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On the limit cycles of aeroelastic systems with quadratic nonlinearities

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Abstract. Limit cycle oscillations of a two-dimensional airfoil with quadratic and cubic pitching nonlinearities are investigated. The equivalent stiffness of the pitching stiffness is obtained by combining the linearization and harmonic balance method. With the equivalent stiffness, the equivalent linearization method for nonlinear flutter analysis is generalized to address aeroelastic system with quadratic nonlinearity. Numerical example shows that good approximation of the limit cycle can be obtained by the generalized method. Furthermore, the proposed method is capable of revealing the unsymmetry of the limit cycle; however the ordinary equivalent linearization method fails to do so.

Keywords: aeroelastic system; quadratic nonlinearity; limit cycle; equivalent linearization method.

1. Introduction

Aeroelastics considers structures subjected to structural, aerodynamic and inetia forces. Structures such as airfoils (Lee *et al.* 1999a, Chowdhury and Sarkar 2004), bridges (Gu *et al.* 2001, Ding *et al.* 2002a) and beam structures (Wang 2003) can oscillate stably or unstably induced by the wind. The complex statics and dynamic behaviors of structures in the wind/air have stimulated the interests of many reseachers and engineers (Gu *et al.* 2001, Ding *et al.* 2002b, Moon and Lee 2002).

A nonlinear aeroelastic system of airfoil, mainly considered in this paper, is one of the typical self-excited systems. As the flow velocity increases beyond the linear flutter speed, the airfoil oscillates with a limited amplitude, namely, the limit cycle oscillation. Predicting amplitude and frequency of flutter oscillations via analytical or numerical techniques has been an active research area for many years. Harmonic balance (HB) method has been extensively used to analyze nonlinear aeroelastic systems (Lee *et al.* 1999a). The equivalent linearization method (ELM), as another version of the HB method (Liu *et al.* 2006), was also widely applied in nonlinear flutter analysis. Using the ELM, linearized systems of the considered nonlinear problems can be obtained, and then the traditional methods for linear flutter analysis can be employed. The effectiveness of the ELM has been validated by many authors. Using this method, Liu and Zhao (1992) studied the bifurcation of an airfoil with a cubic pitching nonlinearity. Yang (1995) studied the flutter models of a two-dimensional wing and a delta wing with external stores. Tang *et al.* (1998) addressed the limit cycle behavior of an airfoil with a control surface. Also, Shahrzad and Mahzoon (2002) employed

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the ELM to investigate three flutter models, i.e., the steady linear, steady nonlinear and unsteady nonlinear models.

Note that all the above-mentioned studies considered only cubic nonlineaities. Actually, the ELM can work well for the aeroelastic systems with only odd nonlinearities. Well known, even nonlinearities exist in many aeroelastic systems. For example, they may arise as structural nonlinearities (Coller and Chamara 2004, Chamara and Coller 2004) or come from aerodynamic forces, e.g., the nonlinear steady flow flutter model (Shahrzad and Mahzoon 2002). However, the ELM fails to deal with the systems containing even nonlinearities because the equivalent stiffness corresponding to even stiffnesses cannot be obtained at its present state (Shahrzad and Mahzoon 2002). To this end, it is necessary and worthwhile to generalize the ELM for studying aeroelastic systems with both quadratic and cubic nonlinearities.

In this paper, we consider an aeroelastic system containing both quadratic and cubic pitching nonlinearities. Firstly, we employ a new analytical method for nonlinear oscillators i.e., practicing the linearization prior to the harmonic balancing (Lim and Wu 2003), to propose an approach for obtaining the equivalent stiffness of the pitch stiffness. This pitch stiffness consists of a linear part and quadratic as well as cubic nonlinearities. Then, the equivalent stiffness is introduced to modify the ELM for the limit cycle of the studied nonlinear aeroelastic system.

2. Equation of motions

An aeroelastic model of a two-dimensional airfoil in unsteady flow is inherently infinite dimensional. Dimension-reducing of the aeroelastic model has stimulated the interest and curiosity of many authors. For example, Lee *et al.* (1999b) reduced the aeroelastic model as a system of eight first order ordinary differential equations. Coller and Chamara (2004) derived a lower order flutter model with only six first order ordinary differential equations, which is equivalent to the one obtained by Lee *et al.* (1999b). Besides, Hall *et al.* (2000) and Thomas *et al.* (2004) developed an orthogonal decomposition technique for transonic unsteady flows. In order to present the generalized ELM more explicitly, we consider only the steady flow model (Liu and Zhao 1992).

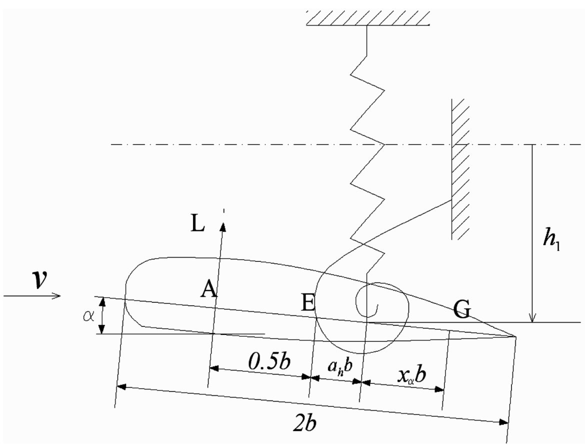

Fig. 1 Sketch of a two-dimensional airfoil

The physical model shown in Fig. 1 corresponds to a two-dimensional airfoil oscillating in pitch and plunge. The pitch angle about the elastic axis is denoted by α, positive with the nose up, and the plunge deflection is denoted by h_1, positive in the downward direction. The elastic axis is located at a distance $a_h b$ from the mid-chord, while the mass center is located at a distance $x_\alpha b$ from the elastic axis. Both distances are positive when measured towards the trailing edge of the airfoil. Let $r_\alpha b$ be the radius of gyration of the airfoil with respect to the elastic axis, and ω_h, ω_α be the eigenfrequencies of the constrained one-degree-of-freedom system associated with the linear plunging and pitching springs, respectively. c_α and c_h are the coefficients of damping in pitch and plunge, respectively.

In terms of non-dimensional time $t = \omega_\alpha t_1$ (t_1 is the real time) and non-dimensional plunge displacement $h = h_1/b$, the coupled motions of the airfoil in incompressible steady flow can be described as (Zhao and Yang 1989, Liu and Zhao 1992)

$$\begin{cases} \mu\ddot{h} + \mu x_\alpha\ddot{\alpha} + \dfrac{c_h}{\pi\rho b^2\omega_\alpha}\dot{h} + \mu\left(\dfrac{\omega_h}{\omega_\alpha}\right)^2\dfrac{\partial V}{\partial h} = -2\left(\dfrac{v}{b\omega_\alpha}\right)^2\alpha \\ \mu x_\alpha\ddot{h} + \mu r_\alpha^2\ddot{\alpha} + \dfrac{c_\alpha}{\pi\rho b^2\omega_\alpha}\dot{\alpha} + \mu r_\alpha^2\alpha\dfrac{\partial V}{\partial \alpha} = (1+2a_h)\left(\dfrac{v}{b\omega_\alpha}\right)^2\alpha \end{cases} \tag{1}$$

where the superscript denotes the differentiation with respect to t. The symbol V above represents the potential for the elastic restoring forces

$$V = \frac{1}{2}h^2 + \frac{1}{2}\alpha^2 + \frac{1}{3}\delta_2\alpha^3 + \frac{1}{4}\delta_3\alpha^4 \tag{2}$$

Thus, we consider a pitch spring with quadratic and cubic nonlinearities. The values of the parameters are listed as follows: $\mu = 20$, $a_h = -0.1$, $b = 1$ m, $x_\alpha = 0.25$, $r_\alpha^2 = 0.5$, $(\omega_h/\omega_\alpha)^2 =$ 0.2, $\omega_\alpha = 62.8$ Hz and the damping terms $0.1\dot{h}$, $0.1\dot{\alpha}$ (Liu and Zhao 1992). Substitutions of these values into Eq. (1) result in

$$\begin{aligned} &\ddot{h} + 0.25\ddot{\alpha} + 0.1\dot{h} + 0.2h + 0.1Q\alpha = 0 \\ &0.25\ddot{h} + 0.5\ddot{\alpha} + 0.1\dot{\alpha} - 0.04Q\alpha + f(\alpha) = 0 \end{aligned} \tag{3}$$

where $Q = (v/b\omega_\alpha)^2$ is called the generalized flow speed and the restoring force in pitch degree of freedom (DOF) is given by

$$f(\alpha) = r_\alpha^2\frac{\partial V}{\partial \alpha} = r_\alpha^2(\alpha + \delta_2\alpha^2 + \delta_3\alpha^3) \tag{4}$$

Rewrite $f(\alpha)$ as $f(\alpha) = k_1\alpha + k_2\alpha^2 + k_3\alpha^3$ and take $f(\alpha) = 0.5\alpha + 2\alpha^2 + 20\alpha^3$ as an illustrative example to validate the proposed method, that, we have $k_1 = 0.5$, $k_2 = 2$ and $k_3 = 20$.

3. Equivalent stiffness

The free oscillations of the airfoil in the uncoupled pitching mode can be described as

$$\ddot{\alpha} + k_1\alpha + k_2\alpha^2 + k_3\alpha^3 = 0, \quad \alpha(0) = A, \quad \dot{\alpha}(0) = 0 \tag{5}$$

where the coefficients k_1, k_2 and k_3 are the same as mentioned above, the initial displacement A is a constant. Without loss of generality, we suppose that $A > 0$. Letting k_{eq} be the equivalent stiffness and ω be the fundamental frequency of the solution of Eq. (5), we have (Caughey 1963)

$$k_{eq} = \omega^2 \tag{6}$$

Under the transformation $\tau = \omega t$, Eq. (5) becomes

$$\omega^2 \alpha'' + k_1 \alpha + k_2 \alpha^2 + k_3 \alpha^3 = 0, \quad \alpha(0) = A, \quad \alpha'(0) = 0 \tag{7}$$

where the superscript denotes the differentiation with respect to τ. The solution of Eq. (7) is periodic. Considering the initial conditions, we let

$$\alpha_0(\tau) = A\cos\tau \tag{8}$$

be the initial approximation of $\alpha(\tau)$, where $\alpha(\tau)$ is described as (Lim and Wu 2003)

$$\alpha(\tau) = \alpha_0(\tau) + \Delta\alpha(\tau) \tag{9}$$

and $\Delta\alpha(\tau)$ is referred to as the correction part. Note that both $\alpha_0(\tau)$ and $\Delta\alpha(\tau)$ are functions of τ with the period as 2π.

Substituting Eqs. (8) and (9) into (7), then expanding (7) as Taylor series of $\Delta\alpha$ and retaining only the first power, we obtain

$$\begin{aligned}\omega^2\Delta\alpha'' + k_1\Delta\alpha + 2k_2A(\cos\tau)\Delta\alpha + 3k_3A^2(\cos\tau)^2\Delta\alpha - \omega^2A\cos\tau + k_1A\cos\tau \\ + k_2A^2(\cos\tau)^2 + k_3A^3(\cos\tau)^3 = 0\end{aligned} \tag{10}$$

Choose the correction part as

$$\Delta\alpha(\tau) = c_1(1-\cos\tau) + c_2(\cos\tau - \cos 2\tau) \tag{11}$$

where c_1 and c_2 are functions of A. According to Eqs. (8) and (11), obviously, Eq. (9) satisfies the initial conditions described in (5). Substituting Eq. (11) into (10), then expanding (10) as Fourier series of τ and equating the constant term and the coefficients of $\{\cos\tau, \cos 2\tau\}$ to be zeroes, yields

$$\begin{aligned} r_1c_1 + r_2c_2 + s_1 &= 0 \\ (c_1 - c_2 - A)\omega^2 + r_3c_1 + r_4c_2 + s_2 &= 0 \\ 8c_2\omega^2 + r_5c_1 - r_1c_2 + s_1 &= 0 \end{aligned} \tag{12}$$

where $r_1 = 3k_3A^2 + 2k_1 - 2k_2A$, $r_2 = -\frac{3}{2}k_3A^2 + 2k_2A$, $r_3 = -k_1 - \frac{9}{4}k_3A^2 + 2k_2A$, $r_4 = -k_2A + k_1 + \frac{9}{4}k_3A^2$, $r_5 = 3k_3A^2 - 2k_2A$, $s_1 = k_2A^2$ and $s_2 = k_1A + \frac{3}{4}k_3A^3$.

Eliminating c_1 and c_2 and after some algebraic manipulations, we have

$$R_2\omega^4 + R_1\omega^2 + R_0 = 0 \tag{13}$$

where $R_2 = -8(r_1A + s_1)$, $R_1 = s_1(2r_1 + r_2 - r_5 - 8r_3) + A(r_2r_5 + r_1^2) + 8s_2r_1$ and $R_0 = -s_1(r_1r_3 - r_1r_4 + r_2r_3$ $r_4r_5) - s_2(r_1^2 + r_2r_5)$.

Two solutions of Eq. (13) with respect to ω^2 (considered as an independent unknown) can be solved, i.e.,

$$\omega_1^2 = \frac{-R_1 - \sqrt{R_1^2 - 4R_2R_0}}{2R_2}, \quad \omega_2^2 = \frac{-R_1 + \sqrt{R_1^2 - 4R_2R_0}}{2R_2} \tag{14}$$

Substituting r_i, $i = 1, 2, \ldots, 5$ and s_j, $j = 1, 2$ into R_k, $k = 0, 1, 2$, letting A approach 0 we can obtain the following equations

$$R_0 = -[k_1A + O(A^2)][4k_1^2 + O(A)] + O(A^2) = -4k_1^3A + O(A^2) \tag{15}$$

$$R_1 = O(A^2) + A[4k_1^2 + O(A)] + 8[k_1A + O(A^3)][2k_1 + O(A)] = 20k_1^2A + O(A^2) \tag{16}$$

$$R_2 = -8[2k_1A + O(A^2)] = -16k_1A + O(A^2) \tag{17}$$

where the symbol $O(A^i)$, $i = 1, 2, \ldots$ denote small quantities at the same order of A^i. Substitutions of Eqs. (15)-(17) into (14) provide us with

$$\lim_{A\to 0}\omega_1^2 = k_1, \quad \lim_{A\to 0}\omega_2^2 = \frac{k_1}{4} \tag{18}$$

Eq. (18) means that ω_1 converges to the exact frequency solution of Eq. (5) as A approaches 0. It is shown in Fig. 2 that the approximations obtained by ω_1 are in good agreement with the exact frequency solutions. Therefore, $k_{eq} = \omega_1^2$ is chosen as the equivalent stiffness corresponding to the pitching stiffness.

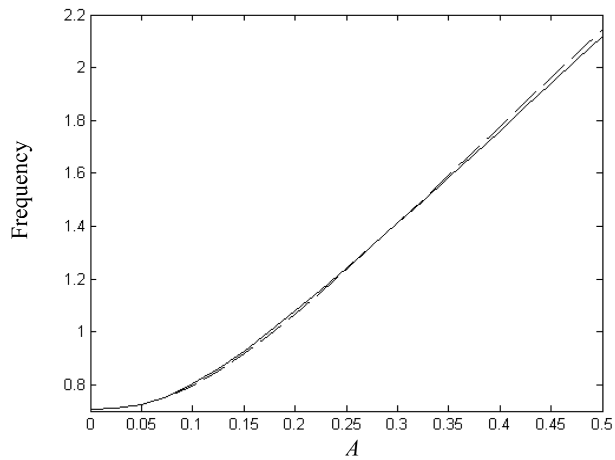

Fig. 2 Frequency of the free oscillation in uncoupled pitching mode. Real line: exact solution; dashed line: $\omega_1(A)$

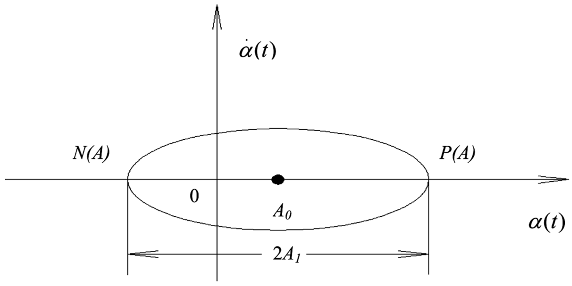

Fig. 3 Sketch of the phase plane of the limit cycle oscillation in pitch DOF

4. Equivalent linearization method

Next, the equivalent stiffness $k_{eq} = \omega_1^2$ is employed to generalize the ELM for system (3). The restoring force $0.5\alpha + 2\alpha^2 + 20\alpha^3$ is replaced by the equivalent restoring force $k_{eq}\alpha = \omega_1^2\alpha$.

A schematic representation of limit cycle oscillation in pitch DOF is presented in Fig. 3, where A_0 denotes the distance from the center of the phase plane to the origin and A_1 denotes one half of the distance from the negative to the positive extreme of α. In addition, we let $P(A) = A$ and $-N(A)$ be the positive and negative extremes of α, and call $P(A)$ and $N(A)$ as the positive and negative amplitudes of the limit cycle oscillation in pitch DOF, respectively. Likewise, we can define H_0 and H_1 for the limit cycle oscillation in plunge DOF. Consequently, the solutions of the limit cycle oscillations can be approximately described as

$$h = H_0 + H_1 e^{i\Omega t}, \quad \alpha = A_0 + H_1 e^{i\Omega t} \tag{19}$$

where Ω is the frequency of the limit cycle oscillation. It is worth pointing out that Ω is different from ω (i.e., the frequency of the free oscillations in uncoupled pitching mode). Here, α denotes the limit cycle solution of pitch while in Section 3 it denotes the free periodic solution of the oscillation of the airfoil in the uncoupled pitching mode. Thus, α is defined as a function of $\tau = \omega t$ above, while of Ωt here.

Substituting Eq. (19) into (3), we obtain

$$\begin{aligned} &[(-\Omega^2 + 0.1i\Omega + 0.2)H_1 + (-0.25\Omega^2 + 0.1Q)A_1]e^{i\Omega t} + 0.1QA_0 + 0.2H_0 = 0 \\ &[-0.25\Omega^2 H_1 + (-0.5\Omega^2 + 0.1i\Omega - 0.04Q + k_{eq})A_1]e^{i\Omega t} + (k_{eq} - 0.04Q)A_0 = 0 \end{aligned} \tag{20}$$

Letting the coefficients of $e^{i\Omega t}$ in Eq. (20) be zeroes, a flutter determinant can be obtained

$$\begin{vmatrix} -\Omega^2 + 0.1i\Omega + 0.2 & -0.25\Omega^2 + 0.1Q \\ -0.25\Omega^2 & -0.5\Omega^2 + 0.1i\Omega + k_{eq} - 0.04Q \end{vmatrix} = 0 \tag{21}$$

Additionally, let the constant terms in the first equation of Eq. (20) be 0, i.e.,

$$0.1QA_0 + 0.2H_0 = 0 \tag{22}$$

Substituting Eq. (22) into the first equation in (20), we have

$$H_1 = -\frac{-0.25\Omega^2 + 0.1Q}{-\Omega^2 + 0.1i\Omega + 0.2} A_1 \tag{23}$$

Separating the imaginary and real parts of Eq. (21) results in

$$\begin{aligned} 0.4375\Omega^4 + (-0.11 + 0.065Q - k_{eq})\Omega^2 + 0.2(k_{eq} - 0.04Q) &= 0 \\ -0.15\Omega^3 + (0.1k_{eq} - 0.004Q + 0.02)\Omega &= 0 \end{aligned} \tag{24}$$

which means

$$\Omega = \sqrt{(k_{eq} - 0.04Q + 0.2)/1.5} \tag{25}$$

Substituting Eq. (25) into the first equation of (24), the relationship between Q and A can be obtained. Thus, the positive amplitude of the limit cycle oscillation in pitch DOF is determined as long as Q is given. However, the negative amplitude $N(A)$ is still not determined. To this end, we assume that $P(A)$ and $N(A)$ still satisfy the solution of Eq. (5), i.e.,

$$P(A) = x(\tau)|_{\tau=0} = A, \quad N(A) = |\alpha(\tau)|_{\tau=\pi}| = |2c_1(A) - 2c_2(A) - A| \tag{26}$$

where $c_1(A)$ and $c_2(A)$ can be obtained by solving Eq. (12) with $\omega^2 = \omega_1^2$.

Moreover, according to Eq. (19), the positive and negative extremes of $\alpha(t)$ are $A_0 + A_1$ and $A_0 - A_1$, respectively, which implies that

$$A_0 + A_1 = A = P(A), \quad A_0 - A_1 = -N(A) \tag{27}$$

According to Eqs. (26) and (27), A_0 and A_1 can be solved. Substituting A_0 into Eq. (22) and A_1 into (23), H_0 and H_1 can then be determined, respectively. Considering that H_0 is a real quantity while H_1 is complex, we know the positive and negative amplitudes of the limit cycle oscillation in plunge DOF are $H_0 + |H_1|$ and $-(H_0 - |H_1|)$, respectively.

5. Results and discussions

Solving Eq. (21), the flutter speed can be found as $Q_f = 4.08$. The following analysis is restricted for the case when the generalized flow speed is lower than the static divergence speed Q_d, because the static divergence has to be avoided in aircraft engineering. Here, Q_d can be found from the second equation of Eq. (3). Letting $\ddot{\alpha}, \dot{\alpha}, \ddot{h}$ and $\dot{h}$ be 0 and α approach 0 results in $-0.04Q + k_1 = 0$, thus the static divergent speed is $Q_d = 12.5$.

When the ordinary ELM is used, the terms $0.5\alpha + 2\alpha^2 + 20\alpha^3$ are replaced by $(0.5 + 15A^2)\alpha$ (Liu and Zhao 1992). However, the quadratic nonlinearity does not play any role in constructing the linearized system. Moreover, it assumes that both the positive and negative amplitudes are A, which is reasonable only if the nonlinear pitch stiffness $f(\alpha)$ represents an odd function (e.g., with even nonlinearities).

Figs. 4 and 5 show the amplitudes in pitch and plunge DOF, respectively. The positive amplitudes in pitch and plunge DOFs obtained by the proposed method are given by $P(A)$ and $H_0 + |H_1|$, and

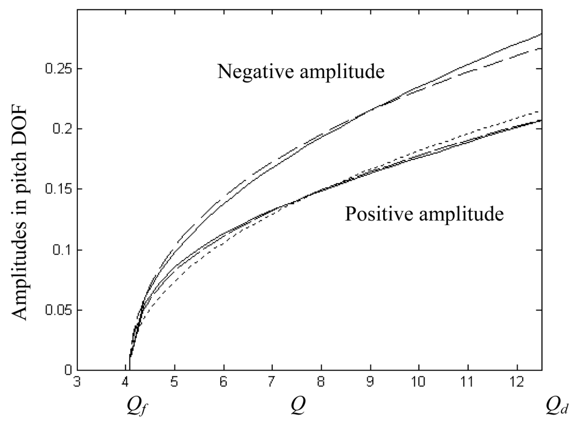

Fig. 4 Amplitudes of limit cycle oscillation in the pitch DOF, real lines: numerical solutions; dashed lines: generalized ELM; dot line: ordinary ELM

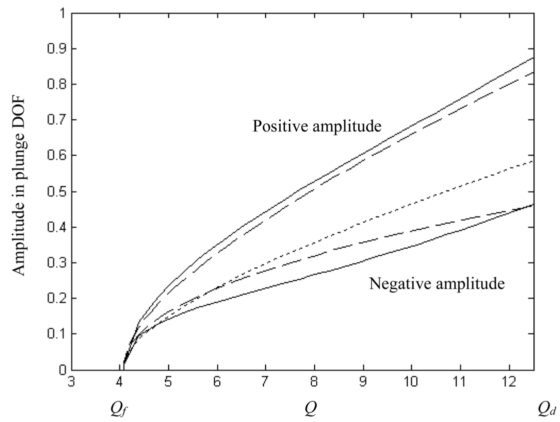

Fig. 5 Amplitudes of limit cycle oscillation in the plunge DOF, real lines: numerical solutions; dashed lines: generalized ELM; dot line: ordinary ELM

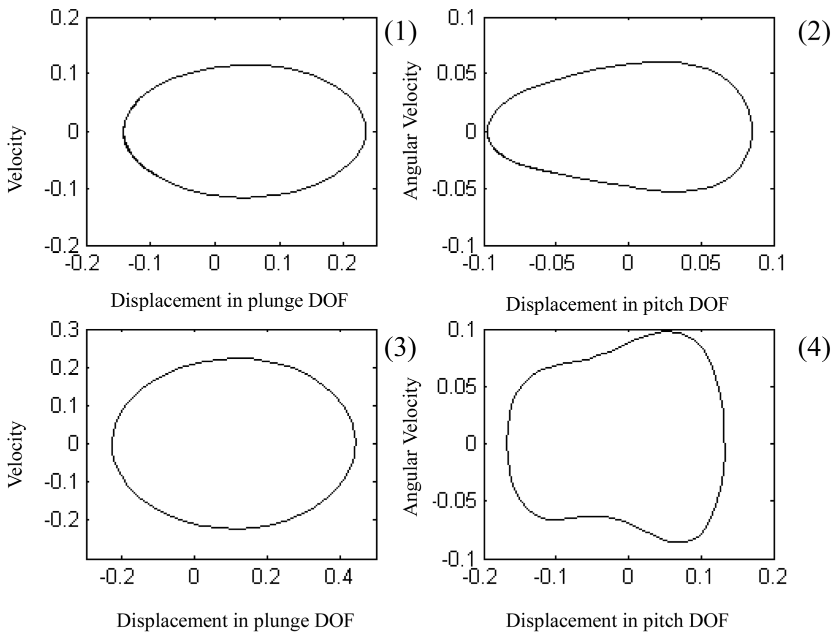

Fig. 6 Phase planes of the limit cycle oscillations of system (3) with Q = 5 for (1-2) and Q = 7 for (3-4)

the negative amplitudes by $N(A)$ and $-(H_0-|H_1|)$. We can see, when Q increases through Q_f, a limit cycle arises. Thus, Q_f is a supercritical Hopf bifurcation point. As is shown by Figs. 4 and 5, both the ordinary ELM and the generalized ELM can predict the bifurcation exactly. But, the generalized method improves the accuracy of the amplitudes both in pitch and plunge DOF. Noticing that, the accuracy of the approximate solutions in pitch DOF is somewhat higher than that of solutions in plunge DOF. One possible reason is that more errors are accumulated in the derivations of the amplitudes in plunge DOF solution, e.g., H_0 and H_1 are roughly determined based on A_0 and A_1, respectively.

Interestingly, the generalized ELM can reveal the fact that the positive amplitude is unequal to the negative one. For example, the negative amplitude in pitch DOF is lager than the positive one as shown in Fig. 4. The situation for plunge DOF, shown in Fig. 5, appears to be just the reverse. However, the ordinary ELM just provides the positive amplitude because the negative amplitude is assumed be the same as the positive one.

Without loss of generality, we obtain the phase planes of the limit cycle oscillation by numerically integrating Eq. (3) with Q = 5 or Q = 7. Fig. 6 shows clearly the positive and negative amplitudes in both pitch and plunge DOFs. The unsymmetries are in accordance with the predictions by the generalized ELM.

6. Conclusions

We have generalized the equivalent linearization method for studying the limit cycle behavior of airfoil with both quadratic and cubic nonlinearities. The equivalent stiffness for the nonlinear stiffness is obtained by combining the linearization and harmonic balance method. With the obtained equivalent stiffness, the ELM is then generalized. The proposed method not only improves the accuracy of the ELM, but also reveals a new characteristic of the limit cycle oscillation caused by the quadratic nonlinearity. These imply that we would expect the proposed method is applicable in other aeroelastic systems, especially those with even nonlinearities.

Acknowledgements

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