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Analysis of the fracture of brittle elastic materials using a continuum damage model

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Abstract. The most known continuum damage theories for brittle structures are suitable to model the degradation of the material due to the deformation process and the consequent initiation of a macro-crack. Nevertheless, they are not able to describe the propagation of the crack that leads, eventually, to the breakage of the structure into parts that undergo rigid body motion. This paper presents a theory, formulated from formal arguments of Continuum Mechanics, that may describe not only the degradation but also the fracture of elastic structures. The modeling of such a discontinuous phenomenon through a continuous theory is possible by taking a cohesion variable, related with the links between material points, as an additional degree of kinematical freedom. The possibilities of the proposed theory are discussed through examples.

Key words: continuum damage mechanics; microstructure theory; brittle materials; strain softening; fracture.

1. Introduction

In the last few years, many different continuum theories have been proposed to describe the behavior of brittle materials. Among others we can refer to the studies developed by Bazant and Pijaudier-Cabot (1988), Bui, *et al.* (1981), Marigo (1985), Quoc Son (1984), Simo and Ju (1987), Florez (1989), Fremond, *et al.* (1990), Costa Mattos, *et al.* (1992).

It is generally accepted that continuum damage theories are suitable for the prediction of the degradation of the material due to the deformation process and the consequent initiation of a macro-crack. Nevertheless, even the theories that perform a mathematically correct and physically realistic description of the degradation of an elastic structure (including the strain-softening and localization behaviors) generally are not able to describe the evolution of a macro-crack and, consequently, the process that leads to its complete rupture.

It is also generally accepted that the evolution of a macro-crack can be reasonably described by theories developed within the framework of Fracture Mechanics, where an initial crack is assumed to exist (Liebowitz 1968-1972, Boeck 1974). Hence, in order to study:

(i) The degradation of a brittle structure due to the elastic deformations;

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(ii) The initiation of a macro-crack;
(iii) The propagation of the crack until the structure is broken into parts that undergo rigid body motion.

It is usually necessary to use a continuum damage model to describe the degradation of the body until a macroscopic crack initiation and then to use a fracture mechanics model to describe the propagation of the crack. The gap between Continuum Damage Mechanics and Fracture Mechanics has been pointed out by many authors (e.g., Lemaitre 1984, Lemaitre and Chaboche 1990).

This paper presents *one* model that describes both the degradation and the fracture of elastic structures which is formulated from formal arguments of Continuum Mechanics. The two main features of this model are:

(i) The governing equations are obtained within the framework of a micro-structure theory since a scalar variable β (called cohesion variable and related with the links between material points) is introduced as an additional kinematic variable.
(ii) The constitutive equations are developed within a thermodynamic framework. The basic assumption is that the thermodynamical state of a given material point at a given instant is a function not only of the strain ε and of the absolute temperature θ but also of the cohesion variable β and of its gradient $\nabla\beta$.

It is worth recalling that, in general, the continuum damage theories are not micro-structure theories. Besides, they do not consider the gradient of the damage variable as a state variable.

The possibilities of the proposed theory are discussed through the simulation of a rectangular plate submitted to prescribed displacements. The role of the material constants that appear in the constitutive equations is discussed and analysed. In this case there is a unique solution of the problem until the complete rupture of the plate into two parts that undergo rigid body motion. A predicted macrocrack is the set of points in the structure where the cohesion has reached its critical value. An approximation of the solution is easily obtained using classical finite element techniques. The proposed model describes the phenomenon of strain localization due to strain-softening. In the present study the presence of the term $\Delta\beta$ in the balance equation that governs the evolution of the variable β prevents the occurrence of discontinuous cohesion gradients and, consequently, the occurrence of discontinuous displacement gradients. This fact allows an adequate simulation of severe local deformations without the numerical difficulties of mesh-dependence (Needleman 1987) that arise in others continuum damage models (Sampaio and Martins 1992).

2. Preliminary definitions

A damageable body is defined as a set of material points B which occupies a region Ω of the euclidean space at the reference configuration. In this theory, besides the classical variables that characterize the kinematics of a continuum medium (displacements, velocities and accelerations of material points), an additional variable $\beta\in[0, 1]$, called cohesion variable, is introduced. This variable is related with the links between material points and can be interpreted as a measure of the local cohesion state of the material. If $\beta=1$, all the links are preserved and the initial elastic properties are also preserved. If $\beta=0$ a local rupture is considered since all the links between material points have been broken. Since the degradation is an irreversible phenomenon, the cohesion variation rate $\dot{\beta}=\frac{D\beta}{Dt}$ $\left(\frac{D(\bullet)}{Dt}\right.$ is the material time derivative of $(\bullet)$) must be negative or equal to zero. It is important to remark that the links between material points can be broken by a deformation (variation of the distance between material points) or by non mechanical actions (chemical or electromagnetical actions, for instance).

In the next sections the basic principles that govern the evolution of this kind of continuum is presented. For the sake of simplicity the hypothesis of small deformation will be assumed throughout this work. Hence, the density ρ is assumed to be constant in time and the conservation of mass principle is automatically satisfied.

The proposed principles may be regarded as a special case of the theories of micro-structures (Mindlin 1964, Toupin 1964). In particular these governing principles are very close to those proposed in the theory of elastic materials with voids (Cowin and Nunziato 1983). Nevertheless, the definition and the physical interpretation of the additional kinematic variable and also the proposed constitutive equations make both theories very different. In the theory of elastic materials with voids the additional variable is related with the change in solid volume fraction. The present theory assumes that the damage is related with micro-cracks and not with micro-voids, and hence the damaged brittle material is not considered a porous medium and the cohesion variable is not directly related with a volume change.

3. Summary of the basic principles

3.1. The virtual power principle

In this work an arbitrary part P of the body B that occupies a region $R \subset \Omega$ at the reference configuration is taken as a mechanical system. By definition, the boundary of the region R will be called Γ. It is considered with respect to P the space V_u of all fields $\hat{\boldsymbol{u}}$ of possible velocities and the space V_β of all fields $\hat{\beta}$ of possible cohesion variation rates. V_u is called the space of virtual velocities and V_β the space of virtual cohesion variation rates.

The power of the external forces $P_{EXT}(P, t, \hat{\boldsymbol{u}}, \hat{\beta})$ for a given virtual field of velocity $\hat{\boldsymbol{u}} \in V_u$ and for a given virtual field of cohesion variation rate $\hat{\beta} \in V_\beta$ is defined as:

$$P_{EXT}(P, t, \hat{\boldsymbol{u}}, \hat{\beta}) = P^{v}_{EXT}(P, t, \hat{\boldsymbol{u}}) + P^{\beta}_{EXT}(P, t, \hat{\beta}) \tag{1a}$$

$$P^{v}_{EXT}(P, t, \hat{\boldsymbol{u}}) = \int_R (\boldsymbol{b} \cdot \hat{\boldsymbol{u}}) dV + \int_\Gamma (\boldsymbol{g} \cdot \hat{\boldsymbol{u}}) dA \tag{1b}$$

$$P^{\beta}_{EXT}(P, t, \hat{\beta}) = \int_R (p\hat{\beta}) dV + \int_\Gamma (q\hat{\beta}) dA \tag{1c}$$

where $(\boldsymbol{b}, \boldsymbol{g})$ are called the external forces and (p, q) the external microscopic forces. The external forces are of two kinds: contact forces $\boldsymbol{g}$ acting on the boundary Γ and volume (or body) forces $\boldsymbol{b}$ acting on R. Similarly, the microscopic forces are of two kinds: contact microscopic forces q acting on Γ and microscopic volume forces p acting on R. The power of the microscopic forces must be introduced in the theory in order to take into account the non mechanical actions that affect the cohesion state of the material even if there is no mechanical deformation.

The power of the inertial forces $P_{IN}(P, t, \hat{\boldsymbol{u}}, \hat{\beta})$ for a given virtual field of velocity $\hat{\boldsymbol{u}} \in V_u$ and for a given virtual field of cohesion variation rate $\hat{\beta} \in V_\beta$ is defined as:

$$P_{IN}(P, \mathrm{t}, \hat{\boldsymbol{u}}, \hat{\beta}) = P^{v}_{IN}(P, t, \hat{\boldsymbol{u}}) + P^{\beta}_{IN}(P, t, \hat{\beta}) \tag{2a}$$

$$P^{v}_{IN}(P, t, \hat{\boldsymbol{u}}) = \int_R (\rho\, \ddot{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}}) dV \tag{2b}$$

$$P_{IN}^{\beta}(P,\ t,\ \hat{\beta})=\int_R(\rho\ l\ \ddot{\beta}\ \hat{\beta})dV \tag{2c}$$

where $\ddot{\boldsymbol{u}}=\dfrac{D^2\boldsymbol{u}}{Dt^2}$ and $\ddot{\beta}=\dfrac{D^2\beta}{Dt^2}$. $\boldsymbol{u}$ is the actual displacement field, β is the actual cohesion field and l is the microscopic inertia. The role of l in this theory is controversial and maybe the term (2c) is unnecessary. Nevertheless, since the microscopic inertia is considered in all the basic works concerned with microstructure theories (Goodman and Cowin 1972, for instance), it will be taken into account in the development.

In Continuum Mechanics it is usual to consider a first order gradient theory in which the power of the internal forces is supposed to be a function of the velocity and its gradient. In a continuum with microstructure the power of the internal forces is also supposed to be a function of β and $\nabla\beta$. On the assumption that, for a fixed instant t, the power of the internal forces can be expressed as a linear functional on $V_u\times V_\beta$, it may be shown that $P_{INT}(P,\ t,\ \hat{\boldsymbol{u}},\ \hat{\beta})$ must have the following form:

$$P_{INT}(P,\ \mathrm{t},\ \hat{\boldsymbol{u}},\ \hat{\beta})=P_{INT}^{v}(P,\ t,\ \hat{\boldsymbol{u}})+P_{INT}^{\beta}(P,\ t,\ \hat{\beta}) \tag{3a}$$

$$P_{INT}^{v}(P,\ t,\ \hat{\boldsymbol{u}})=\int_R(\sigma\cdot\nabla\hat{\boldsymbol{u}})dV \tag{3b}$$

$$P_{INT}^{\beta}(P,\ t,\ \hat{\beta})=\int_R(\boldsymbol{H}\cdot\nabla\hat{\beta}+F\hat{\beta})dV \tag{3c}$$

with σ being a second order tensor, $\boldsymbol{H}$ a vector and F a scalar. σ is the stress tensor and $(\boldsymbol{H},\ F)$ are the internal microscopic forces. Using the previous definitions, the virtual power principle may be stated as follows:

(P1) For a given part P of B that occupies region R of the space at the reference configuration, the stress tensor σ, the internal microscopic forces $(\boldsymbol{H},\ F)$, the external forces $(\boldsymbol{b},\ \boldsymbol{g})$, the microscopic external forces $(p,\ q)$, the actual acceleration $\ddot{\boldsymbol{u}}$ and $\ddot{\beta}$ must be such that:

$$\begin{aligned}&P_{IN}(P,\ \mathrm{t},\ \hat{\boldsymbol{u}},\ \hat{\beta})+P_{INT}(P,\ t,\ \hat{\boldsymbol{u}},\ \hat{\beta})\\&=P_{EXT}(P,\ t,\ \hat{\boldsymbol{u}},\ \hat{\beta})\ \ \forall\,\hat{\boldsymbol{u}}\in V_v\ \text{and}\ \forall\,\hat{\beta}\in V_\beta\end{aligned} \tag{4}$$

(P2) $P_{INT}(P,\ t,\ \hat{\boldsymbol{u}},\ \hat{\beta})$ is zero for a rigid body motion, i.e.:

$$P_{INT}(P,\ t,\ \hat{\boldsymbol{u}},\ \hat{\beta})=0\ \text{when}\ \hat{\boldsymbol{u}}(\boldsymbol{x})=\boldsymbol{A}(x-\boldsymbol{x}_0)+\boldsymbol{c}_0 \tag{5}$$

where $\boldsymbol{A}$ is an antisymmetrical tensor and $\boldsymbol{c}_o$ is the velocity of a reference point $\boldsymbol{x}_o\in R$.

Under suitable regularity assumption, it can be proved that (P1) implies the following local expressions:

$$\rho\ddot{\boldsymbol{u}}=\mathrm{div}\,\sigma+\boldsymbol{b}\ \text{in}\ R \tag{6a}$$

$$\rho\, l\ddot{\beta}=\mathrm{div}\boldsymbol{H}-F+p\ \text{in}\ R \tag{6b}$$

$$\sigma\boldsymbol{n}=\boldsymbol{g}\ \text{and}\ \boldsymbol{H}\cdot\boldsymbol{n}=q\ \text{in}\ \Gamma \tag{7}$$

where $\boldsymbol{n}$ is the unit outward normal to the surface Γ. It can also be proved that (P2) implies the symmetry of the stress tensor.

3.2. The first law of thermodynamics

In order to postulate the energy balance it is necessary to define the internal energy $U(P, t)$, the kinetic energy $K(P, t)$ and the thermal energy $P_{TH}(P, t)$ of the system:

$$U(P, t)=\int_R (\rho e)dV \tag{8}$$

$$K(P, t)=\int_R \left(\frac{\rho}{2}[\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}}+l\dot{\beta}^2]\right)dV \tag{9}$$

$$P_{TH}(P, t)=-\int_\Gamma (\boldsymbol{q}\cdot\boldsymbol{n})dA+\int_R (\rho r)dV \tag{10}$$

where e is the internal energy per unity of mass, $\boldsymbol{u}$ and β are the actual displacement and cohesion fields, $\boldsymbol{q}$ is the heat flux vector and r is a heat supply or source per unity of mass and time. The first law of thermodynamics can then be stated as:

$$\frac{D}{Dt}U(P, t)+\frac{D}{Dt}K(P, t)=P_{EXT}(P, t, \dot{\boldsymbol{u}}, \dot{\beta})+P_{TH}(P, t) \tag{11}$$

If the kinetic energy theorem is assumed to hold:

$$\frac{D}{Dt}K(P, t)=P_{IN}(P, t, \dot{\boldsymbol{u}}, \dot{\beta}) \tag{12}$$

it is possible to prove that $\dot{l}=0$ and thus the first law of thermodynamics may be written as:

$$\frac{D}{Dt}U(P, t)=P_{INT}(P, t, \dot{\boldsymbol{u}}, \dot{\beta})+P_{TH}(P, t) \tag{13}$$

The local form of the principle stated above is given by:

$$\rho\dot{e}=\sigma\cdot\nabla\dot{\boldsymbol{u}}+\boldsymbol{H}\cdot\nabla\dot{\beta}+F\dot{\beta}-div\,\boldsymbol{q}+\rho r \tag{14}$$

3.3. The second law of thermodynamics

If the first law of thermodynamics states the possibility of convertion of mechanical work into thermal energy and vice-versa, the second law of thermodynamics makes a distinction between possible and impossible processes. In order to postulate a second law restriction it is assumed the existence of the entropy per unity of mass s and of the absolute temperature θ. The total entropy $S_T(P, t)$ and the entropy flux $S_F(P, t)$ are then defined as:

$$S_T(P, t)=\int_R (\rho s)dV \tag{15}$$

$$S_F(P, t)=-\int_\Gamma \left(\frac{\boldsymbol{q}\cdot\boldsymbol{n}}{\theta}\right)dA+\int_R \left(\frac{\rho r}{\theta}\right)dV \tag{16}$$

The second law of thermodynamics can then be stated as:

$$\frac{D}{Dt}S_T(P, t)\geq S_F(P, t) \tag{17}$$

Using Eq. (14) to eliminate r and introducing the free energy per unity of volume $\Psi=\rho(e-\theta s)$, and the entropy per unity of volume $S=\rho s$, it is possible to obtain the following local form of the entropy inequality Eq. (17):

$$d=\sigma\cdot \nabla\dot{u}+\boldsymbol{H}\cdot \nabla\dot{\beta}+F\dot{\beta}-(\dot{\Psi}+S\dot{\theta})-\frac{1}{\theta}\boldsymbol{q}\cdot \nabla\theta\geq 0 \tag{18}$$

4. Constitutive equations

The balance Eq. (6), (14) and the second law restrictions Eq. (18) are valid for any kind of process in a damageable continuum. A complete modeling requires additional informations in order to characterize the behavior of each kind of material. In this section it is presented constitutive equations for an elastic damageable material.

4.1. State variables

Under the hypothesis of small deformations, the local state of a brittle elastic material is supposed to be a function of the absolute temperature θ, of the total strain $\varepsilon=(1/2)[\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T]$, of the cohesion variable β and of its gradient $\nabla\beta$.

4.2. Free energy-state laws

Following the classical assumption of the thermodynamic of irreversible processes, the free energy Ψ is supposed to be a function of the state variables with the following form:

$$\Psi(\varepsilon,\ \beta,\ \nabla\beta,\ \theta)=\hat{\Psi}(\varepsilon,\ \beta,\ \nabla\beta,\ \theta)+I(\beta) \tag{19}$$

where $\hat{\Psi}$ is a differentiable function and $I(\beta)$ is the indicator function of the set $[0,\ 1]$:

$$I(\beta)=\begin{cases}0, & \text{if } \beta\in[0,\ 1];\\ +\infty, & \text{otherwise.}\end{cases} \tag{20}$$

The constraint $0\leq\beta\leq 1$ is taken into account in the theory by the term $I(\beta)$ in the free energy function. This non-differentiable term is not classic and follows the idea developed in Fremond (1987, 1989) and Costa Mattos (1992).

In this paper, the following particular expression is chosen for the energy $\hat{\Psi}$:

$$\hat{\Psi}(\varepsilon,\ \beta,\ \nabla\beta,\ \theta)=\beta\hat{\Psi}_1(\varepsilon,\ \theta)+\hat{\Psi}_2(\beta)+\hat{\Psi}_3(\nabla\beta)+\hat{\Psi}_4(\theta) \tag{21}$$

with

$$\hat{\Psi}_1(\varepsilon,\ \theta)=\mu\varepsilon\cdot\varepsilon+\frac{1}{2}\lambda(tr\varepsilon)^2-(3\lambda+2\mu)\alpha(\theta-\theta_o)tr\varepsilon \tag{22a}$$

$$\hat{\Psi}_2(\beta)=w(1-\beta) \tag{22b}$$

$$\hat{\Psi}_3(\nabla\beta)=\frac{k}{2}\nabla\beta\cdot\nabla\beta \tag{22c}$$

$$\hat{\Psi}_4(\theta)=-\frac{C_e}{2\theta_o}(\theta-\theta_o)^2 \tag{22d}$$

where λ, μ are the Lamè constants, α, C_e, w, k, c are positive constants of the material and θ_o is a reference temperature. The term $\hat{\Psi}_1$ is the classical expression for the free energy of an isotropic elastic material. The potential $\hat{\Psi}_4$ is also commonly adopted in the studies of thermo-elasticity. The term $\hat{\Psi}_2$ is introduced to assure that $\dot{\beta}=0$ if $\hat{\Psi}_1 \leq w$ (see Eq. (40)) and, finally, the term $\hat{\Psi}_3$ is considered so as to give to β a diffusive behavior, thus smoothing the field β on Ω.

The here called thermodynamic forces $(\sigma, F^R, \boldsymbol{H}, S)$ related to the state variables $(\varepsilon, \beta, \nabla\beta, \theta)$ are defined from the free energy by the state laws:

$$\sigma=\frac{\partial\Psi}{\partial\varepsilon}=\frac{\partial\hat{\Psi}}{\partial\varepsilon}\Rightarrow\sigma=\beta[\lambda tr(\varepsilon)\mathbf{1}+2\mu\varepsilon]-\beta(3\lambda+2\mu)\alpha(\theta-\theta_o)\mathbf{1} \tag{23}$$

$$\boldsymbol{H}=\frac{\partial\Psi}{\partial(\nabla\beta)}=\frac{\partial\hat{\Psi}}{\partial(\nabla\beta)}\Rightarrow\boldsymbol{H}=k\nabla\beta \tag{24}$$

$$S=-\frac{\partial\Psi}{\partial\theta}=-\frac{\partial\hat{\Psi}}{\partial\theta}\Rightarrow S=\beta(3\lambda+2\mu)\alpha tr\,\varepsilon+\frac{C_e}{\theta_o}(\theta-\theta_o) \tag{25}$$

$$F^R=\frac{\partial\hat{\Psi}}{\partial\beta}+h_\beta=\hat{\Psi}_1(\varepsilon,\ \theta)-w+h_\beta;\ h_\beta\in\partial I(\beta)$$

$$\Rightarrow F^R=\mu\varepsilon\cdot\varepsilon+\frac{1}{2}\lambda(tr\,\varepsilon)^2-(3\lambda+2\mu)\alpha(\theta-\theta_o)tr\,\varepsilon-w+h_\beta \tag{26}$$

It must be noted that the state variables are taken as independent parameters in Eqs. (23)-(26). The set $\partial I(\beta)$ is the subdifferential of I at β (Ekeland and Teman 1976) given by:

$$\partial I(\beta)=\{h_\beta\in(-\infty,\ +\infty);\ h_\beta(\beta-\hat{\beta})\geq(I(\beta)-I(\hat{\beta}))\ \ \forall\hat{\beta}\in(-\infty,\ +\infty)\} \tag{27}$$

It is also interesting to remark that the state law Eq. (23) can be written in a alternative form using the Young's modulus E and the Poisson's ratio ν instead of the Lamè constants:

$$\sigma=\left(\frac{\beta E}{1+\nu}\right)\left(\frac{\nu}{1-2\nu}tr(\varepsilon)\mathbf{1}+\varepsilon\right)-\left(\frac{\alpha\beta E}{1-2\nu}\right)(\theta-\theta_o)\mathbf{1}$$

To complete the constitutive equations additional informations about the dissipative behavior must be given. These informations can be obtained from a pseudo-potential of dissipation and are called complementary laws.

4.3 Pseudo-potential of dissipation - Complementary laws

The pseudo-potencial of dissipation Φ for an elastic damageable material is function of $\dot{\beta}$ and $\nabla\theta$. Since the internal constraint $\dot{\beta}\leq 0$ is a physical property, it must be taken into account by the constitutive theory. Hence, the potential Φ is supposed to have the following form:

$$\Phi(\dot{\beta},\ \nabla\theta,\ \theta)=\hat{\Phi}(\dot{\beta},\ \nabla\theta,\ \theta)+I_-(\dot{\beta}) \tag{28}$$

where $\hat{\Phi}$ is a differentiable function and $I_-(\dot{\beta})$ is the indicator function of the set $(-\infty,\ 0)$:

$$I_-(\dot{\beta})=\begin{cases}0, & \text{if } \dot{\beta}\leq 0;\\ +\infty, & \text{otherwise.}\end{cases} \tag{29}$$

The constraint $\dot{\beta}\leq 0$ is taken into account in the theory by the term $I_-(\dot{\beta})$ in the pseudo-potential of dissipation.

In this paper, the following particular expression is chosen for $\hat{\Phi}$:

$$\hat{\Phi}=\frac{c}{2}\dot{\beta}^2+\frac{1}{2\theta}(\boldsymbol{K}\nabla\theta)\cdot\nabla\theta \tag{30}$$

where c is a positive constant and $\boldsymbol{K}$ is a second order symmetric and positive definite tensor called thermal conductivity tensor.

The thermodynamic force F^{IR}:

$$F^{IR}=F-F^R \tag{31}$$

and the heat flux vector $\boldsymbol{q}$ are related respectively to $\dot{\beta}$ and $\nabla\theta$ by the complementary laws:

$$F^{IR}=\frac{\partial\hat{\Phi}}{\partial\dot{\beta}}+h_{\dot{\beta}}=c\dot{\beta}+h_{\dot{\beta}};\ h_{\dot{\beta}}\in\partial I_-(\dot{\beta}) \tag{32}$$

$$\frac{\boldsymbol{q}}{\theta}=-\frac{\partial\hat{\Phi}}{\partial(\nabla\theta)}\Rightarrow\boldsymbol{q}=-\boldsymbol{K}\nabla\theta \tag{33}$$

The variables $\dot{\beta}$, $\nabla\beta$ and θ are taken as independent parameters in Eqs. (32) and (33). The state laws Eqs. (23) - (26) and the complementary laws Eqs. (32) - (33) define a complete set of constitutive equations for a damageable elastic material.

These constitutive equations are an extension of the theory presented in Fremond, *et al.* (1990). The introduction of a "unilateral" behavior in the theory (damage only under tensile loading) is possible if an additional term $[(1-\beta)\Psi_M(\varepsilon)]$ is considered in the free energy (similarly as it was proposed in Costa Mattos 1992):

$$\Psi_M(\varepsilon)=\sup_{\sigma\in\Sigma}\left\{\sigma\cdot\varepsilon-\frac{\lambda^C}{2}(tr(\sigma))^2-\mu^C\sigma\cdot\sigma-\alpha\theta tr(\sigma)\right\} \tag{34}$$

where λ^C and μ^C are given by:

$$\lambda^C=-\frac{\lambda}{2\mu(3\lambda+2\mu)}\ ; \qquad \mu^C=\frac{1}{4\mu} \tag{35}$$

and Σ is the set of all second order symmetric tensors with nonpositive eigenvalues. This additional term will be neglected in the present work because it is not necessary to understand the basic features of the model and, besides, because it makes the numerical approximation of the resulting problems much more complex.

5. Examples

In this section the study is restricted to quasi-static and isothermal processes. The purpose is to analyse the role of the material parameters w, k and c in the theory.

Under these hypothesis, the balance equations and the constitutive equations can be reduced to:

$$div\,\sigma+b=0 \tag{36}$$

$$div\,\boldsymbol{H}-F+p=0 \tag{37}$$

$$\sigma=\left(\frac{\beta E}{1+\nu}\right)\left(\frac{\nu}{1-2\nu}tr(\varepsilon)\mathbf{1}+\varepsilon\right) \tag{38}$$

$$H=k\,\nabla\beta;\quad F=\hat{\Psi}_1(\varepsilon)-w+h_\beta+c\dot{\beta}+h_{\dot{\beta}} \tag{39}$$

The substitution of the constitutive Eq. (39) in Eq. (37) gives the balance equation that govern the evolution of the cohesion variable β:

$$k\,\Delta\beta+w-\hat{\Psi}_1-h_\beta+p=c\dot{\beta}+h_{\dot{\beta}} \tag{40}$$

5.1. Bar under prescribed displacement

In this example it is considered an elastic damageable bar of lenght L, initially undamaged, free at the ends and submitted to a prescribed axial displacement v: $v(x=0,\ t)=0$ and

$$v(x=L,\ t)=\begin{cases}\alpha t, & \text{if } 0\le t\le t_1;\\ \alpha(2t_1-t), & \text{if } t_1\le t\le t_2;\\ \alpha(t-2t_2+2t_1), & \text{if } t\ge t_2.\end{cases} \tag{41}$$

Neglecting the weight of the bar ($\boldsymbol{b}=0$), assuming that the bar is not submitted to external microscopic forces ($p=0$ and $q=0$) it is possible to obtain an analytical solution for the problem provided an uniaxial state of stress is considered.

Under these assumptions, the equation that governs the evolution of the cohesion variable β can be reduced to:

$$k\,\frac{\partial^2\beta}{\partial x^2}+w-\frac{1}{2}E\left(\frac{v(L,\ t)}{L}\right)^2-h_\beta=c\dot{\beta}+h_{\dot{\beta}} \tag{42a}$$

$$\frac{\partial\beta}{\partial x}(x=0,\ t)=\frac{\partial\beta}{\partial x}(x=L,\ t)=0;\ \beta(x,\ t=0)=1,\ \forall x\in[0,\ L] \tag{42b}$$

This problem admits a unique homogeneous solution $\beta(x,\ t)=\beta_h(t)$ which satisfies:

$$\dot{\beta}_h(t)=\frac{1}{c}\left\langle\frac{1}{2}E\left(\frac{v(L,\ t)}{L}\right)^2-w\right\rangle;\qquad \beta_h(t=0)=1 \tag{43}$$

with $\langle a\rangle=max\{0,\ a\}$. From the Eq. (43) it can be verified that $\dot{\beta}=0$ if the term $\frac{1}{2}E\left(\frac{v(L,\ t)}{L}\right)^2$ is smaller than w. The evolution will begin at the instant t_a where the energy is equal to w. Hence:

$$\beta_h=1,\quad \text{for } t\le t_a=[(2wL^2)/(E\,\alpha)]^{1/2} \tag{44}$$

As the loading process goes on, for $t_a<t<t_1$, the energy increases and $\dot{\beta}$ is strictly negative:

$$\beta_h(t)=\frac{1}{c}\left(wt-\frac{1}{6}E\left(\frac{\alpha}{L}\right)^2t^3\right)+A\qquad \text{for } t_a\le t\le t_1 \tag{45}$$

where

$$A=1-\frac{1}{c}\left(wt_a-\frac{1}{6}E\left(\frac{\alpha}{L}\right)^2t_a^3\right) \tag{46}$$

After the instant $t=t_1$, as the bar is unloaded, ($v(L,\ t)=\alpha(2t_1-t)$, the elastic energy decreases. The cohesion rate $\dot{\beta}$ will be strictly negative while the term $\frac{1}{2}E\left(\frac{v(L,\ t)}{L}\right)^2$ is greater than w.

$$\beta_h(t)=\frac{1}{c}\left(wt-\frac{1}{6}E\left(\frac{\alpha}{L}\right)^2(2t_1-t)^3\right)+B\qquad \text{for } t_1\le t\le t_b=(2t_1-t_a) \tag{47}$$

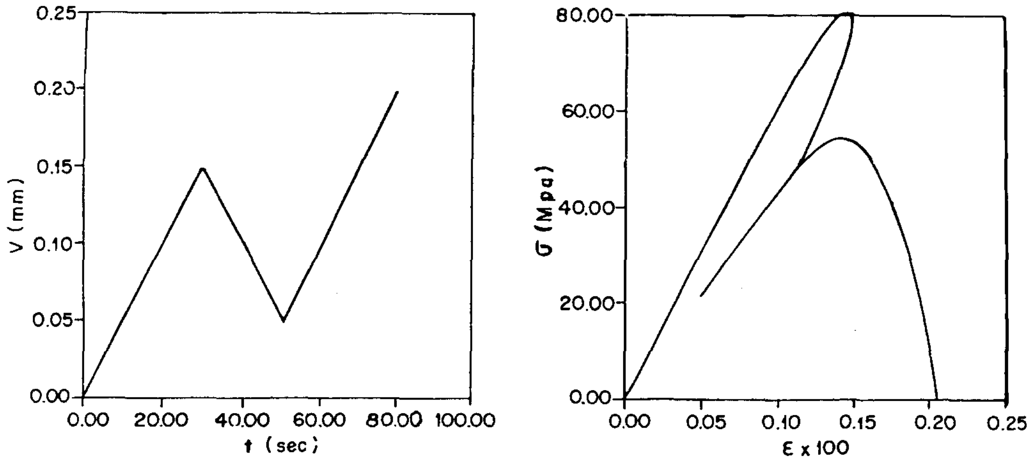

Fig. 1 Hypothetical tensile test.

where

$$B=A-\frac{1}{3c}E\left(\frac{\alpha}{L}\right)^2 t_1^3$$

The loading history is such that, for $t_b \leq t \leq t_c = 2t_2 - t_b$ the value of the term $\frac{1}{2}E\left(\frac{v(L,\ t)}{L}\right)^2$ is smaller than w and thus $\dot{\beta}=0$:

$$\beta_h(t)=\frac{1}{c}\left(wt-\frac{1}{6}E\left(\frac{\alpha}{L}\right)^2(2t_1-t_b)^3\right)+B \qquad \text{for } t_b \leq t \leq t_c=(2t_2-t_b) \tag{48}$$

Finally, after the instant $t=t_c$, the term $\frac{1}{2}E\left(\frac{v(L,\ t)}{L}\right)^2$ is again greater than w and β decreases until the instant t_f where $\beta=0$. The cohesion variable will be zero after t_f:

$$\beta_h(t)=\frac{1}{c}\left(wt-\frac{1}{6}E\left(\frac{\alpha}{L}\right)^2(t-2t_2+2t_1)^3\right)+C \qquad \text{for } t_c \leq t \leq t_f \tag{49}$$

where

$$C=B-\frac{1}{3c}E\left(\frac{\alpha}{L}\right)^2(2t_1-t_2)^3 \tag{50}$$

It is important to remark that the process is always irreversible ($\dot{\beta} \leq 0$). As a con sequence of the evolution of β, the uniaxial stress $\sigma_{xx}=\beta E\frac{v(L,\ t)}{L}$ will reach a maximum value and decrease. The Fig. 1 shows the stress-strain curve for the following material constants: $E=50000$ MPa, $w=0.25$ MPa, $c=1.0$. The parameters E, w are characteristic of some concretes. The results show that both the irreversibility of the damage process and the softening behavior are characterized in the model.

Since it is possible to obtain analytical solutions for uniaxial problems, the constants w and c can then be identified from simple uniaxial tests. w is the elastic energy necessary to begin the damage process and c is related to the viscosity of the material.

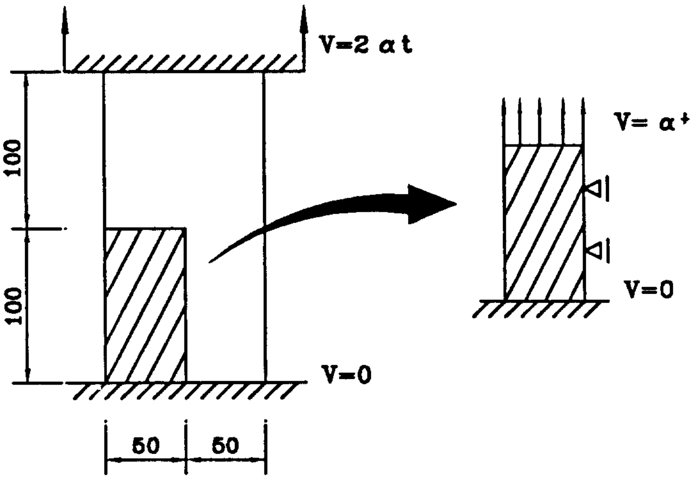

Fig. 2 Plate with prescribed displacement.

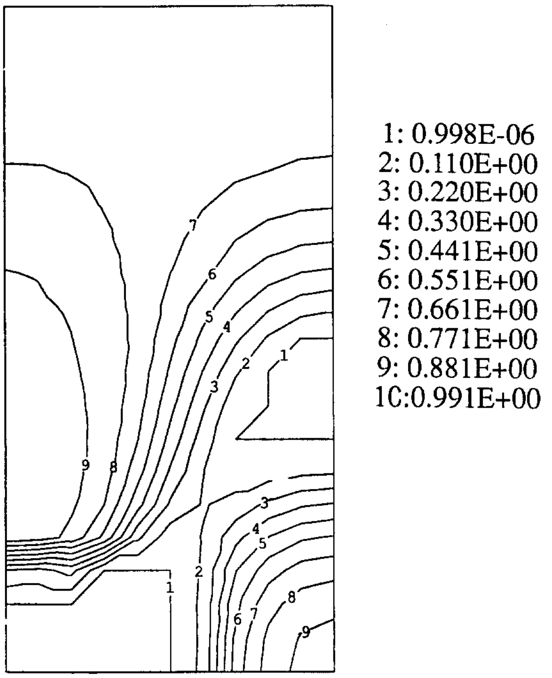

Fig. 3 Isovalues of β. t=25 s.

5.2. Plate under prescribed displacement

In this example it is considered an elastic damageable plate, initially undamaged, clamped at the ends and submitted to a prescribed axial displacement v: $v(x=0)=0$ and $v(x=\mathrm{L})=2\alpha t$. Neglecting the weight of the plate ($\boldsymbol{b}=\mathbf{0}$) and assuming that it is not submitted to external microscopic forces ($p=0$ and $q=0$) it is possible to obtain a numerical approximation of the problem using a Galerkin or Petrov-Galerkin finite element discretization (Fremond, *et al.* 1990).

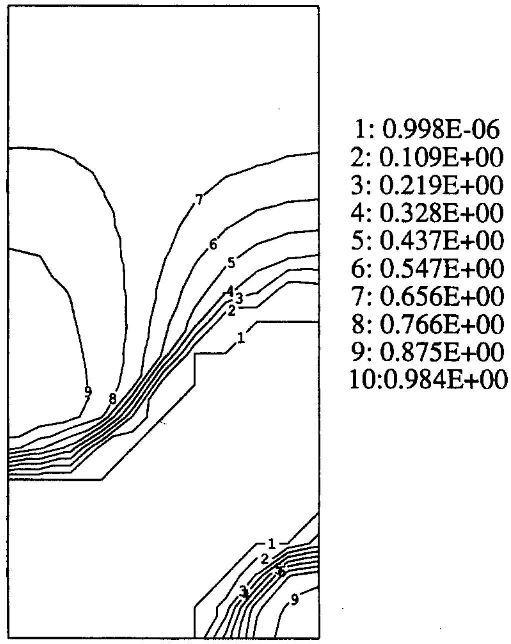

Fig. 4 Isovalues of β. t=30 s.

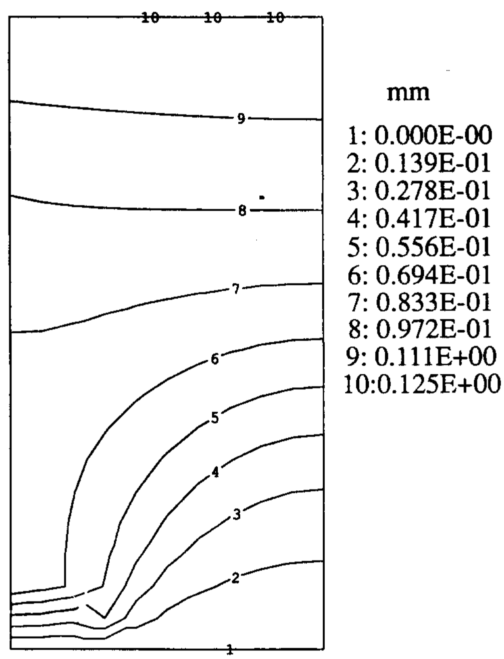

Fig. 5 Isovalues of the vertical displacement v. t=25 s.

In this case a plane state of strain is considered. The symmetry is used to reduce the size of the problem as it is shown in Fig. 2. This problem is adequate to study the role of the parameter k in the theory.

The Figs. 3 and 4 show the isovalues of β at different instants for the plate shown in Fig.

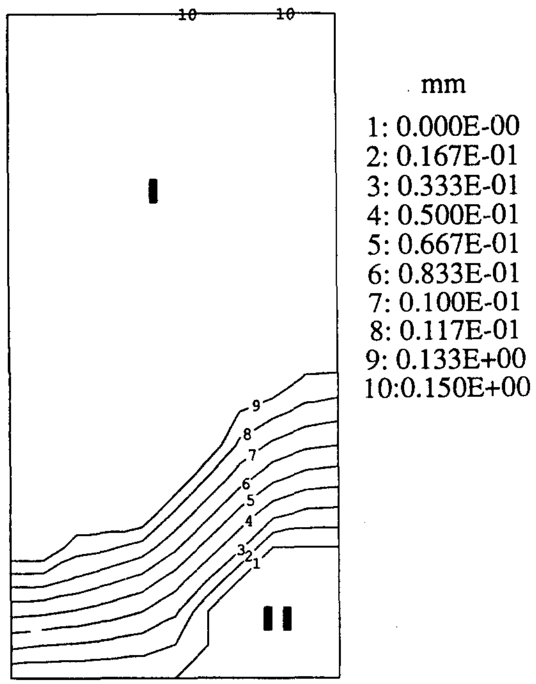

Fig. 6 Isovalues of the vertical displacement v. t=30 s.

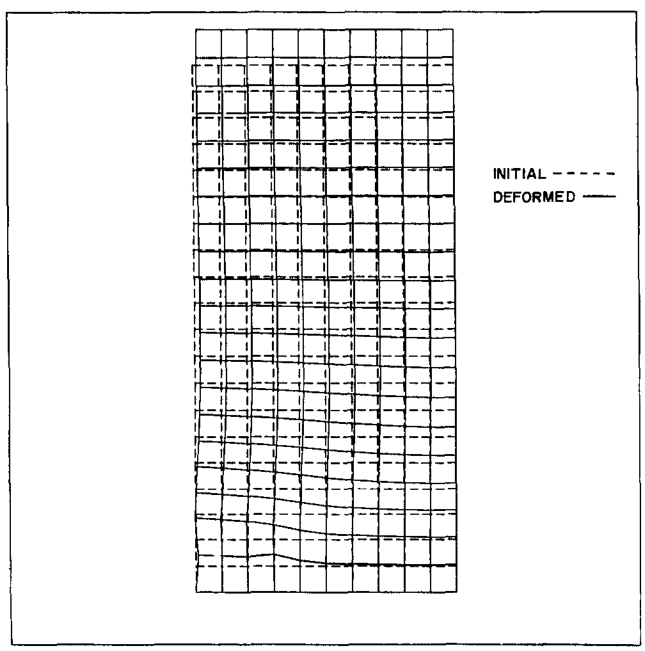

Fig. 7 Deformed mesh. t=25 s.

2 taking α=0.005 mm/s and the following material constants: E=50000 MPa, ν=0.2, w=0.25 MPa, c=1.0. and k=1.0. Due to computational aspects, a minimum value $\beta_{min}=1.E-6$ was imposed.

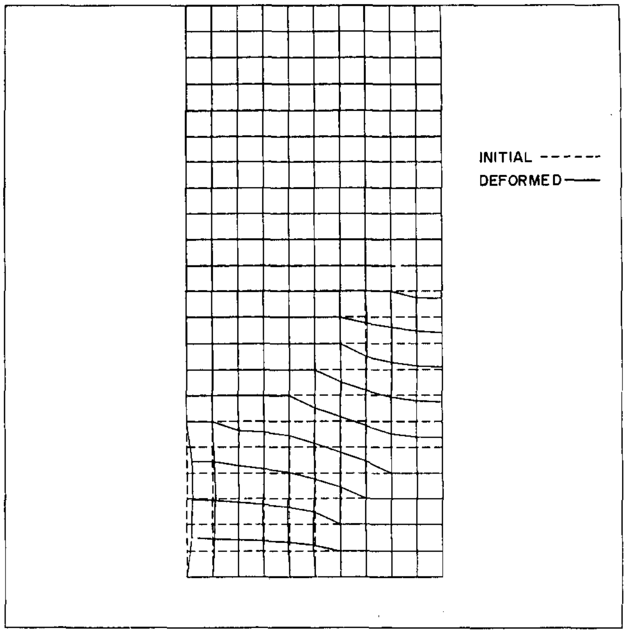

Fig. 8 Deformed mesh. t=30 s.

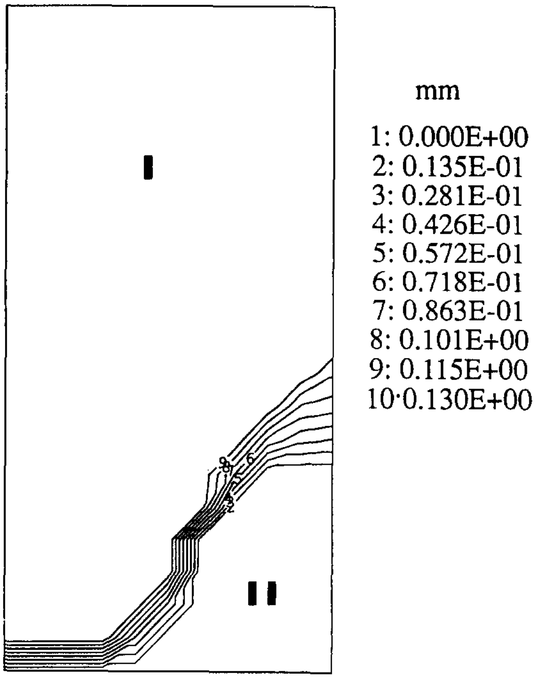

Fig. 9 Isovalues of the vertical displacement v (k=0.5). t=24 s.

As it can be verified in Fig. 4, the plate is completely broken at t=30 s. The analysis of the evolution of the vertical component of the displacement, presented in Figs. 5 and 6, is helpful to understand how the total failure of the structure is described in this theory.

Initially, a macro-crack appears on the left corner at the bottom of the plate ($t=25$ s, Fig. 5). When the structure is completely broken ($t=30$ s, Fig. 6) it is possible to identify two regions undergoing a rigid body motion (region I, with a displacement $v=0.15$ mm and region II with $v=0.$). The "crack" is represented by the zone of transition between the regions I and II where the material cannot resist to any kind of mechanical solicitation. The deformed meshes for $t=25$ s and $t=30$ s are presented, respectively in Figs. 7 and 8.

If a constant $k=0.5$ is considered instead of $k=1$, the fracture is more localized as it is shown in Fig. 9.

Hence, the smaller is the coefficient k, the more localized is the fracture (the transition zone is thinner). The presence of this parameter in the theory allows the modeling of different kind of brittle behaviors:

- Materials in which the fracture is not localized and the transition zone represents the portion of the structure "disintegrated" due to the decohesion of the particles (such as some kinds of rocks and concretes).
- Materials in which the fracture is very localized (such as the glasses and some kind of ceramics).

6. Concluding remarks

In this paper it is proposed a consistent framework in which to model the fracture of an elastic structure (degradation due to the deformation process, crack initiation, crack propagation, total rupture). The theory allows the modeling of different kind of materials (ceramics, concretes, rocks, glasses, etc.) and it is valid for any kind of geometry and for any kind of external loading. The material constants considered in the constitutive equations are simple to be identified experimentally and the resulting mathematical problems can be approximated through standard numerical techniques.

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Appendix

A thermodynamically consistent model

A consistent constitutive theory based on the concept of internal variables must not admit processes where the dissipation rate d (see Eq. (18)) is negative. Thus, it is very important to verify whether or not the constitutive equations satisfy the second law of thermodynamics.

Proposition: The constitutive theory defined in section 4 is consistent with the second law of thermodynamics expressed by Eq. (18) provided the following conditions hold:

(i) $\hat{\phi}(\dot{\beta}, \nabla\theta)$ is a convex and non-negative function.
(ii) $\hat{\phi}(0, \mathbf{0})=0$.

Proof: In order to verify the inequality Eq. (18), it is necessary to compute the term $\dot{\Psi}$. Since Ψ is a function of the state variables $(\varepsilon, \beta, \nabla\beta, \theta)$, we have:

$$\dot{\Psi}=\lim_{\Delta t\to 0}\left[\frac{\Psi(\varepsilon(t+\Delta t), \beta(t+\Delta t), \nabla\beta(t+\Delta t), \theta(t+\Delta t))-\Psi(\varepsilon(t), \beta(t), \nabla\beta(t), \theta(t))}{\Delta t}\right] \tag{A1}$$

This limit is to be computed at instant t with the information which are available, i. e., with the values of the state variables before the instant t. Thus Δt is negative and the derivative with respect to time is a left derivative. Hence, using the definition Eq. (19) of the free energy we have:

$$\dot{\Psi}=\frac{\partial\hat{\Psi}}{\partial\varepsilon}\cdot\dot{\varepsilon}+\frac{\partial\hat{\Psi}}{\partial\beta}\dot{\beta}+\frac{\partial\hat{\Psi}}{\partial(\nabla\beta)}+\nabla\dot{\beta}+\frac{\partial\hat{\Psi}}{\partial\theta}\dot{\theta}+\lim_{\Delta t\to 0}\left[\frac{I(\beta(t)+\Delta t)-I(\beta(t))}{\Delta t}\right] \tag{A2}$$

The subdifferential $\partial I(\beta(t))$ is such that:

$$I(\beta(t+\Delta t))-I(\beta(t))\geq h_\beta(\beta(t+\Delta t)-\beta(t)),\ \forall\, h_\beta\in\partial I(\beta(t)) \tag{A3}$$

The division of Eq. (A3) by $\Delta t<0$ gives:

$$\frac{I(\beta(t+\Delta t))-I(\beta(t))}{\Delta t}\leq h_\beta\frac{(\beta(t+\Delta t)-\beta(t))}{\Delta t},\ \forall\, h_\beta\in\partial I(\beta(t)) \tag{A4}$$

The limit of the inequality Eq. (A4) as Δt approaches the value zero leads to:

$$\lim_{\Delta t\to 0}\left[\frac{I(\beta(t+\Delta t))-I(\beta(t))}{\Delta t}\right]\leq h_\beta\dot{\beta} \tag{A5}$$

and then using the state laws Eqs. (23)-(26), we have:

$$d_1=\sigma\cdot\dot{\varepsilon}+\boldsymbol{H}\cdot\nabla\dot{\beta}+F^R\cdot\dot{\beta}-\dot{\Psi}-S\dot{\theta}\geq 0 \tag{A6}$$

From the definition of $I_-(\dot{\beta})$ it is possible to verify that

$$\partial I_{-}(\dot{\beta})=\{h_{\beta}\in(-\infty,\ +\infty);\ h_{\beta}(\dot{\beta}-\hat{\beta})\geq 0,\ \forall\,\hat{\beta}\leq 0\}$$

And, consequently:

$$d_2=h_{\beta}\dot{\beta}\geq 0 \quad \forall\,\dot{\beta}\leq 0 \tag{A7}$$

Finally, using the following classical result of Convex Analysis:

Let X and Y be elements of a vector space V with an internal product $(X\cdot Y)$.

If $\boldsymbol{\Phi}$: $v\rightarrow[0,\ +\infty]$ is a convex and differentiable function such that $\boldsymbol{\Phi}(0)=0$, then $(X\cdot Y)\geq\boldsymbol{\Phi}(X)\geq 0$ if $Y=\dfrac{d\boldsymbol{\Phi}}{dX}$.

it comes that:

$$d_3=\frac{\partial\hat{\boldsymbol{\Phi}}}{\partial\dot{\beta}}\dot{\beta}+\frac{\partial\hat{\boldsymbol{\Phi}}}{\partial(\nabla\theta)}\cdot\nabla\theta\geq 0,\ \forall\,(\dot{\beta},\ \nabla\theta) \tag{A8}$$

if the conditions (i) and (ii) of the proposition are verified.

Using the complementary laws Eqs. (32), (33) and combining Eqs. (A7) and (A8), we obtain:

$$d_2+d_3=F^{IR}\dot{\beta}-\frac{1}{\theta}\boldsymbol{q}\cdot\nabla\theta\geq 0 \tag{A9}$$

Hence, from Eqs. (A6) and (A9) we conclude that in the proposed model the dissipation $d=d_1+d_2+d_3$ is always non-negative provided the conditions (i) and (ii) hold.