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Dynamic characteristics of structures with multiple tuned mass dampers

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Abstract. Effectiveness of multiple tuned mass dampers (MTMD) in suppressing the dynamic response of base excited structure for first mode vibration is investigated. The effectiveness of the MTMD is expressed by the ratio of the root mean square (RMS) displacement of the structure with MTMD to corresponding displacement without MTMD. The frequency content of base excitation is modelled as a broad-band stationary random process. The MTMD's with uniformly distributed natural frequencies are considered for this purpose. A parametric study is conducted to investigate the fundamental characteristics of the MTMD's and the effect of important parameters on the effectiveness of the MTMD's. The parameters include: the fundamental characteristics of the MTMD system such as damping, mass ratio, total number of MTMD, tuning frequency ratio, frequency spacing of the dampers and frequency content of the base excitation. It has been shown that MTMD can be more effective and more robust than a single TMD with equal mass and damping ratio.

Key words: effectiveness; MTMD; vibration control; stochastic response and system parameters.

1. Introduction

The concept of vibration control has been accepted and is widely applied to civil engineering problems (Brock 1946, Tsumura 1991). The tuned mass damper (TMD) is a classical engineering device consisting of a mass, a spring and a viscous damper attached to a vibrating main system in order to attenuate undesirable vibration at a particular frequency. Because the natural frequency of the damper is tuned to a frequency near to the natural frequency of the main system, the vibration of main system causes the damper to vibrate in resonance, dissipating the vibration energy through the damping in the tuned mass damper. The solution of determining the optimum tuning frequency and the optimum damping of the tuned mass damper for undamped main systems subjected to harmonic external forces over a broad band of forcing frequencies is described in Brock (1946) and Den Hartog (1956). Using Den Hartog's procedure Warburton and Ayorinde (1980) have derived the optimum damper parameters for the undamped main system subjected to harmonic support motion where the acceleration amplitude is fixed for all input frequencies and other kinds of harmonic excitation sources. The explicit formulae for the optimum parameters of a tuned mass damper and its effectiveness are available for different types of excitation (Thompson 1981, Tsai and Lin 1993, Warburton 1982). Using the perturbation technique for optimum TMD parameters under various types of loading were derived by Fujino and Abe (1993).

One of the disadvantages of single tuned mass damper is its sensitivity to the error in the natural frequency of the structure and/or that in the damping ratio of the tuned mass damper.

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The effectiveness of a tuned mass damper is decreased significantly by mistuning or off-optimum damping in TMD. As a result, the use of more than one tuned mass damper with different dynamic characteristics has been proposed in order to improve the robustness to uncertainties in the primary system or the TMD's. Iwanami and Seto (1984) have shown that two tuned mass dampers are more effective than a single tuned mass damper. However, the improvement on the effectiveness was not significant. Recently, a multiple tuned mass damper with distributed natural frequencies was proposed by Xu and Igusa (1992, 1994) and also studied by Yamaguchi and Harnpornchai (1993), Abe and Fujino (1994), Jangid and Datta (1994) and Abe and Igusa (1995). It was shown that the optimally designed MTMD's are more effective and robust than an optimally designed single tuned mass damper. However, there is less discussion on the physical interpretation of the dynamic behaviour of a system with MTMD's under important parametric variations.

In the present paper, the effectiveness of MTMD's in suppressing the response of base excited main system is investigated. The frequency content of the base excitation is modeled as a broad band stationary random process specified by its power spectral density function. In specific terms, the objectives of the study are (i) to study the dynamic behaviour of a system with MTMD, (ii) to distinguish between the response characteristics of a system with MTMD and a single TMD and (iii) to study the effect of important parameters on the effectiveness of MTMD for isolation of the main system. The effectiveness of the MTMD is expressed by the ratio of root mean square (RMS) displacement of the main system with MTMD to the RMS displacement of the main system without MTMD with respect to ground.

2. Structural model

The system configuration consists of a main system supported by n number of multiple tuned mass dampers with different dynamic characteristics as shown in Fig. 1. The main system is characterised by natural frequency ω_s, damping ratio ξ_s and mass m_s. The main system and

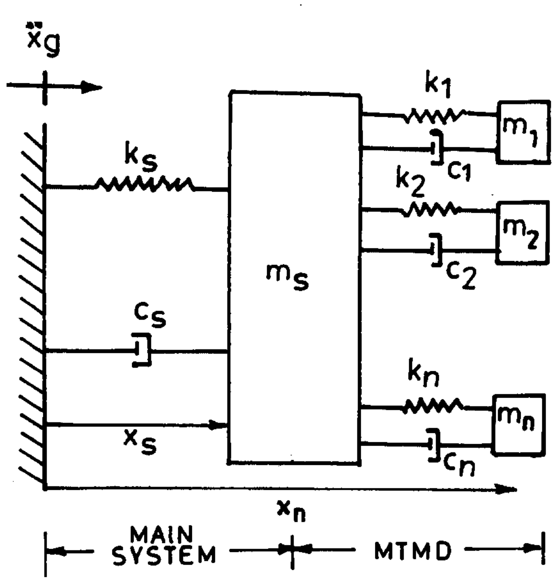

Fig. 1 Structural model.

each TMD is modeled as a single degree-of-freedom system. As a result, the total degrees-of-freedom of the structural system is $n+1$. The natural frequencies of the tuned mass dampers are distributed around the average natural frequency of the MTMD. The various assumptions made are: (i) natural frequencies of the main structure are well separated, (ii) the vibration of the main system to be suppressed is a single mode, (iii) the stiffness and damping constant of each TMD is the same and (iv) the natural frequencies of the MTMD are uniformly distributed around their average natural frequency. The distribution of natural frequencies of the MTMD can be made either by varying the stiffness or mass of each TMD. However, manufacturing of TMD with uniform stiffness and damping constant is simpler than varying stiffness and damping properties (the mass remains unchanged). It is to be noted that MTMD's with identical dynamic characteristics are equivalent to a single TMD. The damping ratio and natural frequency of the equivalent single TMD are the same as those of individual MTMD. However, the mass is the sum of all the MTMD's mass.

Let ω_T be the average frequency of all MTMD's (i.e. $\omega_T = \sum_{j=1}^{n} \omega_j/n$) and n be the total number of MTMD. The natural frequency of j^{th} TMD is expressed as

$$\omega_j = \omega_T\left[1+\left(j-\frac{n+1}{2}\right)\frac{\beta}{n-1}\right] \tag{1}$$

where the parameter β is the non-dimensional frequency spacing of the MTMD defined as

$$\beta = \frac{\omega_n - \omega_1}{\omega_T} \tag{2}$$

If k_T and c_T are the constant stiffness and damping of each TMD, then the mass and damping ratio of j^{th} TMD is expressed as

$$m_j = \frac{k_T}{\omega_j^2} \tag{3}$$

$$\xi_j = \frac{c_T}{2m_j\omega_j} = \frac{c_T}{2k_T}\omega_j \tag{4}$$

The average damping ratio of MTMD is expressed as

$$\xi_T = \sum_{j=1}^{n}\frac{\xi_j}{n} = \frac{\omega_T c_T}{2k_T} \tag{5}$$

The ratio of total MTMD's mass to the main system's mass is defined as the mass ratio i.e.

$$\gamma = \frac{\Sigma\, m_j}{m_s} = \frac{m_T}{m_s} \tag{6}$$

where m_T is the total mass of MTMD; and m_s is the mass of main system. The ratio of total mass of MTMD to mass of main system is taken as 1% which is the standard value of mass ratio for a single TMD in civil engineering problems (Brock 1946, Tsumura 1991).

The constant stiffness required for designed MTMD can be evaluated by the following formulae

$$k_T = \frac{\gamma m_s}{\sum_{j=1}^{n}\frac{1}{\omega_j^2}} \tag{7}$$

The ratio of average frequency of MTMD to the natural frequency of main system is defined as the tuning frequency ratio i.e.

$$f=\frac{\omega_T}{\omega_s} \tag{8}$$

2.1. Equations of motion

The $(n+1)$ equations of motion for the structural model shown in Fig. 1 are expressed in the following matrix form

$$[M]\{\ddot{X}\}+[C]\{\dot{X}\}+[K]\{X\}=-[M]\{1\}\ddot{x}_g \tag{9}$$

in which, $\{X\}=\{x_S, x_1, x_2, \cdots, x_n\}^T$ is the vector of displacements of the structural model; x_S is the displacement of the main system relative to the ground; $x_j (j=1, 2, \cdots, n)$ is the displacement of the j^{th} tuned mass damper relative to the ground; $\ddot{x}_g$ is the ground acceleration; $\{1\}=\{1, 1, \cdots, 1\}^T$; $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrices of size $(n+1)\times(n+1)$. The matrices $[M]$, $[C]$ and $[K]$ are expressed as

$$[M]=\mathrm{diag}[m_s, m_1, m_2, \cdots, m_n] \tag{10}$$

$$[C]=\begin{bmatrix} c_s+\sum c_j & -c_1 & -c_2 & \cdot & \cdot & -c_n \\ & c_1 & 0 & \cdot & \cdot & 0 \\ & & c_2 & \cdot & \cdot & 0 \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \\ \text{sym} & & & & & c_n \end{bmatrix} \tag{11}$$

$$[K]=\begin{bmatrix} k_s+\sum k_j & -k_1 & -k_2 & \cdot & \cdot & -k_n \\ & k_1 & 0 & \cdot & \cdot & 0 \\ & & k_2 & \cdot & \cdot & 0 \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \\ \text{sym} & & & & & k_n \end{bmatrix} \tag{12}$$

The steady state harmonic response of the system to harmonic excitation $\ddot{x}_g=e^{-i\omega t}$ (where ω is the circular frequency and $i=\sqrt{-1}$) will be $\{X\}=X(\omega)e^{-i\omega t}$. The amplitude vector of the steady state response, $X(\omega)$ is given by

$$X(\omega)=\frac{-[M]\{1\}}{-\omega^2[M]+i\omega[C]+[K]} \tag{13}$$

The mass matrix, $[M]$ is diagonal, $[C]$ and $[K]$ matrices have non-zero terms only along the diagonal, the first row and the first column. As a result, the matrix in denominator of Eq. (13) can be inverted using the Cramer's rule. The first element of vector, $X(\omega)$ which is the amplitude of the displacement of the main system is given by

$$x_s(\omega)=\frac{m_s-(i\omega)^{-1}Z(\omega)}{k_s-i\omega c_s-\omega^2 m_s-i\omega Z(\omega)} \tag{14}$$

where,

$$Z(\omega)=-i\omega\sum_{j=1}^{n}\frac{m_j(k_j-i\omega c_j)}{k_j-i\omega c_j-\omega^2 m_j} \tag{15}$$

If the base excitation is modeled as a stationary random process characterised by its power spectral density function (PSDF) then the PSDF of the displacement of main system (Nigam 1983) is given by

$$S_{x_s}(\omega)=|x_s(\omega)|^2 S_{\ddot{x}_g}(\omega) \tag{16}$$

where $S_{\ddot{x}_g}(\omega)$ is the PSDF function of the ground acceleration.

The mean square displacement of the main system is

$$\sigma_{x_s}^2=\int_{-\infty}^{\infty} S_{x_s}(\omega)d\omega \tag{17}$$

3. Numerical study

In this section, the effectiveness of MTMD in suppressing the dynamic response of main system is analysed under the important parametric variations. The parameters include: (i) damping ratio of MTMD (ξ_T), total number of MTMD (n), mass ratio (γ), frequency spacing parameter of MTMD (β), tuning frequency ratio (f) and frequency content of ground excitation. In addition, the effect of the distribution of frequencies of the MTMD around the mean frequency is investigated. The damping ratio of the main system is taken as 2% of critical. The system is subjected to a base acceleration $\ddot{x}_g$ and its PSDF function is modeled as a stationary white-noise random process i.e.

$$E[\ddot{x}_g(t)\ddot{x}_g(t+\tau)]=2\pi S_0\delta(\tau) \tag{18}$$

where E is the expectation operator; S_0 is the intensity of white-noise excitation; $\delta(\cdot)$ is delta-diarc function. However, the effect of frequency content of ground excitation is also investigated.

The response quantity of interest is the root mean square (RMS) displacement of the main system. In order to study the effectiveness of MTMD, it is convenient to express the response in terms of the response ratio, R, defined as:

$$R=\frac{\text{RMS displacement of main system with MTMD}}{\text{RMS displacement of main system without MTMD}} \tag{19}$$

The response ratio R is a measure of the effectiveness of MTMD. The ratio being less than unity implies that the RMS displacement of the main system with MTMD has been reduced in comparison to the response without MTMD and effective in reducing the dynamic response of the system. The effect of parameters, ξ_T, n, γ, β and f is investigated for white-noise excitation and uniform variation of frequencies of the MTMD. However, the effect of the distribution of frequencies is investigated separately.

3.1. Effect of damping ratio of MTMD (ξ_T)

Fig. 2 shows the variation of response ratio R against the damping ratio of the MTMD for $n=1$ and 21. The frequency ratio and frequency spacing parameter of the MTMD are taken as 1 and 0.2, respectively. At sufficiently high damping, the response ratio R slowly increases with the increase of damping. However, at low damping there is a significant difference between

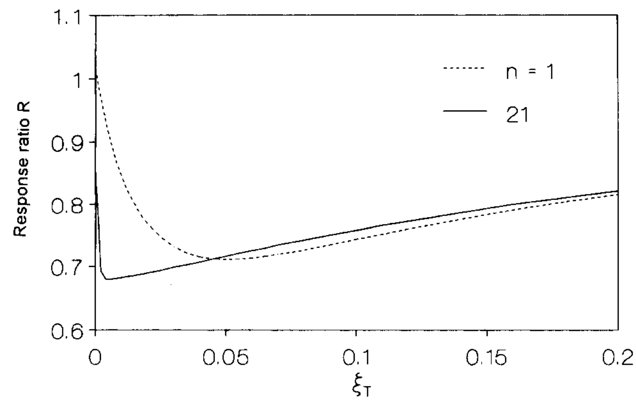

Fig. 2 Variation of response ratio R against damping ratio of MTMD (ξ_T) $\gamma=1\%$, $\beta=0.2$ and $f=1$.

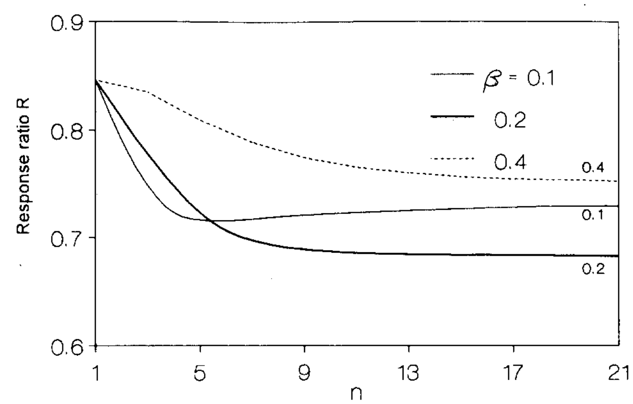

Fig. 3 Variation of response ratio R against total number of MTMD (n) and non-dimensional frequency spacing, β for $\xi_T=1\%$, $\gamma=1\%$ and $f=1$.

the effectiveness of a single TMD and the MTMD. The single TMD provides the minimum response at approximately 5% damping which confirms the conclusion of Warburton (1982). For lower value of damping the single TMD loses all of its effectiveness the response ratio R approaches unity which corresponds to response of the main system without a tuned mass damper. Further, the MTMD is significantly less sensitive than the single TMD at low value of damping ratio. Fig. 2 also shows that the MTMD can be more effective than a single TMD, the difference being more pronounced at low damping. Thus, the MTMD can provide better attenuation of base excited vibrations, and can do so with a smaller damping ratio of each auxiliary mass, than the damping ratio of an optimal single TMD.

3.2. Effect of total number of TMD (n)

In Fig. 3, the variation of response ratio R is plotted against the total number of MTMD for $\beta=0.1$, 0.2 and 0.4. The damping ratio of each TMD is taken as 1% and tuning frequency ratio is unity for all cases. As the total number of MTMD increases the effectiveness of MTMD in suppressing dynamic response of main system increases. It is to be noted that $n=1$ indicates isolation of the main system by a single TMD and the figure shows that the response ratio

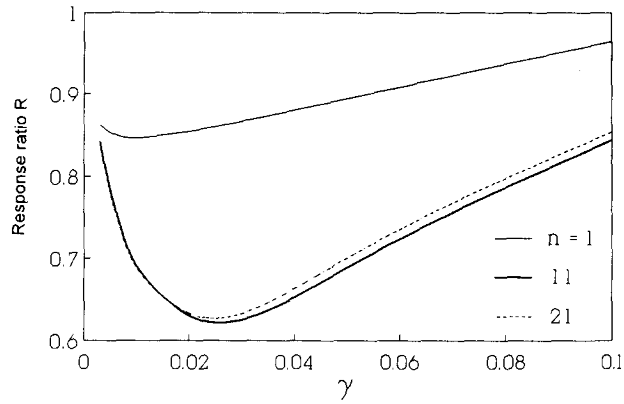

Fig. 4 Variation of response ratio R against mass ratio (γ) $\xi_T=1\%$, $\beta=0.2$ and $f=1$.

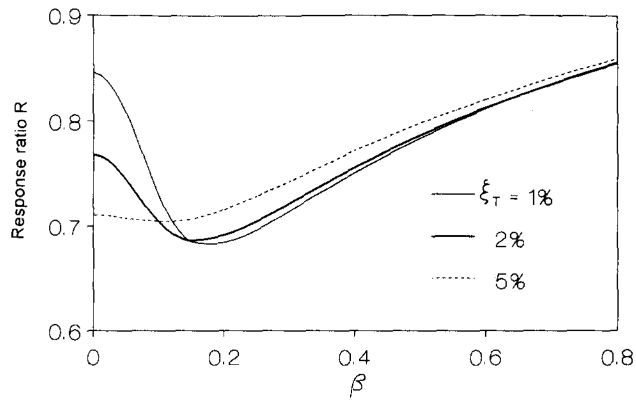

Fig. 5 Variation of response ratio R against frequency range of MTMD (β) $n=21$, $\gamma=1\%$ and $f=1$.

R is maximum for all values of frequency range parameter. Thus, the MTMD is more effective than a single TMD. Further, the increase in the number of TMD's beyond a certain value (in this case $n=11$), the effectiveness of MTMD remains almost invariant.

3.3. Effect of mass ratio (γ)

In Fig. 4 variation of the response ratio R is plotted against the mass ratio γ for $n=1$, 11 and 21. The value of other parameters taken are: $\xi_T=1\%$, $\beta=0.2$ and $f=1$. The figure indicates that the response of the system with MTMD is more sensitive to the mass ratio as compared to the response with a single TMD. The optimum value of mass ratio is around 1% for a single TMD. However, for the MTMD it is in the range of 2 to 3%. Further, for all values of the mass ratio MTMD are more effective than a single TMD.

3.4. Effect of frequency spacing parameter of MTMD (β)

In Fig. 5, the variation of the response ratio R is plotted against the non-dimensional frequency spacing, β of the MTMD for damping ratio ξ_T equal to 1%, 2% and 5%. The total number of MTMD's is 21 and tuning frequency ratio is equal to unity. Fig. 5 shows that the frequency

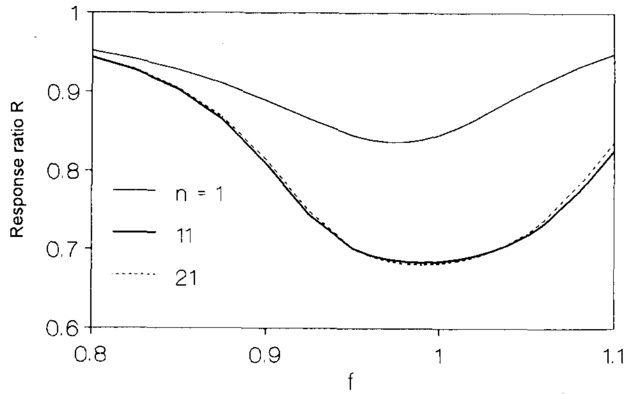

Fig. 6 Variation of response ratio R against tuning frequency ratio (f) ξ_T=1%, γ=1% and β=0.2.

spacing of the MTMD significantly influences the effectiveness of MTMD. There exists an optimum value of the frequency spacing parameter (in the range of $0.1<\beta<0.2$) which provides maximum effectiveness of the MTMD for a given damping ratio of MTMD. This optimum value of frequency spacing parameter decreases with the increase of the damping ratio of the MTMD.

3.5. *Effect of tuning frequency ratio* (f)

In Fig. 6, the variation of response ratio R is plotted against the tuning frequency ratio (f) for total number of TMD equal to 1, 11 and 21. The frequency spacing parameter, β, of the MTMD is taken as 0.2. The damping ratio of MTMD is taken as 1%. Fig. 6 shows that there exists an optimum value of tuning frequency ratio at which the response of main system becomes minimum for both single TMD and MTMD. The optimum value of the frequency ratio occurs in the vicinity of unity (0.99 for single TMD). It is to be noted that response ratio R remains almost invariant for a wider range of tuning frequency ratio where maximum effectiveness occurs. This implies that if there is an error in the natural frequency of the main system (in this case± 5%) the MTMD is still effective.

4. Effects of distribution of frequencies of MTMD

The parametric variation investigated earlier is based on that the natural frequencies of the MTMD are uniformly distributed around the mean frequency of the MTMD. This implies that the natural frequency of MTMD varies uniformly as shown in Fig. 7 by Type-I and expressed by Eq. (1). However, it will be interesting to study where the frequency of MTMD are varied non-uniformly (parabolic) as shown in Fig. 7 by Type-II and Type-III. The natural frequency of the j^{th} TMD for Type-II and III variation of frequencies of the tuned mass damper, respectively is expressed as

$$\omega_j=\omega_T\left[1+\frac{J(2N-|J|)}{N}\ \frac{\beta}{2N}\right] \tag{20}$$

$$\omega_j=\omega_T\left[1+\frac{J|J|}{N}\ \frac{\beta}{2N}\right] \tag{21}$$

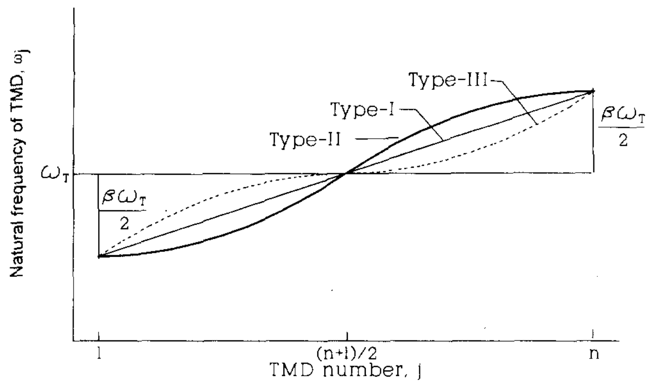

Fig. 7 Distribution of the frequencies of the MTMD.

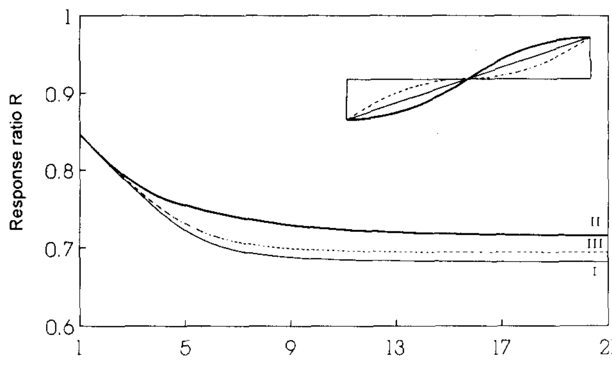

Fig. 8 Variation of response ratio R for different distribution of frequencies against number of MTMD for $\xi_T=1\%$, $\gamma=1\%$ and $\beta=0.2$, and $f=1$.

where the parameter J and N are related to j and n by the relation

$$J=j-\frac{n+1}{2} \tag{22}$$

$$N=\frac{n-1}{2} \tag{23}$$

Fig. 8 shows the variation of the response ratio, R, for the different distributions of TMD natural frequencies against the number of MTMD, n for $\xi_T=1\%$, $\gamma=1\%$, $\beta=0.2$ and $f=1$. Fig. 8 indicates that as the number of MTMD increases the effectiveness increases for the three types of the frequency distributions. The uniform distribution of frequencies of the MTMD (Type-I) performs best of the three. Fig. 9 shows the variation of the response ratio R for different distribution of frequencies against frequency spacing of MTMD for $\xi_T=1\%$, $n=11$, $\gamma=1\%$ and $f=1$. Type-I provides maximum effectiveness for optimum frequency spacing. However, for frequency spacing lesser (greater) than the optimum value Type-II (Type-III) performs better than Type-I, respectively. Further, the effectiveness of MTMD with Type-III distribution is relatively insensitive to the variation of the frequency spacing in the higher range. In Fig. 10 the variation of the ratio R for different frequency distributions is plotted against tuning frequency ratio, f for $\xi_T=1\%$, $n=11$, $\gamma=1\%$ and $\beta=0.2$ which also shows the similar trend. Thus, an optimally designed

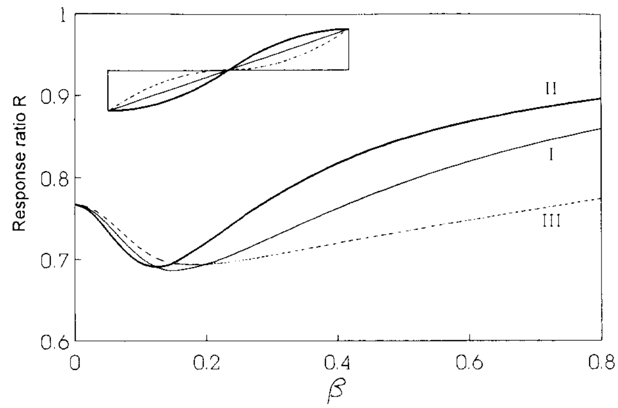

Fig. 9 Variation of response ratio R for different distribution of frequencies against frequency spacing of MTMD for $\xi_T=1\%$, $n=11$, $\gamma=1\%$ and $f=1$.

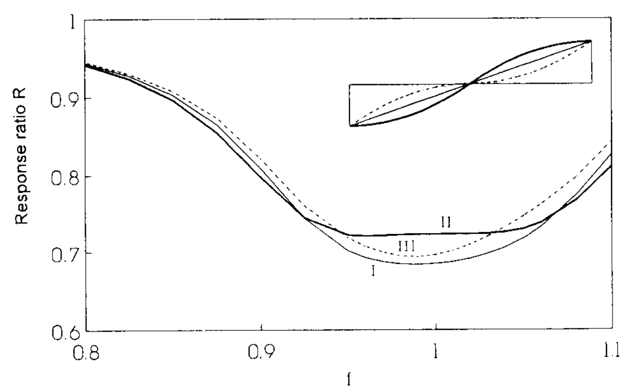

Fig. 10 Variation of response ratio R for different distribution of frequencies against frequency ratio (f) for $\xi_T=1\%$, $n=11$, $\gamma=1\%$ and $\beta=0.2$.

MTMD system with a uniform distribution of frequencies is more effective than other non-uniform distributions.

5. Influence of frequency content of ground excitation

In order to study the effect of frequency content of ground excitation on optimum damping of isolator the ground acceleration is modeled as filtered white-noise (Kanai-Tajimi spectrum). The power spectral density function of the ground acceleration is expressed as

$$S_{\ddot{x}_g}(\omega)=S_0\frac{1+4\xi_g^2(\omega/\omega_g)^2}{[1-(\omega/\omega_g)^2]^2+4\xi_g^2(\omega/\omega_g)^2} \tag{24}$$

where S_0 is the intensity of input white-noise; ξ_g is the damping ratio of the ground filter; and ω_g is the predominant ground frequency.

Fig. 11 shows the effect of excitation frequency on the variation of the response ration R, for different numbers of MTMD for $\xi_T=1\%$, $\gamma=1\%$, $\beta=0.2$ and $f=1$. The damping constant of the ground filter is taken as 0.5. Fig. 11 indicates that the effectiveness of the MTMD is

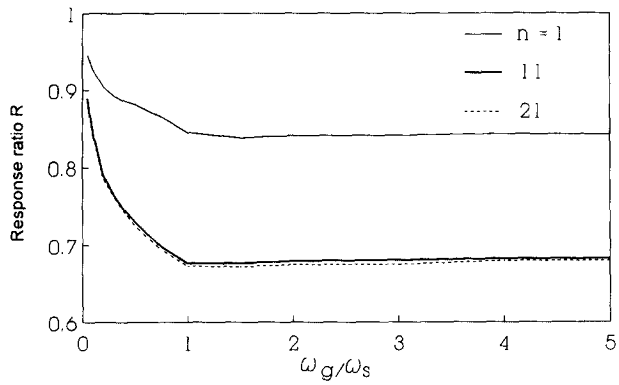

Fig. 11 Effect of excitation frequency on the variation of response ratio R for different number of MTMD for $\xi_T=1\%$, $\gamma=1\%$, $\beta=0.2$ and $f=1$.

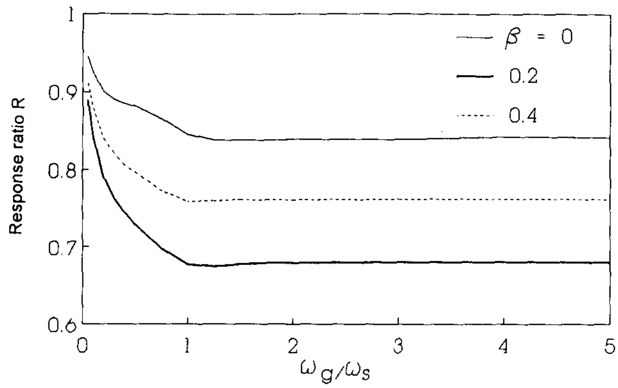

Fig. 12 Effect of excitation frequency on the variation of response ratio R for different frequency spacing of MTMD for $\xi_T=1\%$, $n=21$, $\gamma=1\%$ and $f=1$.

reduced for low frequencies of excitation ($\omega_g/\omega_s<1$). However, it remains constant for the variation of ω_g/ω_s beyond unity. Similar effects of excitation frequency can also be observed in Fig. 12 where the variation of the response ratio R is plotted for different frequency spacings.

6. Conclusions

The effectiveness of MTMD in reducing the dynamic response of a base excited system is investigated. The responses of the system with MTMD are compared with those of the same system without MTMD. A parametric study is conducted to investigate the fundamental characteristics of the MTMD's and the effect of important parameters on the effectiveness of the MTMD's. Further, the difference between the performance of a single TMD and MTMD is also investigated. From the trends of the results of this parametric study, the following conclusions may be drawn:

(1) The optimum designed MTMD is more effective for vibration isolation of a system than the optimum single TMD. Further, the increase in the number of TMD beyond a certain value (in this case $n=11$), the effectiveness of MTMD remains almost invariant.
(2) The optimum damping ratio in MTMD is significantly lower than optimum damping

ratio for a single TMD. For MTMD it is less than 1%.
(3) The mass ratio influences the effectiveness of the MTMD significantly. The optimal mass ratio for the MTMD is in the range of 2 to 3%.
(4) There exists an optimum value of frequency spacing parameter (in the range of 0.1 to 0.2) for which the effectiveness of MTMD is maximum. This optimum value decreases with the increase of the damping ratio of the MTMD.
(5) The optimum value of the tuning frequency ratio for MTMD is in the vicinity of unity.
(6) The effectiveness of MTMD is not much influenced by the changes in, or estimation errors in, the natural frequency of main system (in the range of$\pm$ 5%).
(7) An optimally designed MTMD system with a uniform distribution of frequencies is more effective than other non-uniform distributions.
(8) The effectiveness of MTMD is reduced for low frequency excitations ($\omega_g/\omega_s<1$).

Notations

c_j	damping of j^{th} TMD
c_T	damping constant of each TMD
$[C]$	damping matrix
E	expectation operator
f	tuning frequency ratio
J	parameter defined in Eq. (22)
k_j	stiffness of the j^{th} TMD
k_T	stiffness of each TMD
$[K]$	stiffness matrix
m_j	mass of j^{th} TMD
m_T	total mass of MTMD
m_s	mass of the main system
$[M]$	mass matrix
n	number of the tuned mass dampers
N	parameter defined in Eq. (23)
R	the response ratio
$S_{\ddot{x}_g}(\omega)$	PSDF function of the ground acceleration
S_0	the intensity of white-noise excitation
$S_{x_s}(\omega)$	PSDF of the displacement of main system
x_s	displacement of the main system relative to ground
x_j	displacement of the j^{th} tuned mass damper relative to ground
$\ddot{x}_g$	ground acceleration
$X(\omega)$	amplitude vector of the steady state response
$\{X\}$	the vector of displacement of the structural model
$Z(\omega)$	defined in Eq. (15)
β	non-dimensional frequency spacing
γ	mass ratio
ω	circular frequency
ω_T	average natural frequency of the MTMD
ω_j	natural frequency of the j^{th} TMD
ω_g	predominant ground frequency
ω_s	natural frequency of the main system
ξ_g	damping ratio of the ground filter
ξ_s	damping ratio of the main system
ξ_j	damping ratio of j^{th} TMD

ξ_T damping ratio of each TMD
$\delta(\cdot)$ delta-diarc function
$\sigma^2_{x_s}$ mean square response of the main system
$\{1\}$ vector of unity

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