Structural Engineering and Mechanics, Vol. 30, No. 3 (2008) 369-382
DOI: http://dx.doi.org/10.12989/sem.2008.30.3.369

Dynamics of an elastic beam and a jumping oscillator moving in the longitudinal direction of the beam

Luis Baeza

Dpto. de Ingeniería Mecánica, Universidad Politécnica de Valencia, Camino de Vera, 14 E46022 Valencia, Spain

Huajiang Ouyang†

Department of Engineering, University of Liverpool, Liverpool L69 3GH, U.K.

(Received March 7, 2008, Accepted August 8, 2008)

Abstract. An oscillator of two lumped masses linked through a vertical spring moves forward in the horizontal direction, initially at a certain height, over a horizontal Euler beam and descends on it due to its own weight. Vibration of the beam and the oscillator is excited at the onset of the ensuing impact. The impact produced by the descending oscillator is assumed to be either perfectly elastic or perfectly plastic. If the impact is perfectly elastic, the oscillator bounces off and hits the beam a number of times as it moves forward in the longitudinal direction of the beam, exchanging its dynamics with that of the beam. If the impact is perfectly plastic, the oscillator (initially) sticks to the beam after its first impact and then may separate and reattach to the beam as it moves along the beam. Further events of separation and reattachment may follow. This interesting and seemingly simple dynamic problem actually displays rather complicated dynamic behaviour and has never been studied in the past. It is found through simulated numerical examples that multiple events of separation and impact can take place for both perfectly elastic impact and perfectly plastic impact (though more of these in the case of perfectly elastic impact) and the dynamic response of the oscillator and the beam looks noisy when there is an event of impact because impact excites higher-frequency components. For the perfectly plastic impact, the oscillator can experience multiple events of consecutive separation from the beam and subsequent reattachment to it.

Keywords: jumping, moving oscillator, Euler beam, vibration, separation, impact.

1 Introduction

Vibration of an elastic beam and an oscillator of one mass-one spring or of two masses linked by one spring that traverses a beam is an extensively studied problem (Fryba 1999, Pesterev and Bergman 1997, Yang *et al.* 2000, Metrikine and Verichev 2001). A moving system of multiple masses and springs representing a vehicle (Fryba 1968, Yang *et al.* 2004, Lou 2005) and a moving flexible body (Ouyang and Mottershead 2007) were also studied in the very recent past. It is usually assumed that the contact between the moving-oscillator and the beam is always maintained

† Ph.D., Corresponding author, E-mail: h.ouyang@liverpool.ac.uk

throughout its horizontal travel along the beam. This so-called moving-oscillator problem as a special case of general moving load-problems has been solved and does no longer present any difficulties, when constant contact is assumed.

A few theoretical works indicated that separation between the moving object and the supporting beam was possible (Fryba 1999, Lee 1995). Lee (1996) seems to be the first researcher that investigated separation between a moving object and a supporting structure in the context of a moving-mass problem. A known theorem in matrix theory (Meyer 2000) for the analytical expression of inverting the time-dependent mass matrix of Lee's moving-mass problem was suggested by Pesterev and Bergman (1998). Although separation was considered by Lee (1996), the reattachment of the moving mass to the beam after separation was very simplistic in that the impact between the moving mass and the beam at the instant of reattachment was excluded. Cheng *et al.* (1999) considered separation and in particular reattachment in their moving multi-degree-of-freedom oscillator problem, which was the first investigation that considered both separation and reattachment in a sensible way. The approach to impact at the reattachment was novel but quite complicated. Stancioiu *et al.* (2008) considered separation and also suggested a simple approach to impact at the reattachment in their study of a moving-oscillator problem. There are two major types of complexities in the above investigations. One is the constant monitoring of separation and reattachment and switching between two sets of equations for these two different regimes, and the other is dealing with the reattachment. At reattachment, there is usually an impact because of the different velocities of two colliding bodies (the oscillator and the beam), which is a sophisticated branch of physics in its own right. So how to deal with impact, which may happen several times in a moving-load problem (Stancioiu *et al.* 2008), is an issue of on-going research interest. Impact excites higher frequencies and the velocity at the instant of impact is discontinuous. Moving-load problems considering separation and reattachment belong to non-smooth dynamics systems and are difficult to study.

In this paper, the vibration of a jumping oscillator and its induced vibration of the beam are studied. When a vehicle runs downhill towards a bridge, if the travelling speed is high enough, it may leave the ground for a brief moment and lands onto the bridge from the above. On the other hand, if there is a ramp in front of a bridge, a fast travelling vehicle may also leave the ground briefly and lands on the bridge with a vertical speed. Both situations can be described simplistically by the present dynamic model. The most likely situation would be a shallow downward bend (for example, due to soil settlement) in the road leading up to a bridge. Certain packaging processes in which a robot arm regularly operates on objects on a moving conveyor belt also share some basic features of this model.

If impact between the oscillator and the beam is assumed to be perfectly elastic, the oscillator hits the beam and then bounces off repeatedly while it moves in the longitudinal direction of the beam. This moving-load problem has not been studied in the past to the authors' best knowledge. The real impact should be between a perfectly elastic impact and a perfectly plastic one, the latter of which was studied by Cheng *et al.* (1999) and Stancioiu *et al.* (2008). For the jumping oscillator studied in this paper, there are more events of separation and impact-reattachment than the same moving oscillator simply sliding along the beam. Therefore the numerical integration algorithm used in this paper must be more accurate and be capable of capturing the instants of separation and reattachment in a satisfactory manner.

2. Dynamic model

A moving oscillator, shown in Fig. 1, is initially at a vertical height of H above an Euler beam and moves at a constant speed during its course of horizontal motion. Due to its own weight, it gradually descends and hits the beam below, as shown in Fig. 2. Both the oscillator and the beam then start to vibrate. More than one scenario is possible afterwards. The oscillator may bounce off and continue to travel forward in the air, and then hit the beam again. And this process continues until the oscillator leaves the beam. The oscillator may stick to the beam and moves along the beam like a conventional moving-oscillator problem. Depending on the parameter values involved, the oscillator may then continue its horizontal motion without separation, or may separate from the beam throughout the rest of its horizontal journey, or may separate and then reattach to the beam again and could have repeated events of consecutive separation and reattachment. All these possible scenarios are studied in this paper.

The equation of the transverse motion $w(x,t)$ of the beam under a concentrated contact force f_c (defined as positive when it is compressive) from the oscillator moving on the beam at constant speed v is

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} = -f_c(t)\,\delta(x-vt) \tag{1}$$

where EI is the flexural rigidity and ρA is the mass per unit length of the beam. δ is Dirac's delta function. It should be noted that the horizontal motion of the force (from the oscillator) does not

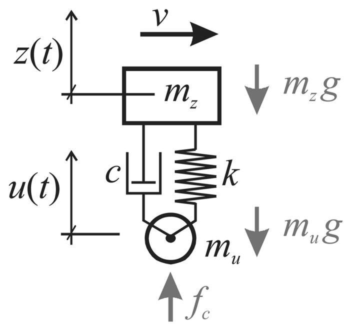

Fig. 1 Moving oscillator

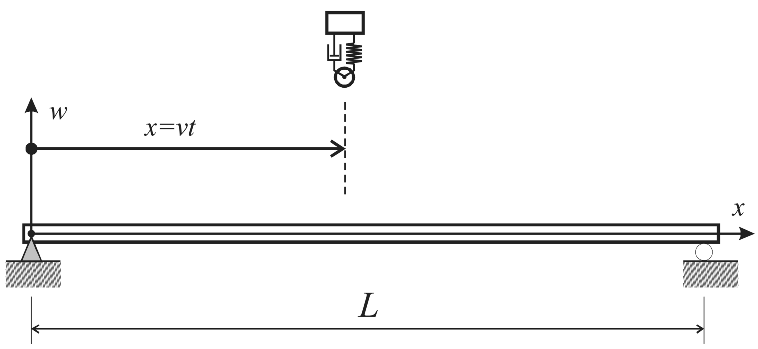

Fig. 2 Moving oscillator and a simply-supported elastic beam

have to be of constant speed and may be an arbitrary known function of time t. When the oscillator is not in contact with the beam, $f_c = 0$, and both the beam and the oscillator undergo free vibration during separation.

The transverse vibration of the beam may be expressed as a modal expansion below

$$w(x,t) = \sum_{j=1}^{\infty} \psi_j(x)q_j(t) = \boldsymbol{\psi}^T(x)\mathbf{q}(t) \tag{2}$$

where q_j is the j-th modal coordinate and ψ_j is the mass-normalised j-th mode of the beam. $\boldsymbol{\psi} = \{\psi_1, \psi_2, \psi_3, \ldots\}^T$ and $\mathbf{q} = \{q_1, q_2, q_3, \ldots\}^T$. T as a superscript stands for matrix or vector transpose. The mass-normalisation of the beam modes means

$$\rho A \int_0^L \boldsymbol{\psi}(x)\boldsymbol{\psi}^T(x)\mathrm{d}x = \mathbf{I}, \quad EI\int_0^L \boldsymbol{\psi}(x)\frac{\partial^4 \boldsymbol{\psi}^T(x)}{\partial x^4}\mathrm{d}x = \mathrm{diag}[\omega_i^2]$$

where $\mathbf{I}$ is identity matrix of appropriate dimension and $\mathrm{diag}[\omega_i^2]$ denotes a diagonal matrix composed of natural frequencies of the beam squared and ranked in ascending order.

Substituting Eqs. (2) into (1) and further mathematical manipulation yield

$$\ddot{q}_i + \omega_i^2 q_i = -\psi_i(vt)f_c(t) \quad (i = 1,2,3,\ldots) \quad \text{or} \quad \ddot{\mathbf{q}} + \mathrm{diag}[\omega_i^2]\mathbf{q} = -f_c(t)\boldsymbol{\psi}(vt) \tag{3}$$

where the overhead dot in this paper represents the differentiation with respect to time t.

The equations of the vertical motion for the sprung and unsprung masses of the oscillator are

$$m_z\ddot{z} + c(\dot{z}-\dot{u}) + k(z-u) = W_z \tag{4}$$

$$m_u\ddot{u} + c(\dot{u}-\dot{z}) + k(u-z) = W_u + f_c \tag{5}$$

where $W_z = -m_z g$ and $W_u = -m_u g$ are the weights of the two masses, in which g is the acceleration due to gravitation.

When the oscillator is in contact with the beam, it is assumed that the vibration of the oscillator and that of the beam at the contact point are equal, which is adopted in most moving-load problems. Therefore whenever there is contact,

$$u(t) = w(vt,t) = \boldsymbol{\psi}^T(vt)\mathbf{q}(t) \tag{6}$$

and hence

$$\begin{aligned} \dot{u}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}[w(vt,t)] = \boldsymbol{\psi}^T(vt)\dot{\mathbf{q}}(t) + v\boldsymbol{\psi}'^T(vt)\mathbf{q}(t) \\ \ddot{u}(t) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}[w(vt,t)] = \boldsymbol{\psi}^T(vt)\ddot{\mathbf{q}}(t) + 2v\boldsymbol{\psi}'^T(vt)\dot{\mathbf{q}}(t) + v^2\boldsymbol{\psi}''^T(vt)\mathbf{q}(t) \end{aligned} \tag{7}$$

where the dash in the paper stands for the differentiation with respect to the horizontal coordinate x. It should be pointed out that when the oscillator is not in contact with the beam, $\dot{u}(t)$ is unrelated to the velocity of the beam and must be found from Eqs. (4) and (5).

3. Separation and reattachment

When the oscillator hits the beam from above, impact takes place. The equation of motion of the beam under this impact at time t_r in the modal coordinates, adapted from Stancioiu *et al.* (2008), is

$$\ddot{\mathbf{q}} + \mathrm{diag}[\omega_i^2]\mathbf{q} = -p\boldsymbol{\psi}(vt_r)\delta(t-t_r) \tag{8}$$

where p is the impulse that occurs at the instant of the impact. It is crucial to derive the modal velocity of the beam and the velocity of the unsprung mass immediately after the impact, as these are the initial velocities for the ensuing vertical vibration that is different from the vertical vibration before the impact.

Time-integration of Eq. (8) in the time-domain before and after the impact yields

$$\dot{\mathbf{q}}(t_r^+) - \dot{\mathbf{q}}(t_r^-) = -p\boldsymbol{\psi}(vt_r) \tag{9}$$

where t_r^- and t_r^+ are the instant just before and just after the impact. It follows from Eq. (9) that

$$\dot{w}(x,t_r^+) - \dot{w}(x,t_r^-) = -p\boldsymbol{\psi}^T(x)\boldsymbol{\psi}(vt_r) \tag{10}$$

For the moving oscillator, similarly

$$\dot{u}(t_r^+) - \dot{u}(t_r^-) = \frac{p}{m_u} \tag{11}$$

Eqs. (9) and (11) reveal that there is a discontinuity in the velocity of the unsprung mass and the velocity of the beam at t_r. The instant of impact t_r should be captured accurately. Otherwise the resultant error will propagate through numerical time-domain integration and lead to poor results over time.

Combining Eqs. (9) and (11) gives

$$\dot{\mathbf{q}}(t_r^+) - \dot{\mathbf{q}}(t_r^-) = -m_u[\dot{u}(t_r^+) - \dot{u}(t_r^-)]\boldsymbol{\psi}(vt_r) \tag{12}$$

Two simplest types of impact, perfectly elastic and perfectly plastic, are studies below. In general, an impact will be neither perfectly elastic nor perfectly plastic, but somewhere in between. Interested readers may refer to Goldsmith (2001) for more general cases of impact. It was found that different types of impact represented by different values of restitution coefficient had some local influence on the dynamic response (Baeza and Ouyang 2008).

3.1 Perfectly elastic impact

If the impact is perfectly elastic, the total energy before and after the impact are the same. The difference between these two energy values is then

$$\Delta W = \frac{\rho A}{2}\int_0^L \{[\dot{w}(x,t_r^+)]^2 - [\dot{w}(x,t_r^-)]^2\}\mathrm{d}x + \frac{m_u}{2}\{[\dot{u}(t_r^+)]^2 - [\dot{u}(t_r^-)]^2\}$$

$$= \frac{1}{2}\dot{\mathbf{q}}^T(t_r^+)\left(\rho A\int_0^L \boldsymbol{\psi}(x)\boldsymbol{\psi}^T(x)\mathrm{d}x\right)\dot{\mathbf{q}}(t_r^+) - \frac{1}{2}\dot{\mathbf{q}}^T(t_r^-)\left(\rho A\int_0^L \boldsymbol{\psi}(x)\boldsymbol{\psi}^T(x)\mathrm{d}x\right)\dot{\mathbf{q}}(t_r^-) + \frac{m_u}{2}\{[\dot{u}(t_r^+)]^2 - [\dot{u}(t_r^-)]^2\}$$
$$= \frac{1}{2}[\dot{\mathbf{q}}^T(t_r^+)\dot{\mathbf{q}}(t_r^+) - \dot{\mathbf{q}}^T(t_r^-)\dot{\mathbf{q}}(t_r^-)] + \frac{m_u}{2}\{[\dot{u}(t_r^+)]^2 - [\dot{u}(t_r^-)]^2\} = 0 \tag{13}$$

where $\left(\rho A\int_0^L \boldsymbol{\psi}(x)\boldsymbol{\psi}^T(x)\mathrm{d}x\right) = \mathbf{I}$ has been used in the mathematical manipulation of Eq. (13). Please notice that just before and just after the impact the values of the beam's potential energy are the same and the values of the kinetic energy of the sprung mass are the same too, and therefore these terms do not appear in Eq. (13).

Substituting Eqs. (12) into (13) and solving for $\dot{u}(t_r^+)$ yields

$$\dot{u}(t_r^+) = \frac{[m_u\boldsymbol{\psi}^T(vt_r)\boldsymbol{\psi}(vt_r) - 1]\dot{u}(t_r^-) + 2\boldsymbol{\psi}^T(vt_r)\dot{\mathbf{q}}(t_r^-)}{m_u\boldsymbol{\psi}^T(vt_r)\boldsymbol{\psi}(vt_r) + 1} \tag{14}$$

and then from Eq. (12) one gets

$$\dot{\mathbf{q}}(t_r^+) = \frac{2m_u\dot{u}(t_r^-)\boldsymbol{\psi}(vt_r)}{m_u\boldsymbol{\psi}^T(vt_r)\boldsymbol{\psi}(vt_r) + 1} + \left[\mathbf{I} - \frac{2m_u\boldsymbol{\psi}(vt_r)\boldsymbol{\psi}^T(vt_r)}{m_u\boldsymbol{\psi}^T(vt_r)\boldsymbol{\psi}(vt_r) + 1}\right]\dot{\mathbf{q}}(t_r^-) \tag{15}$$

Therefore the modal velocity $\dot{\mathbf{q}}(t_r^+)$ and the vertical velocity $\dot{u}(t_r^+)$ of the unsprung mass at t_r^+ can be calculated form the modal velocity $\dot{\mathbf{q}}(t_r^-)$ and the vertical velocity $\dot{u}(t_r^-)$ of the oscillator at t_r^-. At the point of impact, there is usually a velocity discontinuity by the amount of $\dot{\mathbf{q}}(t_r^+) - \dot{\mathbf{q}}(t_r^-)$ and $\dot{u}(t_r^+) - \dot{u}(t_r^-)$ respectively.

After the impact, the oscillator (unsprung mass) bounces off the beam at a newly gained vertical velocity $\dot{u}(t_r^+)$ as its initial velocity and the vertical motion of the unsprung mass is now governed by

$$m_u\ddot{u} + c(\dot{u} - \dot{z}) + k(u - z) = W_u \tag{16}$$

while the equation of motion for the sprung mass remains unchanged as Eq. (4). It should be pointed out that the vertical displacements (motions) of the unsprung mass and of the beam are each continuous throughout the entire time duration of interest and thus the displacements just after and just before the impact remain the same and known. Numerical integration of Eqs. (4) and (16) allows $u(t)$ and $z(t)$ to be found. Whenever $u(t)$ approaches $w(vt, t)$ again at a later time, a new impact takes place in general.

3.2 Perfectly plastic impact

When the impact is perfectly plastic, the unsprung mass sticks to the beam immediately after the impact and hence acquires the same displacement and the same vertical velocity of the beam at t_r^+. That is,

$$\dot{u}(t_r^+) = \left(\frac{\partial w}{\partial t} + v\frac{\partial w}{\partial x}\right)\bigg|_{x = vt_r^+} = \boldsymbol{\psi}^T(vt_r^+)\dot{\mathbf{q}}(t_r^+) + v\boldsymbol{\psi}'^T(vt_r^+)\mathbf{q}(t_r^+) \tag{17}$$

Substituting Eqs. (12) into (17) and noticing that $\mathbf{q}(t_r^+) = \mathbf{q}(t_r^-) = \mathbf{q}(t_r)$ (displacements are always continuous) and effectively $vt_r^+ = vt_r$, one can derive

$$\dot{u}(t_r^+) = \frac{m_u\boldsymbol{\psi}^T(vt_r)\boldsymbol{\psi}(vt_r)\dot{u}(t_r^-) + \boldsymbol{\psi}^T(vt_r)\dot{\mathbf{q}}(t_r^-) + v\boldsymbol{\psi}'^T(vt_r)\mathbf{q}(t_r^-)}{m_u\boldsymbol{\psi}^T(vt_r)\boldsymbol{\psi}(vt_r) + 1} \tag{18}$$

and

$$\dot{\mathbf{q}}(t_r^+) = \frac{\dot{\mathbf{q}}(t_r^-) - m_u v\boldsymbol{\psi}(vt_r)\boldsymbol{\psi}'^T(vt_r)\mathbf{q}(t_r^-) + m_u\boldsymbol{\psi}(vt_r)\dot{u}(t_r^-)}{m_u\boldsymbol{\psi}^T(vt_r)\boldsymbol{\psi}(vt_r) + 1} \tag{19}$$

After the impact, the unsprung mass attaches to the beam and vibrates at vertical initial velocity $\dot{u}(t_r^+)$ given in Eq. (18). Eqs. (3)-(5) must then be solved together by numerical integration. The value of $f_c(t)$ must be constantly checked for possible separation again, which occurs whenever $f_c(t)$ drops to zero.

Whether the impact is perfectly elastic or perfectly plastic (or any sort of impact), separation and reattachment must be constantly monitored. Quite often the chosen time steps will not allow an exact hit at the instant of separation or instant of reattachment at the end of a time step. If during a time step separation or reattachment is found to have taken place, time t must go back to the beginning of this time step and a new (smaller) time step must be used in the numerical integration, so that at the end of this new time step separation or reattachment happens to occur (within a small acceptable error). This new time step is predicted using the idea put forward by the second author of the paper and his colleagues (Ouyang *et al.* 1999). There are two different sets of dynamic equations for these two different regimes (in contact or in separation). This non-smooth dynamic problem takes time to compute and the numerical integration algorithm must be capable of capturing these crucial instants accurately.

Another point to be made is that during perfectly plastic impact, a small amount of energy will be lost. This energy loss should be very small as impact takes place within a very short time interval.

4. Numerical examples

A simply-supported beam with $L = 15$ m, $I = 0.0003$ m^4, $A = 0.03$ m^2, $E = 2.1\times10^{11}$ Pa and $\rho = 7500$ kg·m^{-3}, is considered. For the oscillator, $m_z = 1000$ kg, $m_u = 100$ kg, $k = (2\pi)^2 m_z$ N/m. Height $H = 0.1$ m. $v = 10$ m/s. Given this height and the horizontal velocity, the oscillator lands on the beam at $vt/L = 0.095$. The damping factor ζ is defined as $c/2\sqrt{km_z}$. The two masses together are close to represent a small car.

The natural frequencies and modes of the unloaded beam are $\omega_j = (j\pi/l)^2\sqrt{EI/\rho A}$ and $\psi_j(x) = \sqrt{2/\rho AL}\sin(j\pi x/L)$ $(j = 1, 2, 3, \ldots)$. The critical speed (Fryba 1999) of the beam, defined as the lowest speed at which a moving force of constant magnitude will excite the supporting structure into resonance, is $v_{cr} = \omega_1 L/\pi = 110.8$ m/s. Numerical results of displacements are presented in terms of the displacement ratios of $u(vt)/|w_{st}|$ and $w(vt, t)/|w_{st}|$, where the static mid-span beam deflection $w_{st} = (W_z + W_u)L^3/48EI$. Values of different types of energy involved are also shown.

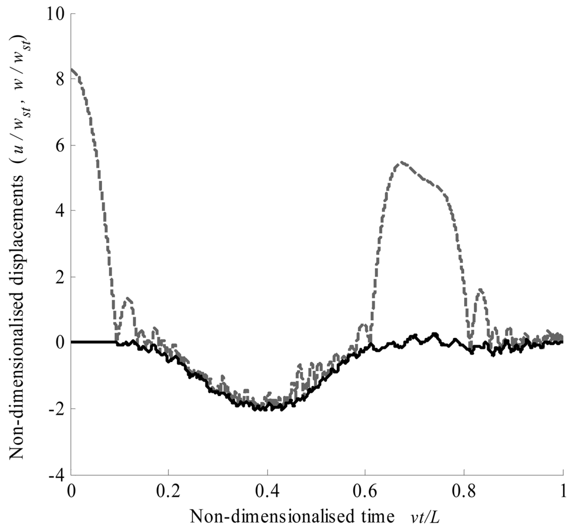

Fig. 3 Displacement ratios: --- $u(vt)/|w_{st}|$; — $w(vt, t)/|w_{st}|$

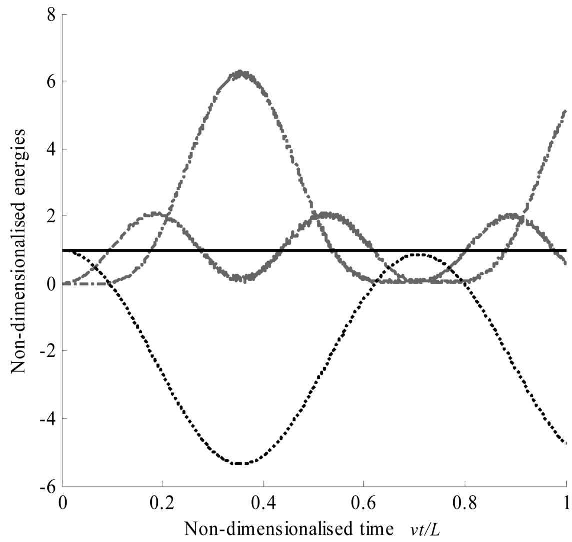

Fig. 4 Energy values: — total energy; --- kinetic energy; –·– elastic energy in the oscillator; ··· gravitational potential energy.

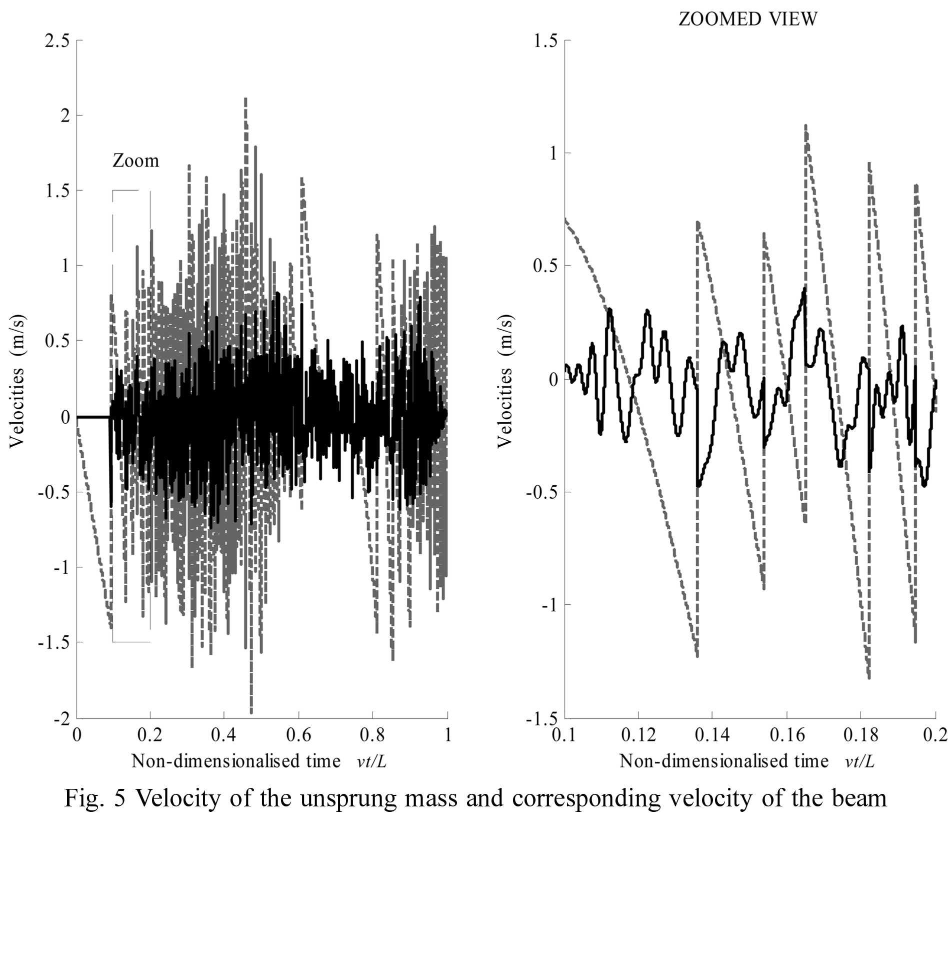

Fig. 5 Velocity of the unsprung mass and corresponding velocity of the beam

Case 1: $\zeta = 0$ and assuming perfectly elastic impact. The results of displacement ratios, values of various energy and velocities are given in Figs. 3-5.

The bounces of the oscillator during x = 0.1 and 0.4 m become smaller and smaller as the beam deflects downward. This decreasing level of bouncing of the oscillator is due to the transfer of kinetic energy from oscillator to the beam. When the beam starts to vibrate upward to meet the oscillator moving downward at a later stage the bounces of the oscillator begin to increase in general. It can be observed from Fig. 3 that the second large separation initiated at about $vt/L = 0.6$ is where the negative deflection of the beam quickly moves upwards towards its un-deformed position of zero deflection and the stiffness of the beam quickly increases. The small-amplitude ripples are high-frequency components excited by impact, giving the appearance of noise. Although the responses at times look noisy, the energy is conserved, as seen in Fig. 4. This serves to validate the model and the numerical results.

Comparing the velocity of the unsprung mass with that of the beam at the corresponding horizontal coordinate shown in Fig. 5 is interesting. The vertical rise or fall (five in total) in the zoomed view indicates impact events in which the velocity of the unsprung mass always rises and that of the beam always falls at any instant of impact, and the two velocity values are always different immediately after impact. The latter is due to perfectly elastic impact assumed so that the two colliding bodies always separate immediately after impact. In contrast, it will be seen later in Cases 2 and 3 that the two velocity values become the same immediately after perfectly plastic contact. As there is repeated switching from separation to impact, the numerical simulation takes time.

Case 2: The same data as in Case 1 are used, but now perfectly plastic impact is assumed. The results are shown in Figs. 6-8.

With the assumption of perfectly plastic impact, two events of impact are clearly seen in Fig. 6. The second impact occurs when the beam vibrates upward after separation. As the oscillator attaches to the beam after an impact, the dynamic responses become much smoother than those of

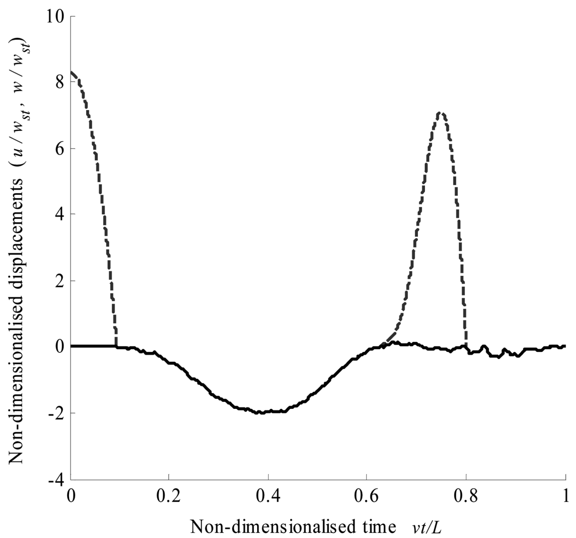

Fig. 6 Displacement ratios: --- $u(vt)/|w_{st}|$; — $w(vt, t)/|w_{st}|$

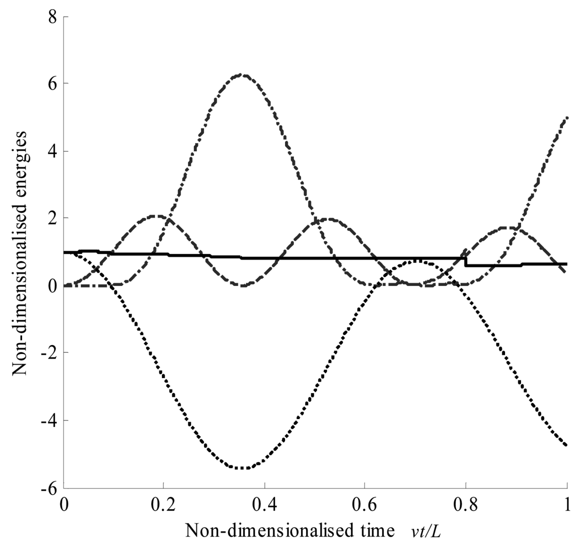

Fig. 7 Energy values: — total energy; --- kinetic energy; –•– elastic energy in the oscillator; ··· gravitational potential energy

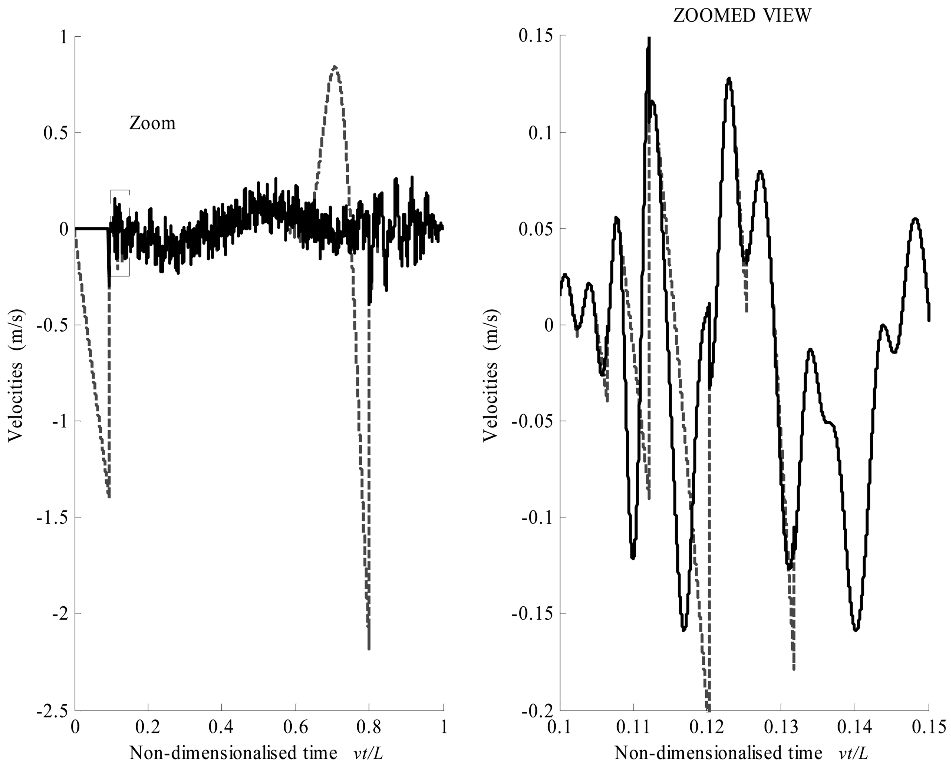

Fig. 8 Velocity of the unsprung mass and corresponding velocity of the beam

Case 1 of perfectly elastic impact. In another word, fewer events of impact or less hard impact (that is, smaller difference in the two velocity values of the colliding bodies just before impact) imply less 'noisy' responses. As a certain amount of energy is lost during perfectly plastic impact, the total energy level, shown in Fig. 6, decreases (slightly) as the oscillator travels forward. The amount of energy lost is very small as the total duration of impact is very short.

A closer look at the velocities (in the zoomed view of Fig. 8) reveals that underneath the appearance of lack of activity between $vt/l = 0.1$ and 0.14, there are actually three events of impact-reattachment. But as the impact is not hard due to a small velocity difference, its influence is effectively not detectable unless zoomed in. Because of the perfectly plastic impact assumed, the velocity of the unsprung mass stays identical to that of the beam for a while immediately after impact. This behaviour is very different from that of perfectly elastic impact studied in Case 1.

Case 3: The same data as in Case 2 are used, except now with a higher $k = (20\pi)^2 m_z$ and damping factor $\zeta = 20\%$ (perfectly plastic impact). The results are presented in Figs. 9-11.

With a much higher spring constant of the oscillator there are more occurrences of bounce and impact in the early stage of the horizontal travel of the oscillator but no further detectable separation at later stage of the travel. The presence of the oscillator damping makes it less oscillatory and less capable of bouncing off, leading to no apparent separation in the later stage of vibration. The beam vibrates in more cycles. Due to the large damping, the total energy drops sharply to nearly zero. There are only two occurrences of impact after the initial one from Fig. 9. Fig. 11 shows that this is indeed the case. The oscillator and the beam maintain contact mostly during its horizontal travel.

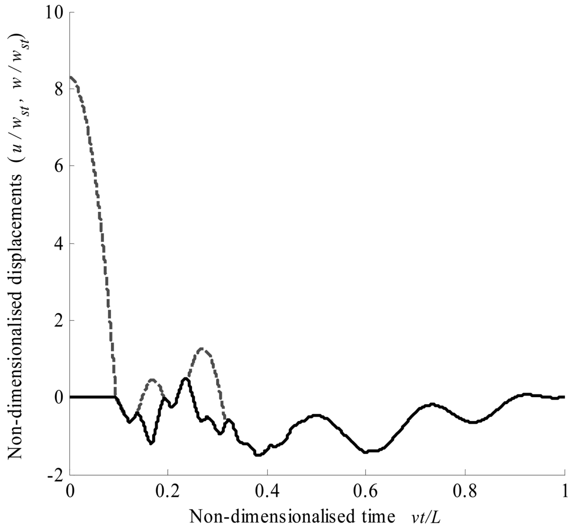

Fig. 9 Displacement ratios: --- $u(vt)/|w_{\mathrm{st}}|$; — $w(vt, t)/|w_{\mathrm{st}}|$

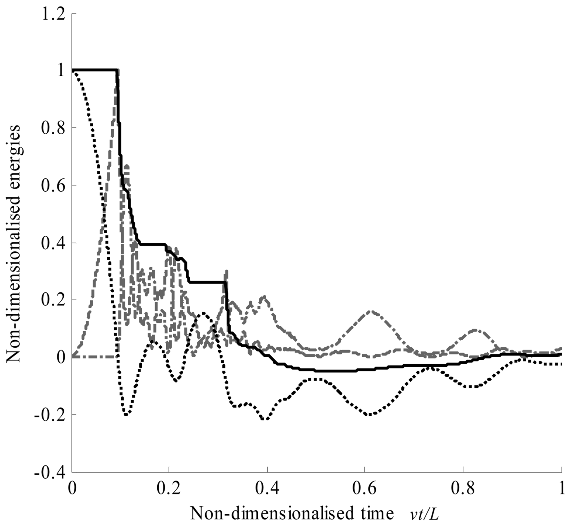

Fig. 10 Energy values: — total energy; --- kinetic energy; –·– elastic energy in the oscillator; ··· gravitational potential energy

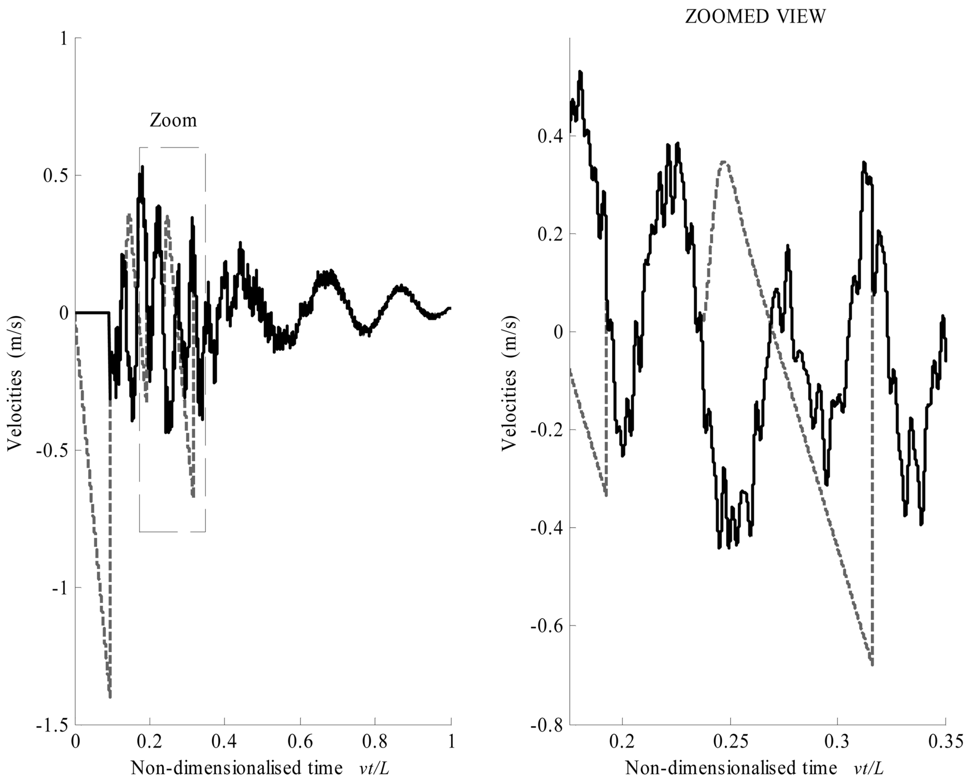

Fig. 11 Velocity of the unsprung mass and corresponding velocity of the beam

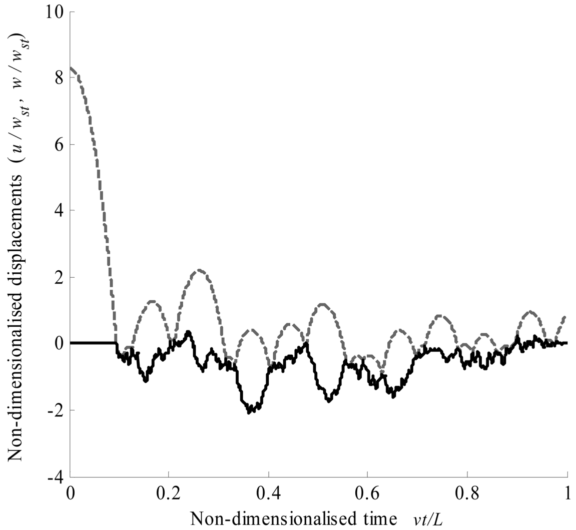

Fig. 12 Displacement ratios: --- $u(vt)/|w_{\mathrm{st}}|$; — $w(vt,t)/|w_{\mathrm{st}}|$

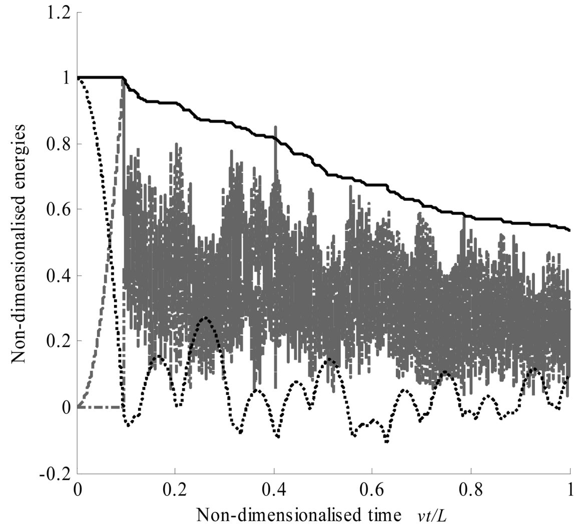

Fig. 13 Energy values: — total energy; --- kinetic energy; –·– elastic energy in the oscillator; ··· gravitational potential energy

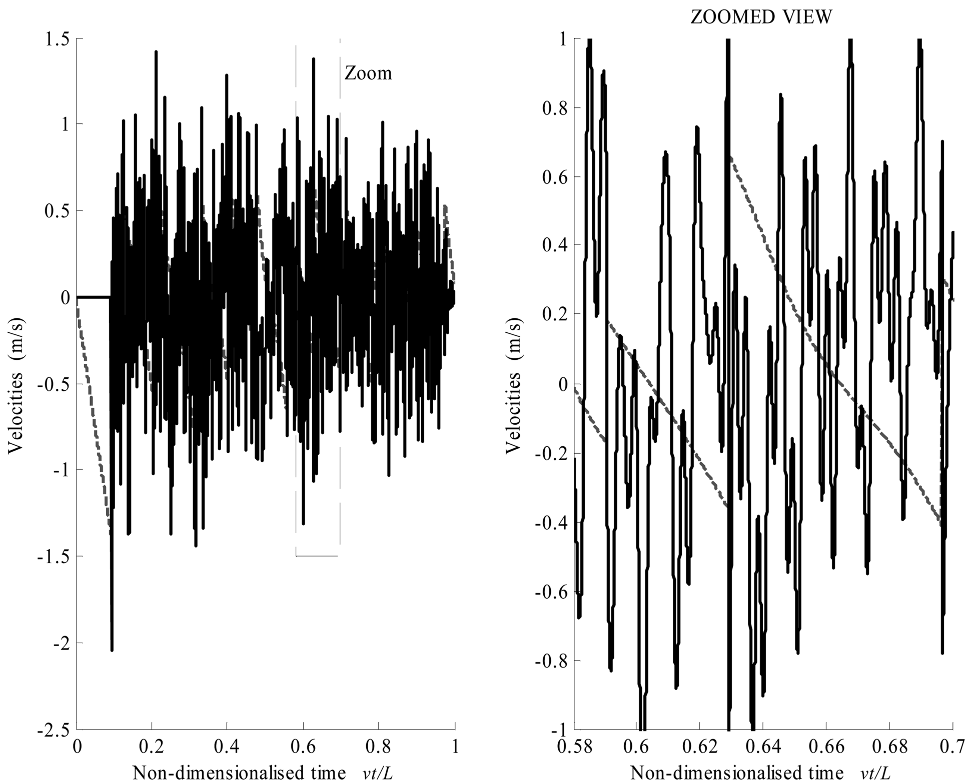

Fig. 14 Velocity of the unsprung mass and corresponding velocity of the beam

Case 4: The same data as in Case 3 are used, except that the two masses are swapped as $m_z = 100$ kg, $m_u = 1000$ kg and perfectly elastic impact is now assumed. The results are shown in Figs. 12-14.

This seems to be the most complicated case so far. There are a number of events of separation and impact. The dynamic response of the beam looks quite noisy because of those impact events exciting high-frequency components. This is also reflected by the oscillating level of kinetic energy, even though the total energy level is fairly smooth. The total energy decreases steadily due to damping of the oscillator. Because of the definition of the damping factor as $c/2\sqrt{km_z}$, effectively a smaller damping of c in Case 4 than in Case 3 is used when $\zeta = 20\%$ in both cases.

As in the previous case of perfectly elastic impact the oscillator never sticks to the beam after impact. It is fairly easy to pinpoint where impact takes place. In Fig. 14, there are three events of impact where the velocity of the unsprung mass undergoes a jump.

Finally, the zoomed views of Figs. 5, 8, 11 and 14 should show one particular challenge of the numerical procedure in which the displacements and velocities of the unsprung mass and the corresponding location of the beam must be tracked in order to decide on whether there is separation and more importantly impact. If there is separation, a different set of equations of motion must be integrated. If there is impact after separation, on the other hand, new initial velocities of the unsprung mass and the beam must be calculated and then the original equations of motion must be integrated.

5. Conclusions

In this paper, an oscillator, comprised of two masses connected through a vertical spring and moving at a constant speed in the horizontal direction, drops onto an elastic beam with impact due to its self-weight and excites vibration in itself and the beam. Two types of impact, perfectly elastic and perfectly plastic, are considered and separation between the oscillator and the beam during ensuing vibration is investigated. The effects of the masses, the spring constant and damping of the oscillator on the vibration of itself and the beam, are studied through four numerical examples. It is found the resultant dynamics of the oscillator and the beam is very complicated. When the impact is perfectly elastic, a number of events of bouncing (separation) and impact may occur. When the impact is perfectly plastic, several events (though fewer than in the perfect elastic impact) of bouncing (separation) and impact may still occur. Whenever there is impact, the vibration looks noisy because of excitation of high-frequency components. When there is damping in the oscillator or the impact is perfectly plastic, there is a loss of total energy as the oscillator moves forward.

Acknowledgements

This research is sponsored by the EPSRC of the UK (Ref. EP/D057671/01) and the RSB funding of the Department of Engineering, University of Liverpool. The first author is sponsored by Projects T79/2006 (Ministerio de Fomento - Metro de Madrid- CDM) and TRA2007-67167 (Ministerio de Educación y Ciencia - FEDER).

References

Baeza, L. and Ouyang, H. (2008), "Dynamics of a truss structure and its moving-oscillator exciter with separation and impact-reattachment", *Proceedings of the Royal Society of London, Series A*, doi: 10.1098/rspa.2008.0057.

Cheng, Y.S., Au, F.T.K., Cheung, Y.K. and Zheng, D.Y. (1999), "On the separation between the moving vehicle and bridge", *J. Sound Vib.*, **222**(5), 781-801.

Fryba, L. (1968), "Impacts of two-axle system traversing a beam", *Int. J. Solid Struct.*, **4**, 1107-1123.

Fryba, L. (1999), *Vibrations of the Solids and Structures under Moving Loads*. Thomas Telford Publishing House.

Goldsmith, W. (2000), *Impact the Theory and Physical Behaviour of Colliding Solids*, Dover Publications, Inc.

Lee, H.P. (1995), "On the separation of a mass travelling on a beam with axial forces", *Mech. Res. Commun.*, **22**(4), 371-376.

Lee, U. (1996), "Revisiting the moving mass problem: onset of the separation between the mass and beam", *J. Vib. Acoust.*, ASME, **118**, 516-521.

Lou, P. (2005), "A vehicle-track-bridge interaction element considering vehicle's pitching effect", *Finite Elem. Anal. Des.*, **41**, 397-427.

Metrikine, A.V. and Verichev, S.N. (2001), "Instability of vibration of a moving oscillator on a flexibly supported Timoshenko beam", *Arch. Appl. Mech.*, **71**(9), 613-624.

Meyer, C.D. (2000), *Matrix Algebra and Applied Linear Algebra*, SIAM.

Ouyang, H., Mottershead, J.E., Cartmell, M.P. and Brookfield, D.J. (1999), "Friction-induced vibration of an elastic slider on a vibrating disc", *Int. J. Mech. Sci.*, **41**, 325-336.

Ouyang, H. and Mottershead, J.E. (2007), "A Numerical-analytical combined method for vibration of a beam excited by a moving flexible body", *Int. J. Numer. Meth. Eng.,* **72**, 1181-1191.

Pesterev, A.V. and Bergman, L.A. (1997), "Response of elastic continuum carrying moving linear oscillator", *J. Eng. Mech.*, ASCE, **123**(8), 878-884.

Pesterev, A.V. and Bergman, L.A. (1998), "A contribution to the moving mass problem", *J. Vib. Acoust.*, ASME, **120**, 824-826.

Stancioiu, D., Ouyang, H. and Mottershead, J.E. (2008), "Vibration of a beam excited by a moving oscillator considering separation and reattachment", *J. Sound Vib.*, **310**(4-5), 1128-1140.

Yang, B., Tan, C.A. and Bergman, L.A. (2000), "Direct numerical procedure for solution of moving oscillator problems", *J. Eng. Mech.*, ASCE, **126**(5), 462-469.

Yang, Y.B., Yau, J.D. and Wu, Y.S. (2004), *Vehicle-Bridge Interaction Dynamics*, World Scientific Publishing Co.