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# Automatic generation of equilibrium and flexibility matrices for plate bending elements using Integrated Force Method

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**Abstract.** The Integrated Force Method (IFM) has been developed in recent years for the analysis of civil, mechanical and aerospace engineering structures. In this method all independent or internal forces are treated as unknown variables which are calculated by simultaneously imposing equations of equilibrium and compatibility conditions. The solution by IFM needs the computation of element equilibrium and flexibility matrices from the assumed displacement, stress-resultant fields and material properties. This paper presents a general purpose code for the automatic generation of element equilibrium and flexibility matrices for plate bending elements using the Integrated Force Method. Kirchhoff and the Mindlin-Reissner plate theories have been employed in the code. Paper illustrates development of element equilibrium and flexibility matrices for the Mindlin-Reissner theory based four node quadrilateral plate bending element using the Integrated Force Method.

**Keywords:** displacement fields; stress-resultant fields; equilibrium matrix; flexibility matrix; Integrated Force Method

## 1. Introduction

The Displacement method of analysis has drawn undivided attention by researchers around the world during the past three to four decades. During the same period, the once popular force method met with its demise due to limitation of redundant analysis as its base. A new novel matrix formulation of the classical force method termed "Integrated Force Method (IFM)" has been

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developed for analyzing civil, mechanical and aerospace engineering structures (Patnaik 1973). The IFM is independent of redundants and the basis determinate structure of the classical force method. Variational formulation for IFM has been established (Patnaik 1986). The application of the IFM to several problems of structural analysis are reported in references (Patnaik *et al.* 1996, Krishnam *et al.* 2000).

In the finite element method of analysis extensive research efforts and time are spent in modeling the behavior of the elements and later deriving the matrices which represent their characteristic behavior. The various matrices are formed with interpolation functions for displacement and sometimes force distribution within or on the boundary of the element. Later on algebraic manipulations, including differentiation and integration, are performed on these matrices describing the characteristics of the element stiffness, flexibility and equilibrium etc. As the number of degrees of freedom of the element increases, the algebraic manipulations become huge and intractable. Therefore automatic generation of these matrices have been attempted by several researchers. Gunderson *et al.* (1971) developed Element Stiffness Matrix Generator (ESMG) for 2D and 3D problems. Luff *et al.* (1971) developed a software for automatic generation of finite element matrices. Cecchi *et al.* (1977) developed a program INTER which generates symbolic closed form expressions for stiffness matrix and could also produce equivalent Fortran statements. Korncoff *et al.* (1979) developed code for symbolic generation of the finite element stiffness matrices. Noor *et al.* (1979) summarized the status and applications of computerized symbolic manipulations to structural mechanics problems giving importance to generation of characteristic arrays of finite elements and evaluation of effective stiffness and mass coefficients of continuum models. Hoa *et al.* (1980), have developed a computer program for automatic generation of stiffness and mass matrices required in finite element analysis. The scope of symbolic manipulations(computer algebra) and their applications are discussed in the references (Foster *et al.* 1989, Beltzer 1990). Chang *et al.* (1990) have presented a symbolic manipulation procedure leading to the analytical integration of a stiffness matrix of hybrid or mixed elements. They used symbolic manipulation package MYCSYMA for the purpose. Closed form of stiffness matrices for a four node quadrilateral element and commonly used hybrid finite elements are given in references (Griffiths 1994, Lee *et al.* 1998). Computer algebra has been applied to classical problems in plate theory in the reference (Mbakogu *et al.* 1999). A symbolic manipulation package MACSYMA is used to generate shape functions of finite elements (Chee *et al.* 2000). A review article on symbolic computations in structural engineering (Pavlovic 2003) explains background of computer algebra and its applications to structural engineering particularly in the conversion of quasi-analytical techniques to truly closed form solutions and to finite elements.

Analogous to automatic generation of stiffness matrices in the displacement based finite element method as cited in the above para, the Integrated Force Method is also in need of the automatic generation of element equilibrium and flexibility matrices, and compatibility conditions for analyzing civil, mechanical and aerospace engineering structures. In this direction, Nagabhushanam, *et al.* (1990), developed a general purpose program to generate compatibility matrix for the Integrated Force Method . Automatic generation of sparse and banded compatibility matrix using the Integrated Force Method is presented by Nagabhushanam *et al.* (1991). A special purpose symbolic processor is developed by Nagabhushanam *et al.* (1992) in Fortran with built-in facilities to perform most of the general symbolic manipulations needed for the generations element stiffness and flexibility matrices.

In this paper, a fully automated and independent general purpose code to generate the element equilibrium and flexibility matrices for the plate bending element is presented. The Kirchhoff and the Mindlin-Reissner plate theories are considered in the code. Code uses the exact integration if the geometry of the plate is square or rectangular in shape and numerical integration for triangular and quadrilateral shapes. Use of the code is illustrated by developing element equilibrium and flexibility matrices for the Mindlin-Reissner theory based 4-node quadrilateral plate bending element. For completeness, brief basic theory of the IFM is given in the next section.

## 2. Basic Theory of IFM

In the Integrated Force Method of analysis, a structure idealized by finite elements is designated as "structure ($n$,$m$)", where $n$ and $m$ are force and displacement degrees of freedom of the discreet model, respectively. The 'structure ($n$,$m$)' has $m$ equilibrium equations (EE) and $r=(n-m)$ compatibility conditions ($CC$).

The equilibrium equation (EE) represents the vectorial summation of the internal forces $\{F\}$ to the external loads $\{P\}$ at the nodes of the finite element discretization. It can be written in symbolized matrix notation as

Equilibrium Equations (EE)

$$[B]\ \{F\} = \{P\} \tag{1}$$

where

$[B]$ = global equilibrium matrix ($m \times n$)
$\{F\}$ = vector of internal forces of the structure ($n \times 1$)
$\{P\}$ = vector of external loads on the structure ($m \times 1$)

The compatibility conditions (CC) are constraints on strains, and for finite element models they are also constraints on member deformations.

In IFM St. Venant's approach has been extended for discrete mechanics to develop the compatibility conditions. Development of CC is briefly explained below

The Deformation-Displacement Relationship (DDR) for discrete mechanics is equivalent to the strain-displacement relationship in elasticity. The DDR for discrete analysis was obtained during the development of the variational energy formulation for the IFM.

According to work energy-conservation theorem, the internal energy ($IE$) stored in the structure is equal to the work done by the external load ($WD$), that is

$$IE = WD$$
$$\frac{1}{2}\{F\}^T\{\beta\} = \frac{1}{2}\{P\}^T\{X\} \tag{2}$$

where $\{X\}$ represents nodal displacements and $\{\beta\}$ represents member deformations. Eq. (2) can be rewritten by eliminating the load $\{P\}$ in favor of forces $\{F\}$, by using Eq. (1) to obtain the following relation

$$\frac{1}{2}\{F\}^T[B]^T\{X\} = \frac{1}{2}\{F\}^T\{\beta\} \tag{3}$$

Eq. (3) can be simplified as

$$\frac{1}{2}\{F\}^T[[B]^T\{X\} - \{\beta\}] = 0$$

Because the $n$ forces can be arbitrary and $\{F\}$ is not a null vector, its coefficient should be zero, which yields the DDR as

$$\{\beta\} = [B]^T\{X\} \tag{4}$$

where $\{\beta\}$ are member deformations.

This equation represents the Deformation-Displacement Relations (DDR) for the discrete structure. The elimination of $m$ displacements from $n$ deformations displacement relations given by the above equation yields $r = (n - m)$ compatibility conditions and the associated matrix $[C]$. It can be symbolized in matrix notations as

$$[C]\{\beta\} = 0 \tag{5}$$

where $[C]$ is the $(r \times n)$ compatibility matrix. It is a kinematics relationship, and is independent of design parameters, material properties and external loads. This matrix is rectangular and banded. The deformation $\{\beta\}$ in the compatibility conditions (CC) given by the Eq. (5) represents the total deformation consisting of an elastic component $\{\beta_e\}$ and the initial component $\{\beta_o\}$ as

$$\{\beta\} = \{\beta_e\} + \{\beta_o\}$$

The CC in terms of elastic deformation can be written as

$$[C]\{\beta\} = [C]\{\beta_e\} + [C]\{\beta_o\}$$

$$[C]\{\beta_e\} = \{\delta R\}$$

where

$$\{\delta R\} = -[C]\{\beta_o\}$$

Using element flexibility characteristics, Eq. (5) with initial deformations $\{\delta R\}$ can be rewritten as

$$[C][G]\{F\} = \{\delta R\} \tag{6}$$

Clubbing of Eq. (1) and Eq. (6) will lead to the IFM governing equation as

$$\begin{bmatrix} [B] \\ [C][G] \end{bmatrix}\{F\} = \begin{Bmatrix} P \\ \delta R \end{Bmatrix}$$

$$[S]\{F\} = \{P^*\} \tag{7}$$

The solution of the Eq. (7) yields $n$ forces $\{F\}$. The $m$ displacements $\{X\}$ are obtained from the forces $\{F\}$ by back substitution as

$$\{X\} = [J]\{[G]\{F\} + \{\beta_o\}\}$$

where $[J] = m$ rows of $[[S]^{-1}]^T$.

## 3. Formulation of element equilibrium and flexibility matrices for plates

The general formulation steps involved in the development of equilibrium and flexibility matrices of the plate bending elements using the IFM are given below.

The strain energy for the plate in general is written as

$$U_p = \iint 1/2\{k\}^T\{M\}dxdy \tag{8}$$

where $\{k\}$ = vector of curvatures
$\{M\}$ = vector of stress resultants

For the Kirchhoff plate theory the vectors $\{k\}$ and $\{M\}$ are described as

$$\{k\} = \left[\frac{\partial^2 w}{\partial x^2} \quad \frac{\partial^2 w}{\partial y^2} \quad 2\frac{\partial^2 w}{\partial x \partial y}\right]^T \tag{9}$$

$$\{M\} = [M_x \quad M_y \quad M_{xy}]^T \tag{10}$$

For the Mindlin-Reissner theory the vectors $\{k\}$ and {M} are described as

$$\{k\} = \left[\frac{\partial \theta_x}{\partial x} \quad \frac{\partial \theta_y}{\partial y} \quad \frac{\partial \theta_x}{\partial y}+\frac{\partial \theta_y}{\partial x} \quad \theta_y - \frac{\partial w}{\partial y} \quad \theta_x - \frac{\partial w}{\partial x}\right]^T \tag{11}$$

$$\{M\} = [M_x \ M_y \ M_{xy} \ Q_y \ Q_x]^T \tag{12}$$

where $w$, $\theta_x$, $\theta_y$ are the transverse displacement and rotations about $x$ and $y$ axes, respectively.

For a discrete plate bending element, the stress-resultants $\{M\}$ and curvature $\{k\}$ can be expressed in terms of assumed stress-resultant and displacement fields respectively. They can be written in the matrix notations as

$$\{M\} = [\psi]\{F_e\} \tag{13}$$

$$\{k\} = [D_{op}][\phi_1]\{\alpha\} = [D_{op}][\phi]\{X_e\} \tag{14}$$

where

$[\psi]$ = matrix of polynomial terms for stress-resultant fields
$\{F_e\}$ = vector of force components of the discrete element
$[\Phi_1]$ = matrix of polynomial terms for displacement fields
$[\Phi] = [\Phi_1][A]^{-1}$
$[A]$ = matrix formed by substituting the coordinates of the element nodes into the polynomial of displacement fields
$\{\alpha\}$ = coefficients of the displacement field polynomial
$\{X_e\}$ = vector of displacements of the discrete element
$[D_{op}]$ = Differential operator matrix. For the Mindlin-Reissner plate theory it can be expressed as

$$[D_{op}] = \begin{bmatrix} 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & 0 & 1 \\ \frac{\partial}{\partial x} & 1 & 0 \end{bmatrix}$$

And for Kirchhoff plate theory the differential operator matrix becomes

$$[D_{op}] = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & 0 & 0 \\ 0 & \frac{\partial^2}{\partial y^2} & 0 \\ 0 & 0 & \frac{\partial^2}{\partial x \partial y} \end{bmatrix}$$

The curvature moment relations for the plate bending element can be written as

$$\{k\} = [H]\{M\} \tag{15}$$

where $[H]$ is the matrix relating curvatures to stress-resultants and it can be written for the Mindlin-Reissner plate with Reissner's shear correction factor of 5/6 as

$$[H] = \frac{1}{D_1}\begin{bmatrix} 1 & -\nu & 0 & 0 & 0 \\ -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & \frac{t^2(1+\nu)}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{t^2(1+\nu)}{5} \end{bmatrix} \tag{16a}$$

where $D_1 = Et^3/12$; $E$ = Young's modulus; $t$ = thickness of the plate; $\nu$ = Poisson's ratio

And for Kirchhoff plate element, the $[H]$ can be expressed as

$$[H] = \frac{1}{D_1}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \tag{16b}$$

where $D_1$ is same as above

Substituting Eqs. (13) and (14) into the Eq. (8), the strain energy for the discrete element can be expressed as

$$U_p = 1/2\{X_e\}^T[B_e]\{F_e\} \tag{17}$$

where $[B_e]$ represents the element equilibrium matrix and is given by

$$[B_e] = \iint[\phi]^T[D_{op}]^T[\psi]dxdy \tag{18}$$

Using the Eq. (16), the complementary strain energy for the element is expressed as

$$U_c = \iint 1/2\{M\}^T[H]\{M\}dxdy = 1/2\{F_e\}^T[G_e]\{F_e\} \tag{19}$$

where $[G_e]$ represents the element flexibility matrix and is given by

$$[G_e] = \iint[\psi]^T[H][\psi]dxdy \tag{20}$$

The Eqs. (18) and (20) are used to obtain element equilibrium and flexibility matrices $[B_e]$ and $[G_e]$ respectively. These element equilibrium matrix $[B_e]$ and element flexibility matrix $[G_e]$ of all elements are assembled to obtain the global equilibrium matrix $[B]$ and global flexibility matrix $[G]$ of the structure and they are used to setup the IFM governing equation to analyze the structure by IFM.

## 4. About the Code

General purpose code for automatic generation of element equilibrium and flexibility matrices for plate bending element accounting shear and non-shear deformation theories is developed in FORTRAN-90. Code requires the element geometry, structural and material parameters and data describing the displacement and stress-resultant fields as the input. All the symbolic manipulation needed for the development of element equilibrium and flexibility matrices are built into the program and hence code does not need any assistance from other symbolic pre-processors. Structured and modular programming techniques are adopted in the code and hence it is amenable for modification and further developments as and when the need arises.

The data describing displacement and stress-resultant fields with coefficient number, numerical part of the term, power of x and power of y of each polynomial term are stored in separate arrays. The differentiation of the polynomial terms specified in the displacement and stress-resultant fields are carried out with built-in code symbolically. Curvatures are developed and stored in the arrays. Multiplication of vector of curvatures and stress-resultant fields is done through implicit symbolic and numerical computations. Further if the element geometry is square or rectangular the product terms of curvatures and stress-resultant fields are integrated exactly over the domain of the plate to obtain the element equilibrium matrix. In triangular or quadrilateral elements the numerical integration is carried out for the product terms of curvatures and stress-resultant fields to obtain the element equilibrium matrix. The matrix [*H*] which relates the curvatures to stress-resultants is formed using structural and material parameters of the element. The product of [*H*] and the stress-resultant fields is carried out using both numerically and symbolically. Similar integration schemes are used to obtain element flexibility matrix also. The flowchart to obtain the element equilibrium and flexibility matrices for two dimensional plate bending elements is shown in the Fig. 1. This code runs on personal, mini or mainframe computers with DOS or Unix operating system.

## 5. Numerical tests and discussions

The Mindlin-Reissner theory based 4-node plate bending element (MQP4) is considered to illustrate the code developed here. This elements considers three degrees of freedom namely a transverse displacement $w$ and two rotations $\theta_x$, $\theta_y$ at each node. The displacement fields for $w$, $\theta_x$ and $\theta_y$ of MQP4 element are assumed in terms of generalized coordinates $\alpha_1, \alpha_2 \ldots\ldots \alpha_{12}$ as

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy; \quad \theta_x = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy; \quad \theta_y = \alpha_9 + \alpha_{10} x + \alpha_{11} y + \alpha_{12} xy \tag{21}$$

and the assumed stress-resultant fields in terms of polynomials with independent generalized force parameters $F_1, F_2 \ldots\ldots\ldots F_9$ are given as

$$M_x = F_1 + F_2 x + F_3 y + F_4 xy; \; M_y = F_5 + F_6 x + F_7 y + F_8 xy;$$
$$M_{xy} = F_9; \quad Q_y = F_7 + F_8 x; \quad Qx = F_2 + F_4 y \tag{22}$$

The coordinates of the element and three degrees of freedom considered at each node i.e., transverse displacement ($w$), and two rotations $\theta_x$ and $\theta_y$ are shown in the Fig. 3. Parameters of this example problem are : thickness of the plate $t = 1$; Young's modulus $E = 200000$; Poisson's ratio $\nu =$ 0.3 and coordinates of nodes are as shown in the Fig. 2.

Considering the displacement and stress-resultant fields as given in the Eqs. (21) and (22)

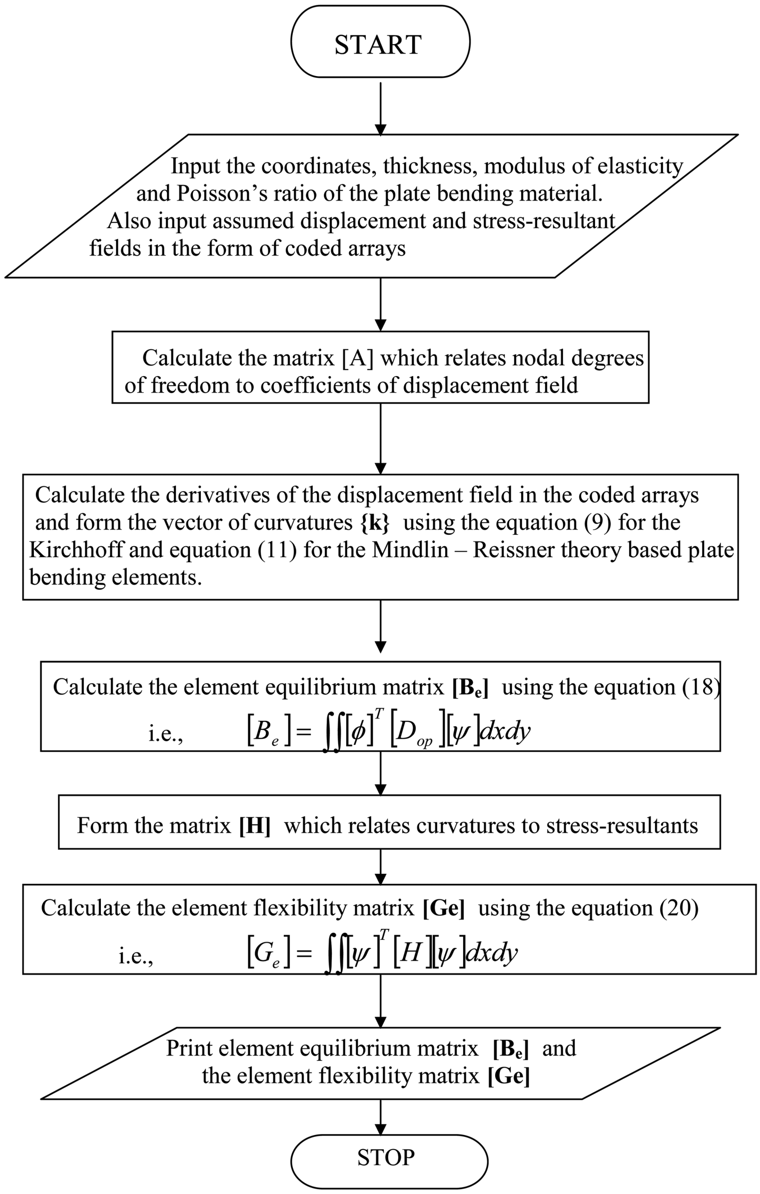


Fig. 1 Flow Chart for the generation of equilibrium and flexibility matrices for plate bending elements

respectively and Eq. (16), element equilibrium and flexibility matrices for the Mindlin-Reissner theory based 4 node quadrilateral plate bending element (MQP4) are obtained using Eqs. (18) and (20) and they are shown in Tables 1 and 2 respectively.

Following example problem has been solved using the equilibrium and flexibility matrices developed by this code for four node plate bending element MQP4 and the results are compared with similar displacement based elements 4-node plate bending elements available in the literature (Chen and Cheung 2000)

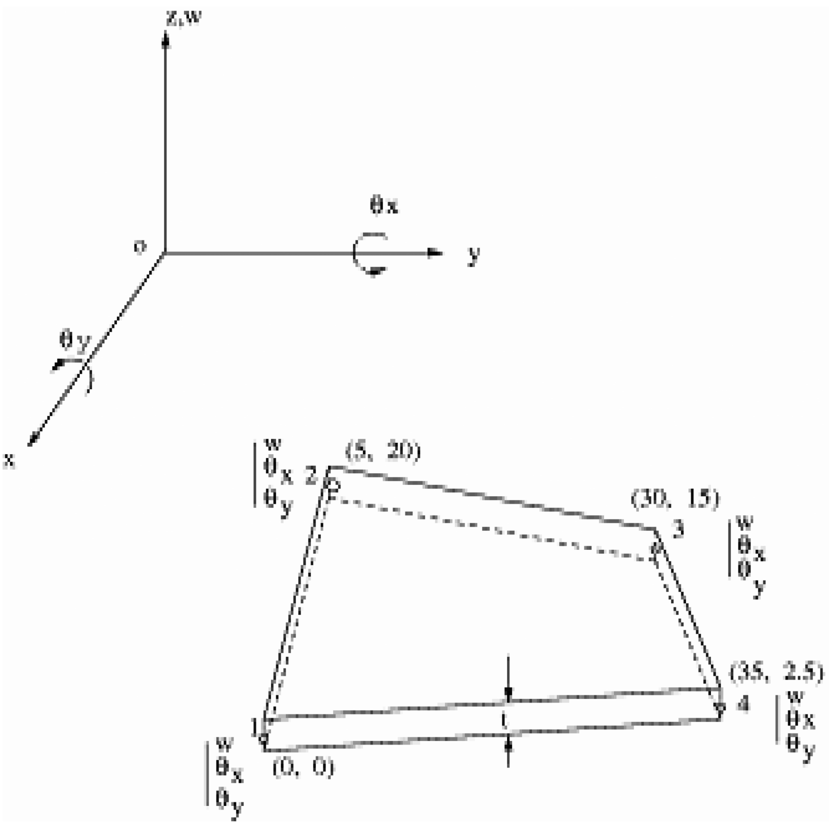


Fig. 2 A typical 4 - node quadrilateral plate bending element

1. A square thin/thick plate with simply supported/clamped boundary conditions subjected to uniform load. The parameter of the problem are: size of the plate = $100 \times 100$, $t = 1$ or 20, $E = 1092000$, $\nu = 0.3$, $q = 1$ (Chen and Cheung 2000)

Because of the symmetry of the geometry, loading and boundary conditions of the plate in the above example problems 1, one quadrant of the plate is considered for the analysis. The typical ($4 \times 4$) mesh considered in one quadrant is as shown in the Fig. 5.

Exact displacements and moments for this plate bending problem with various boundary conditions and loadings are calculated from reference (Timoshenko and Krieger 1959, Jane *et al.* 2000).

Tables 3-6 summarize central deflections and moments of the square thin plate ($t/L = 0.01$) of the example problem 1 for various mesh sizes. Central deflections and moments for square moderately thick plate ($t/L = 0.2$) of the example problem 1 for various mesh sizes are given in the Tables 7-10.

In order to study the shear locking behavior using equilibrium and flexibility matrices developed by the code for the element MQP4, a simply supported square plate subjected to uniform load is analyzed via IFM considering various thickness-span ratios for the mesh size $10 \times 10$ in one quadrant of the plate to estimate the central deflections and moments. The parameters of the problem considered are: $L = 50$, $B = 50$, $t = 5$, 0.5, 0.05, 0.005, 0.0005 and 0.00005, $E = 200000$, $\nu$

Table 1 Equilibrium Matrix [*B*] for the Mindlin-Reissner theory based 4-node quadrilateral plate bending element

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0.0000E+00 | 0.8051E+01 | 0.0000E+00 | 0.5740E+02 | 0.0000E+00 | 0.0000E+00 | 0.1368E+02 | 0.1743E+03 | 0.0000E+00 |
| -0.8051E+01 | 0.6143E+01 | -0.5740E+02 | -0.4721E+02 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | -0.1368E+02 |
| 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | -0.1368E+02 | -0.1743E+03 | 0.1442E+02 | 0.2573E+02 | -0.8051E+01 |
| 0.0000E+00 | 0.8594E+01 | 0.0000E+00 | 0.1004E+03 | 0.0000E+00 | 0.0000E+00 | -0.1294E+02 | -0.1348E+03 | 0.0000E+00 |
| -0.8594E+01 | -0.1288E+02 | -0.1004E+03 | -0.6470E+02 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.1294E+02 |
| 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.1294E+02 | 0.1348E+03 | 0.2432E+03 | 0.2719E+04 | -0.8594E+01 |
| 0.0000E+00 | -0.1042E+02 | 0.0000E+00 | -0.1290E+03 | 0.0000E+00 | 0.0000E+00 | -0.1815E+02 | -0.4115E+03 | 0.0000E+00 |
| 0.1042E+02 | 0.2718E+03 | 0.1290E+03 | 0.3250E+04 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.1815E+02 |
| 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.1815E+02 | 0.4115E+03 | 0.2649E+03 | 0.5848E+04 | 0.1042E+02 |
| 0.0000E+00 | -0.6223E+01 | 0.0000E+00 | -0.2873E+02 | 0.0000E+00 | 0.0000E+00 | 0.1740E+02 | 0.3719E+03 | 0.0000E+00 |
| 0.6223E+01 | 0.2225E+03 | 0.2873E+02 | 0.1237E+04 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | -0.1740E+02 |
| 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | -0.1740E+02 | -0.3719E+03 | -0.3503E+02 | -0.6557E+03 | 0.6223E+01 |

Table 2 Flexibility Matrix [*G*] for Mindlin-Reissner theory based 4-node quadrilateral plate bending element

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0.2925E-01 | 0.4762E+00 | 0.2625E+00 | 0.4214E+01 | -0.8775E-02 | -0.1429E+00 | -0.7875E-01 | -0.1264E+01 | 0.0000E+00 |
| 0.4762E+00 | 0.9973E+01 | 0.4214E+01 | 0.8546E+02 | -0.1429E+00 | -0.2990E+01 | -0.1264E+01 | -0.2562E+02 | 0.0000E+00 |
| 0.2625E+00 | 0.4214E+01 | 0.3033E+01 | 0.4693E+02 | -0.7875E-01 | -0.1264E+01 | -0.9098E+00 | -0.1408E+02 | 0.0000E+00 |
| 0.4214E+01 | 0.8546E+02 | 0.4693E+02 | 0.9144E+03 | -0.1264E+01 | -0.2562E+02 | -0.1408E+02 | -0.2741E+03 | 0.0000E+00 |
| -0.8775E-02 | -0.1429E+00 | -0.7875E-01 | -0.1264E+01 | 0.2925E-01 | 0.4762E+00 | 0.2625E+00 | 0.4214E+01 | 0.0000E+00 |
| -0.1429E+00 | -0.2990E+01 | -0.1264E+01 | -0.2562E+02 | 0.4762E+00 | 0.9966E+01 | 0.4214E+01 | 0.8539E+02 | 0.0000E+00 |
| -0.7875E-01 | -0.1264E+01 | -0.9098E+00 | -0.1408E+02 | 0.2625E+00 | 0.4214E+01 | 0.3040E+01 | 0.4705E+02 | 0.0000E+00 |
| -0.1264E+01 | -0.2562E+02 | -0.1408E+02 | -0.2741E+03 | 0.4214E+01 | 0.8539E+02 | 0.4705E+02 | 0.9162E+03 | 0.0000E+00 |
| 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.0000E+00 | 0.7605E-01 |

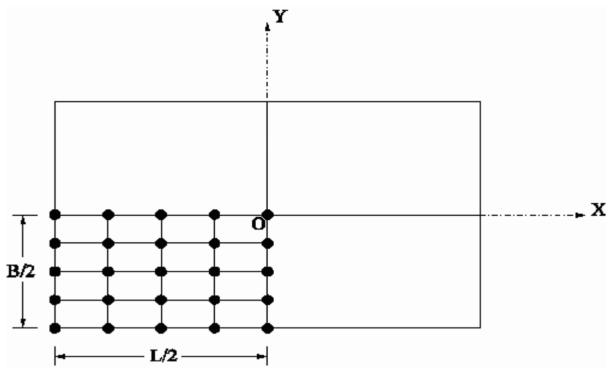


Fig. 3 A typical (4 × 4) mesh in one quadrant of the rectangular plate

Table 3 Central deflection for a simply supported square plate with uniform load ($t/L$ = 0.01)

| | | $W_c$ ($10^{-5}qL^4/D$) | | | (Example Problem 1) | |
|---|---|---|---|---|---|---|
| Elements | Q4 | Q4-R | DKQ | MITC4 | RDKQM | MQP4 |
| 1 × 1 | 1.1 | 319.2 | 378.5 | 319.1 | 378.7 | 496.5 |
| 2 × 2 | 4.5 | 397.1 | 404.6 | 397.1 | 404.8 | 440.9 |
| 4 × 4 | 17.3 | 404.4 | 406.0 | 404.4 | 406.2 | 415.5 |
| 6 × 6 | 36.9 | 405.5 | 406.1 | 405.5 | 406.3 | 409.9 |
| 8 × 8 | 61.3 | 405.9 | 406.2 | 405.9 | 406.3 | 408.8 |
| 10 × 10 | 88.3 | 406.1 | 406.2 | 406.1 | 406.4 | 407.9 |

Exact = 407

Table 4 Central moment for a simply supported square plate with uniform load ($t/L$ = 0.01)

| | | $M_c$ | | | (Example Problem 1) | |
|---|---|---|---|---|---|---|
| Elements | Q4 | Q4-R | DKQ | MITC4 | RDKQM | MQP4 |
| 1 × 1 | 0.833 | 331.6 | 603.1 | 331.6 | 603.6 | 899.3 |
| 2 × 2 | 5.119 | 477.0 | 501.0 | 477.1 | 501.5 | 580.7 |
| 4 × 4 | 20.81 | 479.0 | 483.9 | 479.0 | 484.2 | 502.4 |
| 6 × 6 | 44.74 | 478.9 | 405.5 | 478.9 | 481.4 | 488.7 |
| 8 × 8 | 74.38 | 478.9 | 480.1 | 478.9 | 480.4 | 484.6 |
| 10 × 10 | 107.0 | 478.9 | 479.6 | 478.9 | 480.0 | 482.6 |

Exact = 479

Table 5 Central deflection for a clamped square plate with uniform load ($t/L$ = 0.01)

| | | $W_c$ ($10^{-5}qL^4/D$) | | | (Example Problem 1) | |
|---|---|---|---|---|---|---|
| Elements | Q4 | Q4-R | DKQ | MITC4 | RDKQM | MQP4 |
| 1 × 1 | 0.3 | 0.4 | 156.2 | 0.3 | 156.5 | 182.0 |
| 2 × 2 | 1.0 | 121.4 | 146.1 | 121.3 | 146.3 | 158.3 |
| 4 × 4 | 3.7 | 125.3 | 131.9 | 125.3 | 132.2 | 135.3 |
| 6 × 6 | 7.8 | 126.1 | 129.0 | 126.1 | 129.2 | 130.5 |
| 8 × 8 | 13.2 | 126.4 | 127.9 | 126.4 | 128.1 | 129.0 |
| 10 × 10 | 19.5 | 126.6 | 127.4 | 126.5 | 127.6 | 128.2 |

Exact = 126.6

Table 6 Central moment for a clamped square plate with uniform load ($t/L$ = 0.01)

| | | $M_c$ | | | (Example Problem 1) | |
|---|---|---|---|---|---|---|
| Elements | Q4 | Q4-R | DKQ | MITC4 | RDKQM | MQP4 |
| 1 × 1 | 0.0 | 0.0 | 487.5 | 0.0 | 487.6 | 566.8 |
| 2 × 2 | 1.57 | 251.4 | 287.3 | 251.7 | 287.7 | 324.6 |
| 4 × 4 | 6.90 | 232.9 | 243.3 | 233.1 | 243.6 | 255.7 |
| 6 × 6 | 15.44 | 230.7 | 235.4 | 230.9 | 235.6 | 240.7 |
| 8 × 8 | 25.62 | 230.0 | 232.6 | 230.1 | 232.9 | 235.7 |
| 10 × 10 | 37.71 | 229.7 | 231.3 | 229.7 | 231.6 | 233.3 |

Exact = 231

Table 7 Central deflection for a simply supported square plate with uniform load ($t/L$ = 0.2)

| | | $W_c(10^{-5}qL^4/D)$ | | | (Example Problem 1) | |
|---|---|---|---|---|---|---|
| Elements | Q4 | Q4-R | DKQ | MITC4 | RDKQM | MQP4 |
| 1 × 1 | 267 | 461.7 | 378.5 | 426.0 | 468.6 | 612.8 |
| 2 × 2 | 411 | 492.1 | 404.6 | 485.5 | 485.8 | 541.2 |
| 4 × 4 | 468 | 490.5 | 406.0 | 489.4 | 488.9 | 506.7 |
| 6 × 6 | 480 | 490.5 | 406.1 | 490.0 | 489.7 | 497.5 |
| 8 × 8 | 485 | 490.4 | 406.2 | 490.2 | 490.0 | 495.1 |
| 10 × 10 | 487 | 490.4 | 406.2 | 490.3 | 490.2 | 493.5 |

Exact = 490

Table 8 Central moment for a simply supported square plate with uniform load ($t/L$ = 0.2)

| | | $M_c$ | | | (Example Problem 1) | |
|---|---|---|---|---|---|---|
| Elements | Q4 | Q4-R | DKQ | MITC4 | RDKQM | MQP4 |
| 1 × 1 | 166 | 331.6 | 603.1 | 331.6 | 685.5 | 884.7 |
| 2 × 2 | 386 | 471.1 | 501.0 | 475.0 | 543.0 | 586.9 |
| 4 × 4 | 454 | 479.0 | 483.9 | 478.9 | 497.3 | 507.2 |
| 6 × 6 | 468 | 478.9 | 481.1 | 478.9 | 487.3 | 491.6 |
| 8 × 8 | 473 | 478.9 | 480.1 | 478.9 | 483.7 | 486.4 |
| 10 × 10 | 475 | 478.9 | 479.6 | 478.9 | 482.0 | 483.7 |

Exact = 479

Table 9 Central deflection for a clamped square plate with uniform load ($t/L$ = 0.2)

| | | $W_c(10^{-5}qL^4/D)$ | | | (Example Problem 1) | |
|---|---|---|---|---|---|---|
| Elements | Q4 | Q4-R | DKQ | MITC4 | RDKQM | MQP4 |
| 1 × 1 | 107.1 | 147.2 | 156.2 | 107.1 | 260.4 | 289.9 |
| 2 × 2 | 175.7 | 217.7 | 146.1 | 209.4 | 235.0 | 247.5 |
| 4 × 4 | 204.7 | 217.4 | 131.9 | 215.7 | 221.7 | 224.4 |
| 6 × 6 | 211.4 | 217.3 | 129.0 | 216.6 | 219.2 | 220.4 |
| 8 × 8 | 213.9 | 217.2 | 127.9 | 216.9 | 218.3 | 219.1 |
| 10 × 10 | 215.1 | 217.2 | 127.4 | 217.0 | 217.9 | 218.4 |

Exact = 217

Table 10 Central moment for a clamped square plate with uniform load ($t/L$ = 0.2)

| | | $M_c$ | | | (Example Problem 1) | |
|---|---|---|---|---|---|---|
| Elements | Q4 | Q4-R | DKQ | MITC4 | RDKQM | MQP4 |
| 1 × 1 | 0.00 | 0.00 | 487.5 | 0.00 | 524.7 | 555.8 |
| 2 × 2 | 158.2 | 213.8 | 287.3 | 220.1 | 316.7 | 326.7 |
| 4 × 4 | 215.9 | 235.4 | 243.3 | 235.1 | 256.5 | 258.0 |
| 6 × 6 | 226.8 | 235.7 | 235.4 | 235.5 | 245.0 | 245.3 |
| 8 × 8 | 230.7 | 235.7 | 232.6 | 235.6 | 241.0 | 241.2 |
| 10 × 10 | 235.7 | 235.7 | 231.3 | 235.7 | 239.1 | 239.3 |

Exact = 231

= 0.3, $q$ = 1. The exact central displacements and moments are calculated from the Kirchhoff theory (Timoshenko and Krieger 1959) and the Mindlin plate theory (Jane Liu *et al.* 2000) solutions for thin and moderately thick plate problems respectively. These results are shown in the Figs. 4 and 5. These Figs. indicate that there is no shear locking behaviour using equilibrium and flexibility matrices developed by the code for the element MQP4.

Above results indicate that equilibrium and flexibility matrices developed by the code for the 4-node element MQP4 can be used for the analyzing thin and moderately thick plate bending elements using Integrated Force Method.

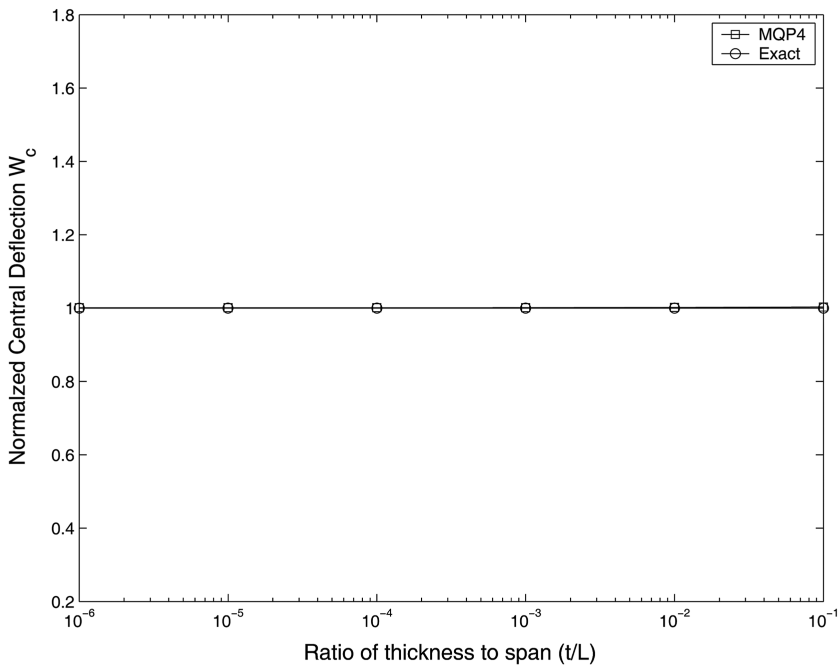


Fig. 4 Normalized central deflections for simply supported plate with uniform load ($t/L$ = 0.1, 0.01, 0.001, 0.0001, 0.00001, 0.000001)

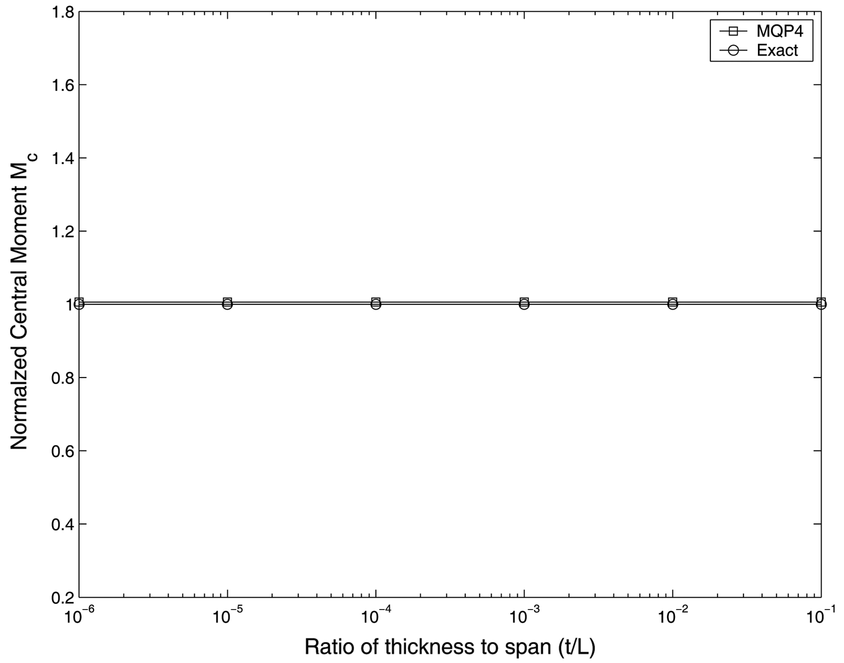


Fig. 5 Normalized central moments for simply supported plate with uniform load ($t/L = 0.1$, 0.01, 0.001, 0.0001, 0.00001, 0.000001)

## 6. Conclusions

Fully automated and independent general purpose code for the generation of element equilibrium and flexibility matrices of the plate bending elements using Integrated Force Method has been developed in FORTRAN-90. The Kirchhoff and the Mindlin-Reissner plate theories have been used in the formulation while developing the code. Structured and modular programming techniques are adopted. Results indicate that element equilibrium and flexibility matrices obtained from code can be used for analyzing thin and moderately thick plate bending problems using Integrated Force Method.

## Notations

| | |
|---|---|
| $[A]$ | = matrix relating nodal degrees of freedom and coefficients of the polynomial |
| $[B]$ | = global equilibrium matrix ($m \times n$) |
| $[B_e]$ | = element equilibrium matrix ($m_e \times n_e$) |
| $[C]$ | = compatibility matrix ($r \times n$) |
| $[D_{op}]$ | = differential operator matrix |
| $E$ | = Young's modulus |
| $\{F\}$ | = vector of internal forces of the structure ($n \times 1$) |
| $\{F_e\}$ | = vector of internal forces of the discrete element ($n_e \times 1$) |
| $[G]$ | = global flexibility matrix ($n \times n$) |

| | |
|---|---|
| $[G_e]$ | = element flexibility matrix ($n_e \times n_e$) |
| $[H]$ | = matrix relating the curvatures to stress-resultants |
| $[J]$ | = deformation coefficient matrix ($m \times n$) |
| $L$, $B$ | = Length and breadth of the plate |
| $M_c$ | = central moment of the plate |
| $\{M\}$ | = vector of stress – resultants |
| $P$ | = point load at the center or tip of the plate |
| $\{P\}$ | = vector of external loads ($m \times 1$) |
| $q$ | = uniform load over the plate |
| $[S]$ | = IFM governing matrix ($n \times n$) |
| $W_c$ | = Central deflection of the plate |
| $\{X\}$ | = vector of displacements of the structure ($m \times 1$) |
| $\{X_e\}$ | = vector of displacements of the discrete element ($m_e \times 1$) |
| $a$, $b$ | = length and breadth of the plate bending element |
| $\{k\}$ | = vector of curvatures |
| $n$, $m$ | = force and displacement degrees of freedom of the structures respectively |
| $n_e$, $m_e$ | = element force and displacement degrees of freedom respectively |
| $t$ | = thickness of the plate |
| $\{\alpha\}$ | = generalized coordinates of the polynomial in the displacement field. |
| $\{\beta\}$ | = vector of elastic deformations |
| $\{\beta_o\}$ | = vector of initial deformations |
| $\nu$ | = Poisson's ratio |
| $[\Phi_1]$ | = matrix of polynomial terms for displacement fields |
| $[\Psi]$ | = matrix of polynomial terms for stress-resultants fields |

## References


Beltzer A.L. (1990), “Engineering analysis via symbolic computations a break through”, *Appl. Mech. Review*, **43**, 119-127.

Chang, T.Y., Tan H.Q., Zheng D. and Yuan M.W. (1990), “Application of symbolic method to hybrid or mixed finite elements and computer implementation”, *Comput. Struct.*, **35**(4), 293-299.

Cecchi, M.M. and Lami, C. (1977, online 2005), “Automatic generation of stiffness matrices for the finite element analysis”, *Int. J. Numer. Meth. Eng.*, **11**, 396-400.

Chee, C., Tong, L., and Steven, G. (2000), “Shape function generations using MACSYMA”, *Commun. Numer. Meth. Eng.*, **16**, 705-719.

Chen Wanji and Cheun Y.K. (2000), “Refined quadrilateral element based on Mindlin/Reissner plate theory”, *Int. J. Numer. Meth. Eng.*, **47**, 605-627.

Foster, K.R. and Bau, H.H. (1989), “Symbolic computation programs for the personal computers”, *Science*, **243**, 679-684.

Griffiths, D.V. (1994), “Stiffness matrix of the four node quadrilateral element in closed form”, *Int. J. Numer. Meth. Eng.*, 1994, **37**, 1028-1038.

Gunderson, R.H. and Cetiner (1971), “Element stiffness matrix generator”, *J. Struct. Div.*, ASCE, **97**, 363-375.

Hoa, S.V. and Shankar, S. (1980), “A computer program for automatic generation of stiffness and mass matrices in finite element analysis”, *Comput. Struct.*, **11**, 147-161.

Jane Liu, Riggs, H.R. and Alexander Tessler. (2000), “A four node shear-deformable shell element developed via explicit Kirchhoff constraints”, *Int. J. Numer. Meth. Eng.*, **49**, 1065-1086.

Korncoff, A.R. and Fenves, S.J. (1979), “Symbolic generation of finite element stiffeness matrices,” *Comput. Struct.*, **10**, 119-124.

Lee, C.K. and Hobbs, R.E. (1998), “Closed form stiffness matrix solutions for some commonly used hybrid finite elements”, *Comput. Struct.*, **67**, 463-482.

Luft, R.W., Roesset, J.M. and Connor, J.J. (1971), “Automatic generation of finite element matrices”, *J. Struct.*

*Div.*, ASCE, **97**, 349-362.
Krishnam Raju, N.R.B. and Nagabhushanam, J. (2000), "Non-linear structural analysis using integrated force method", *Sadhana*, **25**(4), 353-365.
Mbakogu, F.C. and Pavlovic, M.N. (1999), "Some applications of computer algebra to classical problems in plate theory", *Int. J. Mech. Eng. Edu.*, **27**, 348-365.
Nagabhushanam, J. and Patnaik, S.N. (1990), "General purpose program to generate compatibility matrix for the integrated force method", *AIAA J.*, **28**, 1838-1842.
Nagabhushanam, J. and Srinivas, J. (1991), "Automatic generation of sparse and banded compatibility matrix for the integrated force method", *Computer Mechanics '91, International Conference on Computing in Engineering Science*, Patras, Greece, 20-25.
Nagabhushanam, J., Srinivas, C.S. and Gaonkar, G.H. (1992), "Symbolic generation of elemental matrices for finite elemental analysis", *Comput. Struct.*, **42**(3), 375-380
Noor, A.K. and Andersen, C.M. (1979), "Computerized symbolic manipulations in structural mechanics—progress and potential", *Comput. Struct.*, **10**, 95-118.
Patnaik, S.N. (1973), "An integrated force method for discrete analysis", *Int. J. Numer. Meth. Eng.*, **41**, 237-251.
Patnaik, S.N. (1986), "Integrated force method verses standard force method", *Comput. Struct.*, **22**(2), 151-163.
Patnaik, S.N., Hopkins, D.A. and Coroneos, R. (1996), "Structural optimization with approximate sensitivities", *Comput. Struct.*, **58**, 407-418.
Pavlovic, M.N. (2003), "Review article on symbolic computation in structural engineering", *Comput. Struct.*, **81**, 2121-2136.
Timoshenko, S.P. and Krieger, S.W. (1959), *Theory of Plates and Shells*, Second Edition, McGraw Hill, New York.