion

The building of perforated interpolation functions and the properties of hybrid displacement principles were highlighted. We can now discuss the possibility of building a perforated super-element dedicated to non-linear (crashworthiness) computations. It concerns:

- the choice or the building of a variational principle,
- the building of interpolation functions taking into account the hole,
- the extension of material law (Eq. (17)) and strain-displacement (Eq. (16)) descriptions, and the formulation of a mass matrix.

4.1 Choice or building of a variational principle

Concerning the choice or building of an adequate variational principle, the inter-element displacement continuity still needs to be ensured. Thus the variational principle has to be chosen among the "hybrid-displacement" kind. However, it should be noticed that depending on the hybrid-displacement principle chosen, the integration may change. Indeed, there are new terms to integrate when dealing with hybrid and hybrid-mixed principles in comparison with hybrid-Trefftz principles. Hybrid-Trefftz principles only require integration on the perforated element boundary S_i (1D and continuous, Eq. (36)), while hybrid and hybrid-mixed principles also require inner domain integration (2D domain featuring a hole, Eqs. (33), (34)). To integrate these new terms, it is thus necessary to develop an integration that is able to evaluate the chosen interpolation functions on perforated domains. The proposed strategy for the integration is the same as in XFEM, where the quantities need to be evaluated in an element domain featuring a crack: the domain is divided into squares and triangles, and the summation of integrals computed into each sub-domain is equal to the evaluation of the integral in the domain. It is highlighted here that this is only a matter of integral decomposition (to simplify its evaluation) and that no new degrees of freedom are added with the sub-domain decomposition.

4.2 Building of interpolation functions taking into account the hole

The building of suitable shape functions depends on the hybrid-displacement principle chosen. (1) If a hybrid-Trefftz principle is chosen, then the interpolation functions need to satisfy a priori the inner domain non-linear equations. Thus a new analytical solution satisfying the non-linear inner domain equations needs to be built. However, recent developments (Leconte and Di Paola 2007) have shown that it is even troublesome to formulate this kind of solution for plane strain perfectly plastic bodies in the same fashion as Kolosov-Muskhelishvili framework. Needless to say that building this kind of solution for a perforated plate under impact is nearly impossible. Thus the possibility of building a hybrid-Trefftz perforated element for non-linear computations is put aside. This is in accordance with the comments of Freitas who indicates that the hybrid and hybrid-mixed principles are easier to extend to non-linear problems than hybrid-Trefftz principles (Freitas 1999).

(2) If a hybrid or hybrid-mixed principle is used, then the chosen interpolation functions are not required to satisfy pointwise all inner domain equations. There is thus more flexibility in the choice of interpolation functions (Freitas 1999). However, in our case, the interpolation functions have to take into account the hole's presence and to be suitable whatever the kind of loading applied, so that a perforated element can be built. Nevertheless, the only interpolation functions featuring these

characteristics are the interpolation functions built from Kolosov-Muskhelishvili formalism, which is dedicated to elasticity only. The only remaining possibility is thus the use of the existing interpolation functions built on Kolosov-Muskhelishvili formalism. However, these interpolation functions need to be modified, so that their "exact perforated plate elastic solution" characteristics can be discarded. It is proposed to modify these functions thanks to the following comments.

It can be highlighted from Eqs. (14), (15) that the displacement and stress fields u_i and σ_{ij} are function of the same vector of parameters c_j, which features the unknowns α_j, β_j. The fact that the displacement and stress fields are functions of the same parameter is due to Eq. (1), where it can be noticed that the displacement and stress fields are function of the same holomorphic potentials Φ and Ψ (that are functions of α_j, β_j). The fact that the displacement and stress fields are function of the same parameter is thus a consequence of the use of Eq. (1), which is valid for elasticity only. It is then concluded that if these interpolation functions are used for non-linear computations, there is no specific reason why, in general, the linkage should be the same than in elasticity.

The following definition is thus proposed for the interpolation functions to be used in non-linear computations

$$\mathbf{u} = \begin{bmatrix}\mathbf{U}\\ \mathbf{V}\end{bmatrix}\begin{Bmatrix}\alpha_j^1\\ \beta_j^1\end{Bmatrix} = \frac{1}{2\mu}N_{ij}c_j^1 \tag{37}$$

$$\mathbf{T} = \begin{bmatrix}\mathbf{T}_x\\ \mathbf{T}_y\end{bmatrix}\begin{Bmatrix}\alpha_j^2\\ \beta_j^2\end{Bmatrix} = \frac{1}{r_0}P_{ij}c_j^2 \tag{38}$$

with $c_j^1 \neq c_j^2$ a priori.

Moreover, one should notice that Eqs. (37), (38) can be recast this way

$$\mathbf{u} = \begin{bmatrix}\mathbf{U}\\ \mathbf{V}\end{bmatrix}\begin{Bmatrix}\alpha_j^1\\ \beta_j^1\end{Bmatrix} = N_{ij}\left(\frac{1}{2\mu}c_j^1\right) \approx N_{ij}u_j^c \tag{39}$$

$$\mathbf{T} = \begin{bmatrix}\mathbf{T}_x\\ \mathbf{T}_y\end{bmatrix}\begin{Bmatrix}\alpha_j^2\\ \beta_j^2\end{Bmatrix} = P_{ij}\left(\frac{1}{r_0}c_j^2\right) \approx P_{ij}\sigma_j^c \tag{40}$$

with no particular a priori linkage between u_j^c and σ_j^c.

Indeed, as R, k, j are dimensionless, it can be concluded from Eqs. (9)-(13) that a parameter of unknowns c is homogenous to N/m (SI units). Then $c_i/2\mu$ is homogenous to a displacement, and c_i/r_0 is homogenous to a stress. Then, it can be concluded that Eq. (37) is equivalent to a matrix of dimensionless shape functions multiplied by a vector of displacement values (Eq. (39)). Similarly, Eq. (38) is equivalent to a matrix of dimensionless interpolation functions multiplied by a vector of stress values (Eq. (40)). Moreover, there is no particular linkage between Eqs. (39), (40), contrary to Eqs. (14), (15). Eqs. (39)-(40) are thus falling into the standard finite element definition of shape functions that can be used with mixed principles. Hence, there is thus no specific reason why these interpolation functions should not be associated with an extended mixed principle (i.e., hybrid or hybrid-mixed) to perform non-linear computations.

The interest of the reader is called upon refinement strategies too. In standard displacement FE, the interpolation functions are not built depending on the kind of computation performed (linear or non-linear). The mesh is only refined so that the behaviour of the structure is adequately "captured". As more super-elements featuring holes will lead to more holes in the structure, the refinement of the mesh cannot be performed with super-element featuring defects. As a consequence, one tries to capture the behaviour of the structure by increasing the number of terms in the truncated series of the interpolation functions of the perforated super-element (Zienkiewicz *et al.* 1977). Indeed, if a series (Eqs. (9)-(13) truncated to $N = M = 4$ show satisfactory results in elasticity (Leconte *et al.* 2008b), increasing the number of terms could be necessary to "capture" the localisation of non-linear fields around the hole of the super-element.

4.3 Extension of material law and strain-displacement descriptions and formulation of a mass matrix

Several authors have given clues for the extension of material law and strain-displacement descriptions and the formulation of a mass matrix. For the extension of mixed principles to materially non-linear problems, Grimaldi *et al.* proposed a non-linear elastic law (Grimaldi *et al.* 2004), which can be expressed for plane strain

$$\begin{cases} \boldsymbol{\sigma} = \mathbf{E}\omega = \dfrac{E(\varepsilon)}{1-\nu^2}\begin{vmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{vmatrix}\boldsymbol{\varepsilon} \\ E(\varepsilon) = \dfrac{E_0}{1+\sqrt{\varepsilon_1^2+\varepsilon_2^2}} \end{cases} \tag{41}$$

where E_0 is Young's modulus, $E(\varepsilon)$ is the non-linear elastic modulus, whose variation is governed by an equivalent measure of the principal strains ε_1 and ε_2.

If more complicated material laws need to be described then, the work of Horrigmoe *et al.* (Horrigmoe and Bergan 1976) can be followed for the formulation of suitable incremental hybrid-displacement principles.

For the extension of mixed principles to geometrically non-linear problems, Liu *et al.* (Liu and To 1995) proposed an extension of the kinematic description Eq. (16) (and gave hints to linearize variational principle if required)

$$\begin{cases} e_{ij} = \varepsilon_{ij} + \eta_{ij} \\ \varepsilon_{ij} = \dfrac{1}{2}(u_{i,j}+u_{j,i});\ \eta_{ij} = \dfrac{1}{2}u_{k,i}u_{k,j} \end{cases} \tag{42}$$

Finally, Darilmaz *et al.* (Darilmaz and Kumbasar 2006) proposed a way to formulate the mass matrix of a hybrid stress element from the statement of the kinetic energy. All this clues can be followed so that the long-term goal of formulating a hybrid-displacement element featuring a hole for crashworthiness can be achieved.

4.4 Synthesis

To sum up, hybrid and hybrid-mixed displacement principles are shown to be the only variational principles suitable to formulate a perforated finite element dedicated to non-linear (impact) computations. As these principles allow flexibility in the choice of the interpolation functions and that the only known way to formulate interpolation functions taking into account the hole whatever the kind of loading is to use Kolosov-Muskhelishvili formalism, the interpolation functions are chosen according to Eqs. (37), (38). The chosen formulations for displacement and traction fields take into account the hole and fall under the common definition of interpolation functions, and do not feature any specific linkage between displacement and traction field. There is thus no specific reason why the chosen interpolation shouldn't be used for non-linear computations, as mixed or hybrid principles have been proven to be successfully applicable in these cases. However, the descriptions of material law, strain-displacement and equilibrium need to be adequately extended. Finally, the choice of hybrid or hybrid-mixed principles instead of a hybrid-Trefftz leads to the need to integrate quantities in the inner perforated domain. The integration method of X-FEM could be used to perform such computations.

5. Mass matrix formulation

As the elastic energy stored in the element is known, it remains to determine the kinetic energy stored in the system so that a system $M_{ij}\dot{q}_j + K_{ij}q_j = F_i$ can be constituted (Darilmaz and Kumbasar 2006).

The kinetic energy expression is the following one

$$E^k = \frac{1}{2}\int_V \dot{\tilde{u}}_i I_{ij} \dot{\tilde{u}}_j dV \tag{43}$$

where $\dot{\tilde{u}}_i$ denotes the velocity components and I_{ij} is the inertia matrix.

From Eq. (32), $\dot{\tilde{u}}_i$ is defined as:

$$\dot{\tilde{u}}_i = \tilde{N}_{ij}\dot{q}_j \tag{44}$$

By substituting Eq. (44) into Eq. (43), we obtain

$$E^k = \frac{1}{2}\int_V \dot{q}_i \tilde{N}_{ji} I_{jl} \tilde{N}_{lm} \dot{q}_m dV \tag{45}$$

$$E^k = \frac{1}{2}\dot{q}_i\left(\int_V \tilde{N}_{ji} I_{jl} \tilde{N}_{lm} \dot{q}_m dV\right)\dot{q}_m \tag{46}$$

It is thus concluded that the element mass matrix is provided by the following expression

$$M = \int_V \tilde{N}_{ji} I_{jl} \tilde{N}_{lm} dV \tag{47}$$

6. Conclusions

The paper focuses on the building of super-elements featuring holes, and in particular on a hybrid-Trefftz displacement element featuring a hole that is restricted to elasticity. In order to formulate a super-element featuring a hole for non-linear problems, it is proposed to switch from a hybrid-

Trefftz displacement principle to a hybrid or hybrid-mixed displacement principle that does not require the interpolation functions to satisfy a priori the governing equations and that allow the extension of the constitutive and kinematic descriptions. A modification of the interpolation functions built through Kolosov-Muskhelishvili formalism is then proposed, so that they fit into the standard definition of interpolation functions to be used with mixed principles, while keeping the ability to localise fields caused by the hole's presence. Finally, the formulation of the mass matrix is proposed as a first step to the formulation of an element featuring a hole for non-linear crashworthiness problems.

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