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Elasticity solutions for a uniformly loaded annular plate of functionally graded materials

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Abstract. The axisymmetric problem of a functionally graded annular plate is considered by extending the theory of functionally graded materials plates suggested by Mian and Spencer (1998). In particular, their expansion formula for displacements is adopted and the hypothesis that the material parameters can vary along the thickness direction in an arbitrary continuous fashion is retained. However, their analysis is extended here in two aspects. First, the material is assumed to be transversely isotropic, rather than isotropic. Second, the plate is no longer tractions-free on the top and bottom surfaces, but subject to uniform loads applied on the surfaces. The elasticity solutions are given for a uniformly loaded annular plate of functionally graded materials for a total of six different boundary conditions. Numerical results are given for a simply supported functionally graded annular plate, and good agreement with those by the classical plate theory is obtained.

Keywords: functionally graded materials; annular plates; transversely isotropic; elasticity solutions.

1. Introduction

Functionally graded materials (FGMs) are a new type of inhomogeneous material wherein the volume fractions of constituent materials vary continuously in some specific directions, such as thickness direction. Therefore, their macroscopic material properties exhibit a smooth and continuous change in these directions, which satisfy the different requirements for material service performance at different locations in the structures. The concept of FGM (Yamanouchi *et al*. 1990, Koizumi 1993) was initiated for super heat resistant materials used in aerospace engineering so as to overcome the interface problems due to the mismatch of materials in traditional composites. Now FGMs have been used in many fields, such as electronics, chemistry, nuclear energy and

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biomedicine, etc, and have shown a wonderful application prospects.

The problem of a homogeneous plate subject to uniform loads is one of important classical research topics in elasticity. Timoshenko and Woinowsky-Krieger (1959) and Lekhnitskii (1968) adopted the classical plate theory (CPT), which employs a number of simplifying assumptions about the stress and displacement fields, to study isotropic and anisotropic plates. Love (1927), Timoshenko and Goodier (1970) and Ding *et al*. (2006) presented a series of analytical solutions for isotropic and transversely isotropic circular and annular plates subject to uniform loads directly based on the three-dimensional (3D) elasticity theory.

A number of works have been conducted on the analysis of elastic behavior of FGM plates. Reddy *et al*. (1999) examined the axisymmetric bending of functionally graded circular and annular plates by developing the exact relationships between the solution of CPT and that of the first-order shear deformation plate theory (FSDT). The asymptotic method was used to analyze the thermo-elastic coupled deformation of isotropic FGM rectangular plate by Reddy and Cheng (2001). Vel and Batra (2002) presented an exact 3D solution for the thermo-elastic deformation of simply-supported isotropic FGM plates. A 3D elasticity solution for an isotropic FGM rectangular plate with simply supported edges subject to transverse loading was developed by Kashtalyan (2004). Soldatos (2004), on the basis of the refined plate theory, developed Stroh-like complex formalisms and studied the bending problem of inhomogeneous anisotropic rectangular plates. Prakash and Ganapathi (2005) investigated the dynamic instability characteristics of aero-thermo-mechanically stressed functionally graded plates using finite element procedure. Ootao and Tanigawa (2005) analyzed exactly the transient problem of thermoelasticity involving an orthotropic functionally graded thick strip due to nonuniform heat supply in the width direction. Based on the CPT, Chi and Chung (2006a, 2006b) presented analytical solutions for simply supported isotropic FGM plates. Employing a discrete layer theory in combination with the Ritz method, Ramirez *et al*. (2006) presented an approximate 3D analysis of anisotropic functionally graded rectangular plates. Based on the piezoelasticity method and state space formulations, a simply supported hybrid plate consisting of top and bottom functionally graded elastic layers and an intermediate actuating or sensing homogeneous piezoelectric layer was investigated by Bian *et al*. (2006). Using the stress function method, Li *et al*. (2006, 2008) obtained elasticity solutions for transversely isotropic FGM circular plates subject to pure bending as well as a load in form of qr^k (k is zero or a finite even number). Mian and Spencer (1998) developed an ingenious method and obtained a class of 3D solutions for isotropic FGM plates with traction-free surfaces; in their analysis, the material properties are assumed to vary arbitrarily with the thickness-coordinate. Efforts on finding similar analytic solutions to this class of problems were firstly made by Kaprielian *et al*. (1988) and Rogers (1990). Using complex function method, England (2006) made a further extension by accounting for particular pressures, which satisfy the biharmonic equation or higher-order ones, applied to the top surface.

In this paper, the method suggested by Mian and Spencer (1998) is extended, through appropriate modifications, to consider a transversely isotropic FGM plate subject to uniform loads applied on the top and bottom surfaces, and the simplified two-dimensional (2D) governing equations are derived. The material properties can vary arbitrarily in a continuous fashion along the thickness of the plate. Elasticity solutions for FGM annular plates are presented for a total of six different boundary conditions at the cylindrical edges.

2. Basic equations for axisymmetric problem

Consider the axisymmetric problem in cylindrical coordinates (r, θ, z). Denote u_r and w as the displacement components in the r- and z- directions, respectively; σ_r, σ_θ, σ_z and τ_{rz} as the stress components; ε_r, ε_θ, ε_z and γ_{rz} as the strain components. The equations of equilibrium in absence of body forces are

$$\frac{\partial \sigma_r}{\partial r}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0, \qquad \frac{\partial \tau_{rz}}{\partial r}+\frac{\tau_{rz}}{r}+\frac{\partial \sigma_z}{\partial z}=0. \tag{1}$$

The stress-strain relations for transversely isotropic material can be expressed as (Ding *et al.* 2006)

$$\sigma_r=c_{11}\frac{\partial u_r}{\partial r}+c_{12}\frac{u_r}{r}+c_{13}\frac{\partial w}{\partial z}, \qquad \sigma_\theta=c_{12}\frac{\partial u_r}{\partial r}+c_{11}\frac{u_r}{r}+c_{13}\frac{\partial w}{\partial z},$$

$$\sigma_z=c_{13}\frac{\partial u_r}{\partial r}+c_{13}\frac{u_r}{r}+c_{33}\frac{\partial w}{\partial z}, \qquad \tau_{rz}=c_{55}\left(\frac{\partial u_r}{\partial z}+\frac{\partial w}{\partial r}\right), \tag{2}$$

where c_{ij} are the elastic constants. For FGM, c_{ij} are functions of z, i.e., $c_{ij}=c_{ij}(z)$.

According to Mian and Spencer (1998), we seek the following solutions of Eqs. (1) and (2)

$$u_r(r,z)=\bar{u}_r(r)+F\frac{d\Delta}{dr}+A\frac{d\bar{w}}{dr}+B\frac{d}{dr}\nabla^2\bar{w},$$

$$w(r,z)=\bar{w}(r)+G\Delta+C\nabla^2\bar{w}+D \tag{3}$$

where A, B, C, D, F and G are functions of z, and

$$\Delta=\frac{d\bar{u}_r}{dr}+\frac{\bar{u}_r}{r}=\frac{1}{r}\frac{d}{dr}(r\bar{u}_r), \nabla^2=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}=\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right). \tag{4}$$

Substituting Eq. (3) into Eq. (2) gives

$$\sigma_r=(c_{11}+c_{13}G')\frac{d\bar{u}_r}{dr}+(c_{12}+c_{13}G')\frac{\bar{u}_r}{r}+c_{11}F\frac{d^2\Delta}{dr^2}+(c_{11}A+c_{13}C')\frac{d^2\bar{w}}{dr^2}+(c_{12}A+c_{13}C')\frac{1}{r}\frac{d\bar{w}}{dr}$$

$$+c_{11}B\frac{d^2}{dr^2}\nabla^2\bar{w}+c_{12}F\frac{1}{r}\frac{d\Delta}{dr}+c_{12}B\frac{1}{r}\frac{d}{dr}\nabla^2\bar{w}+c_{13}D',$$

$$\sigma_\theta=(c_{12}+c_{13}G')\frac{d\bar{u}_r}{dr}+(c_{11}+c_{13}G')\frac{\bar{u}_r}{r}+c_{12}F\frac{d^2\Delta}{dr^2}+(c_{12}A+c_{13}C')\frac{d^2\bar{w}}{dr^2}+(c_{11}A+c_{13}C')\frac{1}{r}\frac{d\bar{w}}{dr}$$

$$+c_{12}B\frac{d^2}{dr^2}\nabla^2\bar{w}+c_{11}F\frac{1}{r}\frac{d\Delta}{dr}+c_{11}B\frac{1}{r}\frac{d}{dr}\nabla^2\bar{w}+c_{13}D',$$

$$\sigma_z=(c_{13}+c_{33}G')\Delta+c_{13}F\nabla^2\Delta+(c_{13}A+c_{33}C')\nabla^2\bar{w}+c_{13}B\nabla^4\bar{w}+c_{33}D',$$

$$\tau_{rz}=c_{55}(F+G)\frac{d\Delta}{dr}+c_{55}(A'+1)\frac{d\bar{w}}{dr}+c_{55}(B'+C)\frac{d}{dr}\nabla^2\bar{w}, \tag{5}$$

where the prime denotes derivative with respect to z.

By substituting Eq. (5) into Eq. (1), and setting

$$[c_{55}(A'+1)]'=0, \qquad (c_{13}+c_{33}G')'=0, \qquad c_{55}(A'+1)+(c_{13}A+c_{33}C')'=0\,, \quad (6),(7),(8)$$

$$c_{11}+c_{13}G'+[c_{55}(F'+G)]'=c_{55}\kappa_1, \qquad c_{11}A+c_{13}C'+[c_{55}(B'+C)]'=c_{55}\kappa_2\,. \quad (9),(10)$$

we have

$$c_{55}\kappa_1\frac{d\Delta}{dr}+c_{11}F\frac{d}{dr}\nabla^2\Delta+c_{55}\kappa_2\frac{d}{dr}\nabla^2\overline{w}+c_{11}B\frac{d}{dr}\nabla^4\overline{w}=0\,, \tag{11}$$

$$H\nabla^2\Delta+I\nabla^4\overline{w}+(c_{33}D')'=0 \tag{12}$$

where κ_1 and κ_2 are arbitrary constants, and

$$H=c_{55}(F'+G)+(c_{13}F)', \qquad I=c_{55}(B'+C)+(c_{13}B)' \tag{13}$$

If H and I are not zero simultaneously, it follows from Eq. (12) that

$$\nabla^2\Delta=\kappa_3, \qquad \nabla^4\overline{w}=\kappa_4 \quad (14),(15)$$

where κ_3 and κ_4 are arbitrary constants. Then Eqs. (11) and (12) become

$$\frac{d}{dr}(\kappa_1\Delta+\kappa_2\nabla^2\overline{w})=0\,, \tag{16}$$

$$H\kappa_3+I\kappa_4+(c_{33}D')'=0 \tag{17}$$

By virtue of Eqs. (14) and (15), it follows from Eq. (16) that

$$\kappa_1\kappa_3+\kappa_2\kappa_4=0\,. \tag{18}$$

Integrating Eqs. (15) and (16) in turn leads to

$$\overline{w}=C_1r^2+C_2r^2\ln r+C_3\ln r+C_4+\frac{1}{64}\kappa_4r^4\,, \tag{19}$$

$$\overline{u}_r=\frac{C_5}{2\kappa_1}r-\frac{2\kappa_2C_1}{\kappa_1}r-\frac{2\kappa_2C_2}{\kappa_1}r\ln r-\frac{\kappa_2C_2}{\kappa_1}r-\frac{\kappa_2C_3}{\kappa_1 r}-\frac{\kappa_2\kappa_4}{16\kappa_1}r^3+\frac{C_6}{r}\,, \tag{20}$$

where C_i ($i=1,2,3,4,5,6$) are integral constants which can be completely determined from the cylindrical boundary conditions.

3. Determination of the functions *A*(*z*), *B*(*z*), *C*(*z*), *D*(*z*), *F*(*z*) and *G*(*z*)

Consider an annular plate of height h, inner radius r_0, outer radius r_1, as shown in Fig. 1, subject to uniform loads q_1 and q_2. Obviously, we have $\tau_{rz}=0$ at $z=\pm h/2$ $\sigma_z=-q_1$, at $z=-h/2$, and $\sigma_z=-q_2$ at $z=h/2$.

By substituting the expressions for τ_{rz} and σ_z in Eq. (5) into the boundary conditions at $z=\pm h/2$, we obtain

$$A'(\pm h/2)+1=0, \qquad c_{13}(\pm h/2)+c_{33}(\pm h/2)G'(\pm h/2)=0\,, \quad (21),(22)$$

$$F'(\pm h/2)+G(\pm h/2)=0, \qquad B'(\pm h/2)+C(\pm h/2)=0\,, \quad (23),(24)$$

$$c_{13}(\pm h/2)A(\pm h/2)+c_{33}(\pm h/2)C'(\pm h/2)=0\,, \tag{25}$$

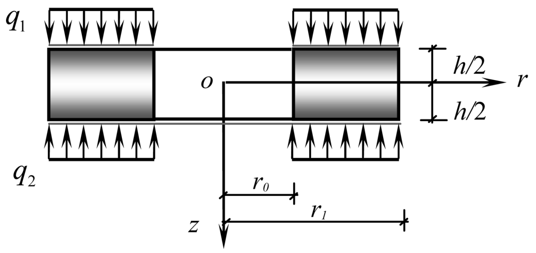

Fig. 1 The geometry and coordinates of annular plate

$$c_{13}(-h/2)[\kappa_3 F(-h/2)+\kappa_4 B(-h/2)]+c_{33}(-h/2)D'(-h/2) = -q_1\,, \tag{26}$$

$$c_{13}(h/2)[\kappa_3 F(h/2)+\kappa_4 B(h/2)]+c_{33}(h/2)D'(h/2) = -q_2\,. \tag{27}$$

Integrating (6)-(10) and (17) and making use of the boundary conditions (21)-(27) and relation (18), yields

$$A(z) = -z + A_0, \qquad G(z) = -g_0^0(z)+G_0\,,$$

$$C(z) = g_1^0(z) - A_0 g_0^0(z) + C_0\,,$$

$$F(z) = \kappa_1 f_6(z) - f_5(z) + g_0^1(z) - (z+h/2)G_0 + F_0\,,$$

$$B(z) = \kappa_2 f_6(z) + f_4(z) + A_0 f_2(z) - g_1^1(z) - (z+h/2)C_0 + B_0\,,$$

$$D(z) = \kappa_3 f_7(z) - \kappa_4 f_8(z) + D_1 f_3(z) + D_0\,, \tag{28}$$

where A_0, B_0, C_0, D_0, D_1, F_0 and G_0 are integral constants, and

$$g_i^n(z) = \int_{-h/2}^{z}(z-\xi)^n \xi^i \frac{c_{13}}{c_{33}} d\xi, \qquad (i,n=0,1)$$

$$h_i^n(z) = \int_{-h/2}^{z}(z-\xi)^n \xi^i \left(c_{11} - \frac{c_{13}^2}{c_{33}}\right) d\xi, \qquad (i,n=0,1)$$

$$g_6(z) = \int_{-h/2}^{z} c_{55}\, d\xi, \qquad f_2(z) = g_0^1(z) - f_5(z)\,,$$

$$f_3(z) = \int_{-h/2}^{z} c_{33}^{-1} d\xi, \qquad f_4(z) = \int_{-h/2}^{z} h_1^0(\xi) c_{55}^{-1}(\xi) d\xi,$$

$$f_5(z) = \int_{-h/2}^{z} h_0^0(\xi) c_{55}^{-1}(\xi) d\xi, \qquad f_6(z) = \int_{-h/2}^{z} g_6(\xi) c_{55}^{-1}(\xi) d\xi,$$

$$f_7(z) = \int_{-h/2}^{z} c_{33}^{-1} \{ h_0^1(\xi) + c_{13}[f_5(\xi) - g_0^1(\xi) + (\xi + h/2)G_0 - F_0] \} d\xi,$$

$$f_8(z) = \int_{-h/2}^{z} c_{33}^{-1} \{ h_1^1(\xi) + c_{13}[f_4(\xi) - g_1^1(\xi) - (\xi + h/2)C_0 + B_0] \} d\xi. \tag{29}$$

$$\kappa_1 = \frac{h_0^0(h/2)}{g_6(h/2)}, \qquad \kappa_2 = \frac{[A_0 h_0^0(h/2) - h_1^0(h/2)]}{g_6(h/2)}, \qquad D_1 = -q_1$$

$$\kappa_4 = \kappa_1(q_2 - q_1)/[\kappa_2 h_0^1(h/2) + \kappa_1 h_1^1(h/2)], \qquad \kappa_3 = -\kappa_2 \kappa_4 / \kappa_1 \tag{30}$$

Given that $\bar{u}_r$ and $\bar{w}$ are the mid-plane displacements, namely

$$\bar{u}_r(r)=u_r(r,0), \qquad \bar{w}_r(w)=w(r,0) \tag{31}$$

we can deduce from Eq. (3) that

$$A(0)=0, \quad B(0)=0, \quad C(0)=0, \quad D(0)=0, \quad F(0)=0, \quad G(0)=0 \tag{32}$$

Substituting Eq. (28) into Eq. (32) gives

$$\begin{aligned} &A_0=0, \qquad C_0=-g_1^0(0), \qquad G_0=g_0^0(0), \\ &F_0=f_5(0)+hg_0^0(0)/2-g_0^1(0)-\kappa_1 f_6(0), \\ &B_0=g_1^1(0)-hg_1^0(0)/2-f_4(0)-\kappa_2 f_6(0), \\ &D_0=-\kappa_3 f_7(0)+\kappa_4 f_8(0)+q_1 f_3(0)\ . \end{aligned} \tag{33}$$

In such a way, the functions $A(z)$, $B(z)$, $C(z)$, $D(z)$, $F(z)$ and $G(z)$ can be determined completely.

4. Resultant forces

By virtue of Eq. (5), the expressions for the radial resultant force $N_r(r)$, bending moment $M_r(r)$ and shear force $Q_r(r)$ are

$$\begin{aligned} N_r(r) = \int_{-h/2}^{h/2}\sigma_r dz = N_0+N_1\frac{d\bar{u}_r}{dr}+N_2\frac{\bar{u}_r}{r}+N_3\frac{d^2\bar{w}}{dr^2}+N_4\frac{1}{r}\frac{d\bar{w}}{dr}+N_5\frac{d^2\Delta}{dr^2} \\ +N_6\frac{1}{r}\frac{d\Delta}{dr}+N_7\frac{d^2}{dr^2}\nabla^2\bar{w}+N_8\frac{1}{r}\frac{d}{dr}\nabla^2\bar{w}, \end{aligned} \tag{34}$$

$$\begin{aligned} M_r(r) = \int_{-h/2}^{h/2}\sigma_r z dz = M_0+M_1\frac{d\bar{u}_r}{dr}+M_2\frac{\bar{u}_r}{r}+M_3\frac{d^2\bar{w}}{dr^2}+M_4\frac{1}{r}\frac{d\bar{w}}{dr}+M_5\frac{d^2\Delta}{dr^2} \\ +M_6\frac{1}{r}\frac{d\Delta}{dr}+M_7\frac{d^2}{dr^2}\nabla^2\bar{w}+M_8\frac{1}{r}\frac{d}{dr}\nabla^2\bar{w}, \end{aligned} \tag{35}$$

$$Q_r(r) = \int_{-h/2}^{h/2}\tau_{rz}dz = Q_1\frac{d\Delta}{dr}+Q_2\frac{d}{dr}\nabla^2\bar{w}, \tag{36}$$

where

$$\begin{aligned} &N_1=h_0^0(h/2), \qquad N_2=\int_{-h/2}^{h/2}(c_{12}-c_{13}^2/c_{33})dz, \qquad N_3=-h_1^0(h/2), \\ &N_4=-\int_{-h/2}^{h/2}(c_{12}-c_{13}^2/c_{33})zdz, \qquad N_5=\int_{-h/2}^{h/2}c_{11}Fdz, \qquad N_6=\int_{-h/2}^{h/2}c_{12}Fdz, \\ &N_7=\int_{-h/2}^{h/2}c_{11}Bdz, \qquad N_8=\int_{-h/2}^{h/2}c_{12}Bdz, \qquad N_0=\int_{-h/2}^{h/2}(c_{13}D')dz\,. \end{aligned} \tag{37}$$

$$\begin{aligned} &M_1=h_1^0(h/2), \qquad M_2=\int_{-h/2}^{h/2}(c_{12}-c_{13}^2/c_{33})zdz, \qquad M_3=-h_2^0(h/2) \\ &M_4=-\int_{-h/2}^{h/2}(c_{12}-c_{13}^2/c_{33})z^2dz, \qquad M_5=\int_{-h/2}^{h/2}c_{11}zFdz, \qquad M_6=\int_{-h/2}^{h/2}c_{12}zFdz \end{aligned}$$

$$M_7 = \int_{-h/2}^{h/2} c_{11} zB dz, \qquad M_8 = \int_{-h/2}^{h/2} c_{12} zB dz, \qquad M_0 = \int_{-h/2}^{h/2} (c_{13} D' z) dz \tag{38}$$

$$Q_1 = \int_{-h/2}^{h/2} c_{55}(F' + G) dz, \qquad Q_2 = \int_{-h/2}^{h/2} c_{55}(B' + C) dz \tag{39}$$

5. Solutions for six kinds of annular plates

For the annular plate, there are three cylindrical boundary conditions at $r = r_0$ and $r = r_1$, respectively, which can be used to determine the six integral constants in the solutions (19) and (20).

5.1 Simply supported-simply supported (SS) annular plate.

The boundary conditions are

$$N_r(r_i) = 0, \qquad M_r(r_i) = 0, \qquad \overline{w}(r_i) = 0\,. \qquad (i = 0,1) \tag{40}$$

5.2 Clamped-Clamped (CC) annular plate. The boundary conditions are

$$\bar{u}_r(r_i) = \overline{w}(r_i) = 0, \qquad \frac{d\overline{w}(r_i)}{dr} = 0\,. \qquad (i = 0,1) \tag{41}$$

5.3 Annular plate with inner edge simply supported and outer edge clamped (SC) or inner edge clamped and outer edge simply supported (CS)

The boundary conditions are comprised of Eqs. $(40)_1$ and $(41)_2$ or Eqs. $(40)_2$ and $(41)_1$, respectively.

5.4 Annular plate with inner edge free and outer edge clamped (FC) or inner edge clamped and outer edge free (CF)

The boundary conditions are, respectively.

$$N_r(r_0) = Q_r(r_0) = 0, \qquad M_r(r_0) = 0, \qquad \bar{u}_r(r_1) = \overline{w}(r_1) = 0, \qquad \frac{d\overline{w}(r_1)}{dr} = 0\,. \tag{42}$$

$$\bar{u}_r(r_0) = \overline{w}(r_0) = 0, \qquad \frac{d\overline{w}(r_0)}{dr} = 0, \qquad N_r(r_1) = Q_r(r_1) = 0, \qquad M_r(r_1) = 0\,. \tag{43}$$

Substituting Eqs. (19), (20), (34), (35) and (36) into the corresponding boundary conditions, the six equations to determine the six integral constants C_i ($i = 1,2,3,4,5,6$) can be readily derived, and the corresponding solutions are obtained completely.

6. Results and discussion

For convenience, the following dimensionless quantities are introduced

$$\overline{W}_1 = wD_0/qr_1^4, \qquad \overline{W}_2 = w/h, \qquad \overline{U}_r = u_r/\bar{r}, \qquad \beta = h/r_2$$

Table 1 Dimensionless deflection $\overline{W}_1(\bar{r}, 0)$

	r_0/r_1			
	0.1	0.3	0.5	0.7
CPT	0.00600	0.00290	0.000800	0.000100
Present	0.00607	0.00286	0.000795	0.000106

Table 2 Elastic constants of Al_2O_3 and Titanium (Unit: GPa)

Materials	c_{11}^0	c_{12}^0	c_{13}^0	c_{33}^0	c_{55}^0
Al_2O_3	460.2	174.7	127.4	509.5	126.9
Titanium	162.4	92	69	180.7	46.7

$$\bar{\sigma}_r = \sigma_r/q, \quad \bar{\sigma}_\theta = \sigma_\theta/q, \quad \bar{\sigma}_z = \sigma_z/q, \quad \bar{\tau}_{rz} = \tau_{rz}/q\,,$$

where $\bar{r} = (r_0 + r_1)/2$, $r_2 = r_1 - r_0$, $D_0 = Eh^3/12(1 - \nu^2)$, $q = 1 \times 10^6\ N/m^2$ and all the results are given at $r = \bar{r}$.

Example 1: Simply supported homogeneous isotropic annular plate.

The present method is first validated by comparison with the CPT solutions (Young and Budynas 2002) for a homogeneous annular plate subject to a uniform load on the top surface, with $h = 0.002$ m, $r_0 = 0.025$ m, $r_1 = 0.1$ m, and $\nu = 0.3$.

It is seen from Table 1 that the present elasticity solution agrees well with the classical plate theory; it is expected for the thin plate considered in this example.

Example 2: Transversely isotropic FGM annular plate

Now consider an FGM annular plate of $r_0 = 0.25$ m, and $r_1 = 1$ m, subject to uniform load acting on the top surface. The material constants are assumed to be in the form of (Reddy *et al.* 1999)

$$C_{ij} = C_{ij}^{0(A)}(0.5 - z/h)^\lambda + C_{ij}^{0(T)}[1 - (0.5 - z/h)^\lambda] \quad (i, j = 1,2,3,4,5,6)$$

where $C_{ij}^{0(A)}$ are those of $\mathbf{Al_2O_3}$ at $z = -h/2$, $C_{ij}^{0(T)}$ are those of **Titanium** at $z = h/2$, both given in Table 2 (Ding *et al.* 2006). The parameter λ is usually referred to as the gradient index since it reflects the degree of material inhomogeneity.

Fig. 2 shows the variations of the elastic constant c_{11} along the thickness direction for different values of λ. Obviously, $\lambda = 0$ corresponds to the case of homogeneous material ($\mathbf{Al_2O_3}$), while $\lambda > 0$ corresponds to FGM with material properties changing continuously from $\mathbf{Al_2O_3}$ at $z = -h/2$ to **Titanium** at $z = h/2$.

Tables 3 and 4 give the dimensionless deflection $\overline{W}_2$, normal stress $\bar{\sigma}_r$ and shear stress $\bar{\tau}_{rz}$ of the FGM annular plate with six kinds of boundary conditions and three values of λ, for thickness-to-span ratio $\beta = 0.2$ and $\beta = 0.3$, respectively. It is found that:

(1) The deflection increases with λ, regardless of the boundary conditions and the value of β. This is simply because the whole rigidity of the FGM plate decreases with λ. The deflections of CF and CC annular plates are, respectively, the biggest and smallest among the plates with the six kinds of boundary conditions;

(2) With the increase of λ, the absolute value of normal stress decreases firstly and then increases gradually, except the CC annular plate with $\beta = 0.3$, for which the normal stress is always

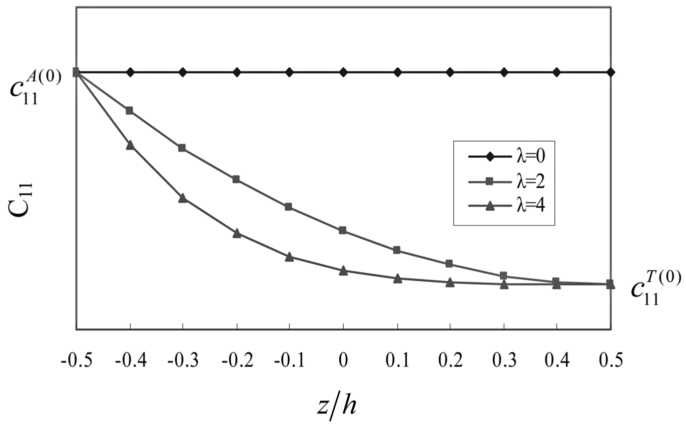

Fig. 2 Variation of elastic constant c_{11} along the thickness

Table 3 Dimensionless deflection and stresses ($\beta = 0.2$)

B.C.	$\overline{W}_2(\bar{r}, 0)$ ($\times 10^{-4}$)			$\overline{\sigma}_r(\bar{r}, h/2)$			$\overline{\tau}_{rz}(\bar{r}, 0)$		
	$\lambda = 0$	$\lambda = 2$	$\lambda = 4$	$\lambda = 0$	$\lambda = 2$	$\lambda = 4$	$\lambda = 0$	$\lambda = 2$	$\lambda = 4$
SS	2.3526	4.5494	5.0404	16.5334	12.2439	13.3335	0.2448	0.1819	0.1673
CC	0.4608	0.8858	0.9642	4.3029	3.3572	3.5276	-0.0686	-0.0663	-0.0648
SC	0.8318	1.6150	1.7771	6.2651	4.8370	5.1709	-0.8898	-0.8788	-0.8776
CS	1.1158	2.1243	2.3318	9.1785	6.6047	7.0679	1.1419	1.0857	1.0769
FC	3.0677	6.0410	6.5850	-2.5217	-1.3547	-1.6069	-2.6250	-2.5401	-2.4811
CF	9.0881	17.1374	18.5306	-14.8677	-12.5627	-13.3890	4.8750	4.7173	4.6078

Table 4 Dimensionless deflection and stresses ($\beta = 0.3$)

B.C.	$\overline{W}_2(\bar{r}, 0)$ ($\times 10^{-4}$)			$\overline{\sigma}_r(\bar{r}, h/2)$			$\overline{\tau}_{rz}(\bar{r}, 0)$		
	$\lambda = 0$	$\lambda = 2$	$\lambda = 4$	$\lambda = 0$	$\lambda = 2$	$\lambda = 4$	$\lambda = 0$	$\lambda = 2$	$\lambda = 4$
SS	0.5450	1.0449	1.1737	7.5580	5.5350	6.0634	0.1439	0.1083	0.0980
CC	0.0910	0.1750	0.1905	0.9471	0.8668	0.8339	-0.0457	-0.0442	-0.0432
SC	0.1798	0.3467	0.3842	2.1212	1.6621	1.7292	-0.7088	-0.6901	-0.6971
CS	0.2421	0.4608	0.5112	3.3945	2.4297	2.5553	0.8968	0.8533	0.8604
FC	0.5925	1.1653	1.2659	-1.5341	-0.9088	-1.0614	-1.7500	-1.6934	-1.6541
CF	1.7191	3.2490	3.4952	-6.9001	-5.7269	-6.0943	3.2500	3.1448	3.0719

decreasing. The normal stresses at $z = h/2$ are compressive for FC and CF boundary conditions, and tensile for other kinds of boundary conditions, regardless of the value of β;

(3) With the increase of λ, the absolute value of shear stress decreases, except the SC and CS annular plates with $\beta = 0.3$, for which the shear stress first decreases and then gradually increases. Shear stress for CC, SC and FC boundary conditions shows a reverse sign compared with other three kinds of boundary conditions, regardless of the value of β;

(4) The constraints imposed on the outer boundary have more obvious influence on the deflection and stress than those on the inner boundary, as can be seen from the comparison between SC plate and CS plate or that between FC plate and CF plate. With β increasing, the absolute values of

deflection and stress decrease correspondingly.

Figs. 3-8 depict the distributions of dimensionless displacements and stresses along the thickness direction of the SS FGM annular plate for different values of λ, with the thickness-to-span ratio being $\beta = 0.3$.

Figs. 3 and 4 depict the distributions of dimensionless radial and axial displacements $\overline{U}_r$ and $\overline{W}_2$. Fig. 3 indicates that $\overline{U}_r$ changes almost linearly along the thickness direction, regardless of the magnitude of λ. From Fig. 4, one can see that $\overline{W}_2$ almost keeps invariant along the thickness direction for $\lambda = 0$; but for $\lambda = 2$ or 4, $\overline{W}_2$ is no longer linearly distributed. The absolute value of $\overline{U}_r$ or $\overline{W}_2$ at the same point of the plate increases with λ.

Figs. 5 and 6 show the distributions of dimensionless radial and hoop stresses $\overline{\sigma}_r$ and $\overline{\sigma}_\theta$, respectively. It is found that $\overline{\sigma}_r$ and $\overline{\sigma}_\theta$ are almost linearly distributed along the thickness for $\lambda = 0$. However, for $\lambda = 2$ or 4, the distributions of $\overline{\sigma}_r$ and $\overline{\sigma}_\theta$ are no longer linear. For larger λ, the distributions are similar, and at the same point, the magnitudes of $\overline{\sigma}_r$ and $\overline{\sigma}_\theta$ change little with λ.

Fig. 7 gives the distributions of dimensionless axial stress $\overline{\sigma}_z$. It is found that the distributions of $\overline{\sigma}_z$ seem to be similar for different λ. The absolute value of $\overline{\sigma}_z$ for $\lambda = 0$ is bigger than those for $\lambda = 2$ or 4. In fact, for larger λ (say bigger than 2), $\overline{\sigma}_z$ changes little with λ.

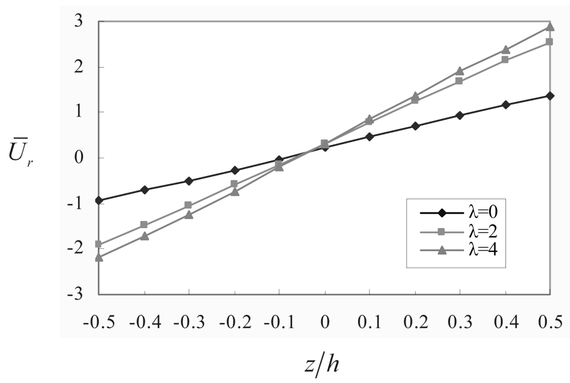

Fig. 3 Dimensionless radial displacement ($\times 10^{-6}$) ($\beta = 0.3$, SS)

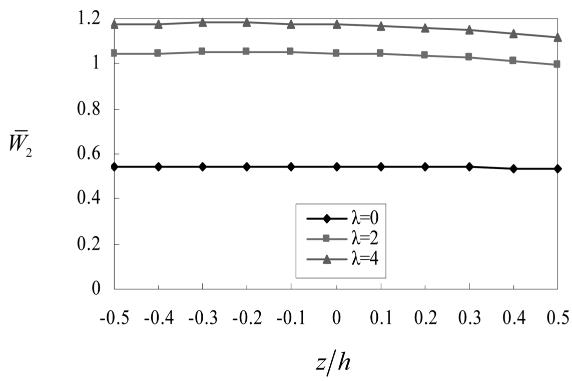

Fig. 4 Dimensionless axial displacement ($\times 10^{-4}$) ($\beta = 0.3$, SS)

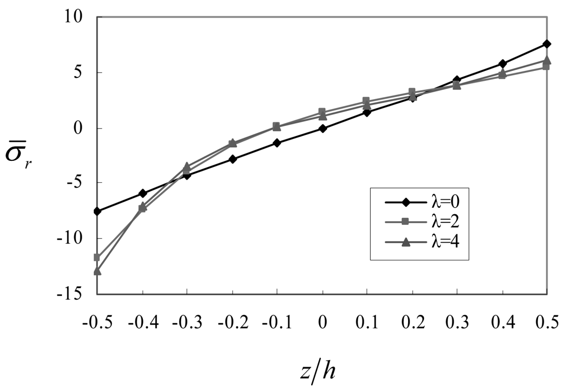

Fig. 5 Dimensionless radial stress ($\beta = 0.3$, SS)

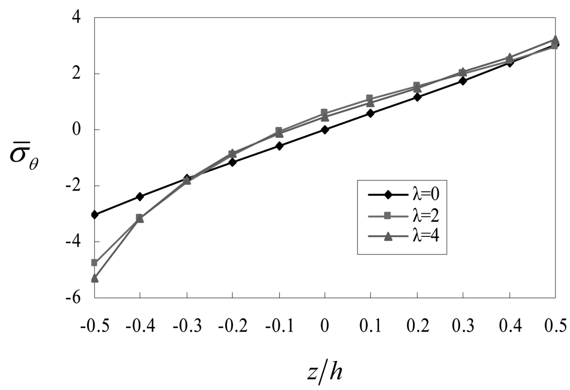

Fig. 6 Dimensionless hoop stress ($\beta = 0.3$, SS)

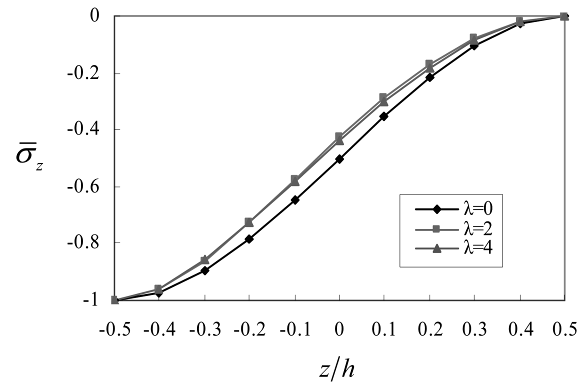

Fig. 7 Dimensionless axial stress ($\beta = 0.3$, SS)

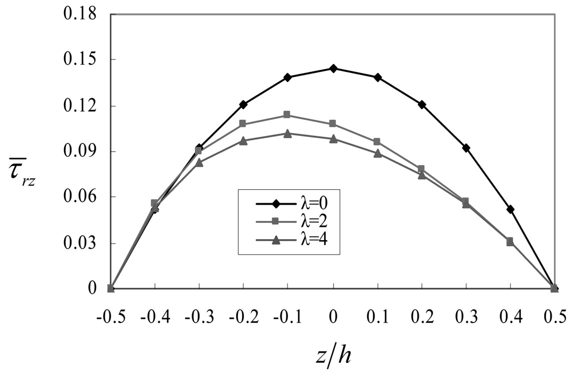

Fig. 8 Dimensionless shear stress ($\beta = 0.3$, SS)

Fig. 8 shows the distributions of dimensionless shear stress $\bar{\tau}_{rz}$. It is found that the distribution curves of $\bar{\tau}_{rz}$ along the thickness direction are analogous to the parabola, regardless of the magnitude of λ. For $\lambda = 0$, the maximum shear stress takes place at $z = 0$, while it shifts to the position approaching to the $z = -h/2$ and decrease gradually with λ.

7. Conclusions

The governing Eqs. (15) and (16) for transversely isotropic FGM plate subject to uniform loads are obtained by extending the method suggested by Mian and Spencer (1998). The material coefficients can vary arbitrarily in a continuous fashion with the thickness-coordinate. Elasticity solutions for FGM annular plate for a total of six different cylindrical boundary conditions are presented. The present theory differs from the CPT mainly because it exactly satisfies the basic 3D elasticity equations and the boundary conditions at the top and bottom surfaces, and employs only the simplified boundary conditions in the CPT at the cylindrical boundary.

The theory is first clarified by comparing with the CPT solution for a thin homogeneous plate. The bending of a transversely isotropic FGM annular plate is then examined. It is shown that the boundary conditions and the material inhomogeneity have important effects on the response of the FGM annular plate, and the effect of constraints imposed on the outer boundary is greater than that of constraints on the inner boundary. Therefore, the mechanical behavior of FGM annular plates can be optimized by adjusting properly the factors mentioned above in engineering applications.

Although there are simplifying assumptions in a Saint-Venant sense made around the edge of the plate, the present analysis does not employ other simplifying hypotheses about the stress and displacement field along the thickness as in usual plate theories. Hence, the proposed elasticity solution can serve as good benchmarks for accessing validity of various approximate plate theories or numerical methods that may be used in the analysis of FGM plates.

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