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An adaptive control of spatial-temporal discretization error in finite element analysis of dynamic problems

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Abstract. The application of adaptive finite element method to dynamic problems is investigated. Both the kinetic and strain energy errors induced by space and time discretization were estimated in a consistent manner and controlled by the simultaneous use of the adaptive mesh generation and the automatic time stepping. Also an optimal ratio of spatial discretization error to temporal discretization error was discussed. In this study it was found that the best performance can be obtained when the specified spatial and temporal discretization errors have the same value. Numerical examples are carried out to verify the performance of the procedure.

Key words: adaptive analysis; dynamic analysis; stress wave propagation; discretization error; finite element analysis; direct time integration; Newmark method.

1. Introduction

Since the early development and application of the finite element method, attempts have been made to obtain the information about finite element discretization errors for better solutions. Accompanying the efforts to evaluate the discretization errors, the adaptive finite element method for more effective solution became one of the popular branches of the finite element method during the last decade. Especially for the finite element analysis of dynamic problems, it is not reasonable to proceed with a fixed mesh and fixed time step as the locations of steep stress regions or damping out of the energy is changing from time to time.

There are two basic issues for the adaptive finite element analysis, *i.e.* the error estimation and the adaptive control of the error. For elliptic problems the discretization error occurs from spatial discretization and the control of the error is effectively achieved by an adaptive mesh generation. To date a considerable success has been achieved on the problems of elliptic type, such as linear elastostatic problems (Zienkiewics and Zhu 1987, Babuska and Reinboldt 1979). Unlike the adaptive methods in elliptic problems in which only the spatial discretization error of displacement is concerned, more error sources such as the truncation in a time integration have to be considered for hyperbolic problems such as dynamic problems. Therefore for hyperbolic type problems a combined posteriori error estimate including both space and time discretization is needed (Zeng and Wiberg 1992). The adaptive control of the error performed by the adaptive mesh generation and automatic time stepping has been suggested by Choi and Chung (1994).

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There is already a rich literature on adaptive methods for transient problems, but almost all works deal with the estimation and the adaptive control of either the spatial or the time discretization error only. Among the works on adaptive control of the spatial discretization error Probert, *et al.* (1991, 1992) proposed an adaptive finite element method which controls the spatial discretization error to the solution of transient heat conduction problems and compressible flow problems. Zeng and Wiberg (1992) extended an a posteriori error estimator developed by Zienkiewicz and Zhu (1987) for elliptic problems to the dynamic analysis to estimate the spatial discretization error at a certain time and attempted to make the adaptive analysis by an automatic remeshing scheme. Bajer, *et al.* (1991) studied an adaptive technique in the dynamic elastic-viscoplastic problem by space-time elements with moving spatial nodes, where the modification of spatial meshes is made according to an interpolation-based error indicator. Joo and Wilson (1988) solved the structural dynamic problems by an adaptive mesh refinement based on Ritz vectors and a posteriori energy norm of residual errors.

For the conditionally stable time integration schemes, such as those used in explicit methods, the proper time step is primarily related to the stability criterion, as represented by a critical time step. Methods for automatic selection of time steps for the central difference method have been proposed by Park and Underwood (1980). The situation is more complicated when an unconditionally stable integration scheme, such as the Newmark method, is used. Bergan and Mollestad (1985) suggested an objective criteria for the performance and guidelines for making an adaptive time stepping algorithm for practical applications. Zienkiewicz and Xie (1991) proposed an error estimator by comparing the Newmark solution with the exact solutions obtained from the expanded Taylor series. They also proposed an adaptive time stepping procedure which uses the time discretization error estimator for dynamic analysis. Wiberg and Li (1993) developed a more precise error estimator which can evaluate the errors of displacement and velocities by a post-processing technique.

There is virtually no works reported in the published literature on the simultaneous consideration of the effects of space and time discretization which is desirable for the analysis of transient problems. In this study both the kinetic and strain energy errors induced by the space and time discretization were estimated in a consistent manner. These temporal and spatial discretization errors are controlled by the simultaneous use of adaptive mesh generation and automatic time stepping at every time stage. The optimal ratio of spatial and temporal discretization error to the total error is also discussed.

2. Discretization errors

To solve a dynamic problem by finite element method, the domain of interest is subdivided first into a number of elements. Then the semidiscrete Galerkin approximation can be used to obtain an integral formulation which is usually referred to as the weak form.

After evaluation of the integrals, a set of algebraic equations with initial conditions in a matrix form is obtained as follows.

$$M\ddot{U}_t+C\dot{U}_t+KU_t=F_t,\ t\in(0,\ T) \tag{1a}$$

$$U_t(x,\ 0)=U_0,\ \dot{U}_t(x,\ 0)=\dot{U}_0 \tag{1b}$$

where M is the mass matrix, C is the viscous damping matrix, K is the stiffness matrix, F_t is the vector of applied forces, and U_t, $\dot{U}_t$, $\ddot{U}_t$ are the displacement, velocity and acceleration vectors, respectively. The aforementioned discretization of continuous spatial domain during the formulation of finite elements induces the spatial discretization error, $e_S(x, t_n)$ which can be written as follows:

$$e_S(x, t_n) = u(x, t_n) - u^h(x, t_n) \tag{2}$$

where $u(x, t_n)$ is the exact solution of dynamic problem and $u^h(x, t_n)$ the solution to semidiscrete Galerkin approximation.

To obtain the transient responses, Eq. (1) is solved with certain time integration scheme. In the direct time integration, the approach is to write Eq. (1) at a specific instant of time $t = n\Delta t$,

$$M\ddot{U}_n + C\dot{U}_n + KU_n = F_n \tag{3}$$

where subscript n denotes the number of time steps, $n\Delta t$ and Δt are the current time and the size of time step, respectively. Generally, when a single-step scheme like the Newmark method is used for the direct time integration, the variation of the acceleration in each time step is assumed to be either constant or linear. This approximation yields a discontinuous distribution for the acceleration in the time domain and induces the temporal discretization error, $e_T(x, t_n)$ as follows:

$$e_T(x, t_n) = u^h(x, t_n) - U_n(x) \tag{4}$$

where $U_n(x)$ is the solution to Eq. (3) at time $t = t_n$.

The total discretization error which contains both the spatial and temporal discretization errors in the finite element solution can be expressed as

$$e(x, t_n) = u(x, t_n) - U_n(x) \tag{5}$$

Then, for any choice of norm (Oden and Reddy 1976)

$$\begin{aligned} \|e(x, t_n)\| &= \|u(x, t_n) - U_n(x) + u^h(x, t_n) - u^h(x, t_n)\| \\ &= \|e_S(x, t_n) + e_T(x, t_n)\| \le \|e_S(x, t_n)\| + \|e_T(x, t_n)\| \end{aligned} \tag{6}$$

Therefore the spatial discretization error and time discretization error can be estimated separately and the upper bound of the total discretization error can be evaluated by adding up both errors. This can be more clearly illustrated by Fig. 1.

3. Error estimates

In this study, for the consistent measure of the temporal and spatial discretization errors, an energy norm is taken. Let the total energy of the body be denoted by

$$E(u, \dot{u}) = \frac{1}{2}[(\dot{u}, \rho\dot{u}) + a(u, u)] \tag{7}$$

And replacing the displacement u by the error of displacement e and the velocity $\dot{u}$ by the error of velocity $\dot{e}$, the total energy of the error is obtained as follows:

$$E(e, \dot{e}) = \frac{1}{2}[(\dot{e}, \rho\dot{e}) + a(e, e)] \tag{8}$$

The square root of $E(\cdot, \cdot)$ defines an energy norm and from Eq. (6) the following energy norm

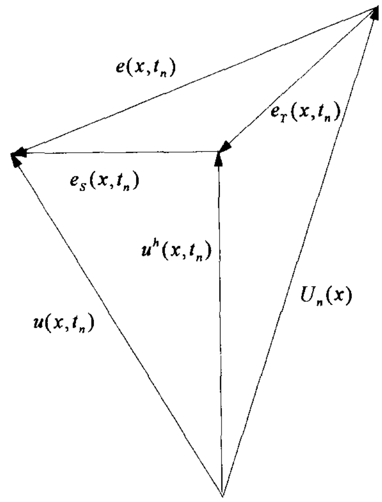


Fig. 1 Errors for dynamic analysis.

of the error is obtained.

$$E(e,\ \dot{e})^{1/2} \leq E(e_S,\ \dot{e}_S)^{1/2} + E(e_T,\ \dot{e}_T)^{1/2} \tag{9}$$

The upper bound of total energy norm of the error can be obtained by adding the energy norms of spatial discretization error and temporal discretization error.

3.1. Estimate of temporal discretization error

In dynamic analysis the methods of direct time integration are popular and the choice of method is strongly problem-dependent. Most of the useful implicit methods including the Newmark method are unconditionally stable and have no restriction on the time step size other than as required for accuracy. The Newmark method uses following two basic assumptions:

$$U_{n+1} = U_n + \dot{U}_n \Delta t + [(1-2\beta)\ddot{U}_n + 2\beta \ddot{U}_{n+1}]\frac{\Delta t^2}{2} \tag{10a}$$

$$\dot{U}_{n+1} = \dot{U}_n + [(1-\gamma)\ddot{U}_n + \gamma \ddot{U}_{n+1}]\Delta t \tag{10b}$$

where U_{n+1}, $\dot{U}_{n+1}$ and $\ddot{U}_{n+1}$ are respectively the displacement, velocity and acceleration vectors at time $t=t_n+\Delta t$, Δt is the time step size, β and γ are parameters. The Newmark method contains, as special cases, many widely used practical methods. When the average acceleration method ($\beta=\frac{1}{4}$, $\gamma=\frac{1}{2}$) is used, the variation of acceleration in each time step is assumed to be constant and equal to the average of the accelerations at the two ends of a time step. And when the linear acceleration method ($\beta=\frac{1}{6}$, $\gamma=\frac{1}{2}$) is used, the acceleration is assumed to vary linearly. In fact, since the acceleration varies continuously at the entire time domain, these assumptions yield a discontinuous distribution for the acceleration and the temporal discretization error which can be reduced by the choice of smaller time step size may occur.

Let us consider a time interval $[t,\ t+\Delta t]$ and assume that $\tau \in [t,\ t+\Delta t]$. The temporal discretization error of acceleration at time τ is

$$\ddot{e}(\tau)=\ddot{u}^{n}-\ddot{u}^{ex}(\tau) \tag{11}$$

where $\ddot{u}^{n}$ is the assumed acceleration and $\ddot{u}^{ex}$ is the real acceleration which varies continuously. Suppose that the solutions at time station t are exact. Then, the time discretization error of velocity solution at time τ and $t+\Delta t$ can be estimated by

$$\dot{e}(\tau)=\int_{t}^{\tau}\ddot{e}(\tau')d\tau' \tag{12a}$$

$$\dot{e}(t+\Delta t)=\ddot{u}^{nv}\Delta t-\int_{t}^{t+\Delta t}\ddot{u}^{ex}(\tau')d\tau' \tag{12b}$$

where

$$\ddot{u}^{nv}=(1-\gamma)\ddot{u}_{t}+\gamma\ddot{u}_{t+\Delta t} \tag{13}$$

And the error of displacement solution can be estimated by

$$e(\tau)=\int_{t}^{\tau}\dot{e}(\tau')d\tau' \tag{14a}$$

$$e(t+\Delta t)=\frac{\ddot{u}^{nd}}{2}\Delta t^{2}-\int_{t}^{t+\Delta t}\int_{t}^{t+\tau}\ddot{u}^{ex}(\tau')d\tau' d\tau \tag{14b}$$

where

$$\ddot{u}^{nd}=(1-2\beta)\ddot{u}_{t}+2\beta\ddot{u}_{t+\Delta t} \tag{15}$$

Since, the exact value of the acceleration $\ddot{u}^{ex}(\tau)$ cannot be obtained in most real problems, it is desirable to approximate the acceleration by a higher order function than the order of the assumed acceleration function in the Newmark method.

Zienkiewicz and Xie (1991) proposed a local error estimator by comparing the Newmark solution with the exact solutions expanded in the Taylor series. However, this error estimator uses the linearly approximated solution as an exact solution. It can only estimate the errors for displacements and cannot measure the errors for velocities since the positive values and negative values of acceleration error cancel out during the integration by Eq. (12) (See Fig. 2a). Accordingly, the error estimate in the total energy norm is not obtainable by this method. Moreover when $\beta=\frac{1}{6}$ and $\gamma=\frac{1}{2}$, the Newmark scheme itself is equivalent to the linear acceleration method in which the acceleration is assumed to be linear. This error estimator can measure neither the strain energy of error nor the kinetic energy of the error. Also if this estimator is used for an adaptive time stepping, the time step size may be changed too frequently. A step size change usually requires an inversion of a new effective stiffness matrix, which is expensive for implicit schemes. Wiberg and Li (1993) derived a formulation for linearly varying third-order derivatives and, based on this, they obtained a posteriori error estimates for displacements and velocities.

In this study, a quadratic function is used for the approximation for $\ddot{u}^{ex}(\tau)$ and the corresponding parameter for the function is obtained from accelerations at three time stations; $t-\Delta t$,

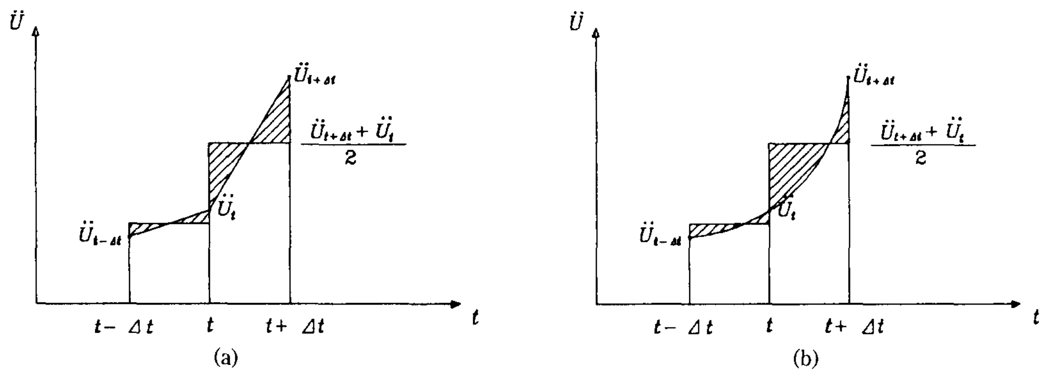


Fig. 2 Accelerations assumed in the Newmark time integration (constant average acceleration) and a post-processed continuous accelerations: (a) Linear and (b) Quadratic approximation.

t and $t+\Delta t$. This approximation for the acceleration gives a more accurate error estimation than a linear approximation of acceleration since it consists of connecting the successive three points as a group on the curve expressed by a second-degree parabola. (See Fig. 2)

A pointwise definition of error, as given in Eq. (11), is generally difficult to use in the measure for adaptive control, and the energy norm is more conveniently adopted. Therefore, as a posteriori error estimate, the energy norm of temporal discretization error can be obtained as follows:

$$E(e_T,\ \dot{e}_T)^{1/2}=\left(\frac{1}{2}\dot{e}_T{}^T M\dot{e}_T+\frac{1}{2}e_T{}^T K e_T\right)^{1/2} \tag{16}$$

3.2. Estimate of spatial discretization error

Among the types of error estimators on spatial discretization, the simpler one presented by Zienkiewicz and Zhu (1987) in plane elasticity problems was shown to be effective. Its application to plate bending analysis using transition element has also been reported (Choi and Park 1992). Ewing (1990) pointed out that the error estimate in strain energy norm for elliptic problems could be extended to the dynamic problems. In addition, Zeng and Wiberg (1992) extended this error estimate to dynamic analysis. According to them, if a sufficiently small time step size is chosen, the total energy of the error is mainly due to the strain energy of the error at most stages. Hence, the total energy norm of the spatial discretization error is given approximately as follows

$$E(e_S,\ \dot{e}_S)^{1/2}\cong E(e_S)^{1/2} \tag{17}$$

Eq. (17) implies that the spatial discretization error in dynamic problems can be approximately estimated by ignoring the kinetic energy of the error. The spatial discretization error estimator $\|e_S^i\|$ in the displacement for an element i is defined as follows:

$$\|e^i_S\|=\left(\int_{\Omega_i} e_S{}^T K e_S d\Omega\right)^{1/2}\left(\int_{\Omega_i}(\bar{\sigma}-\sigma_h)^T D^{-1}(\bar{\sigma}-\sigma_h)d\Omega\right)^{1/2} \tag{18}$$

where $\bar{\sigma}$ is the smoothed stress which is an approximation to the exact stress, σ_h is the finite element approximation for stress and D is the elasticity matrix. To obtain the continuous stress $\bar{\sigma}$ based on σ_h, the global stress smoothing procedure by the least square method was used in this study. The energy norm of the spatial discretization error for the whole solution domain can be calculated approximately by summing up the squares of the local error estimators in Eq. (18) over individual elements, that is,

$$E(e_S)^{1/2}=\left\{\sum_{i=1}^{n}(\|e_S^i\|)^2\right\}^{1/2} \tag{19}$$

where n is the total number of elements.

4. Combined adaptive procedure

An adaptive analysis should find a discretization in the least cost, such that the local error is uniformly distributed and within a given tolerance over the entire spatial/time domain. To control the relative error which is defined as an error norm of energy divided by the total energy norm, the time step size and the mesh distribution should be modified at the same time based on the local error estimate and the prescribed error tolerance.

From Eqs. (9) and (17) the total energy norm of the discretization error can be estimated by simply combining the energy norms of the spatial discretization error and time discretization error, that is,

$$E(e,\dot{e})^{1/2}\cong E(e_T,\dot{e}_T)^{1/2}+E(e_S)^{1/2} \tag{20}$$

On the other hand, nevertheless the user specifies the relative error tolerance η, the portion of the spatial discretization error out of the total error and that of the temporal discretization error should also be specified. Accordingly, parameters can be defined as follows

$$\varepsilon=\frac{\eta_S}{\eta} \quad \text{and} \quad \delta=\frac{\eta_T}{\eta} \tag{21}$$

where η_S and η_T are the prescribed relative energy norm of spatial and temporal discretization errors, respectively, and the sum of two parameters is 1.0 ($\varepsilon+\delta=1.0$). Optimal values for this parameters will be discussed in the numerical studies.

The combined spatial and temporal adaptive procedure proposed in this paper can be summarized as follows:

(1) Step 1

Carry out the finite element analysis with the initial or previous mesh and the time step until the prescribed termination time is reached.

(2) Step 2

Estimate the temporal discretization error and check whether the value of temporal discretization error is within the range of specified error tolerance. If the error is not within that range, change the current time step size and go to Setp 1. Otherwise go to the next step.

(3) Step 3

Estimate the spatial discretization error and check whether the value of the spatial discretization error is within the range of specified error tolerance. If the error is not within the range, change the mesh distribution and go to Step 1. Otherwise forward one time step and go to Step 1. For practical reasons, when there is no change in the mesh distribution after mesh generation, forward to the next time stage.

Overall adaptive procedure which is proposed in this study is symbolically depicted in Fig. 3.

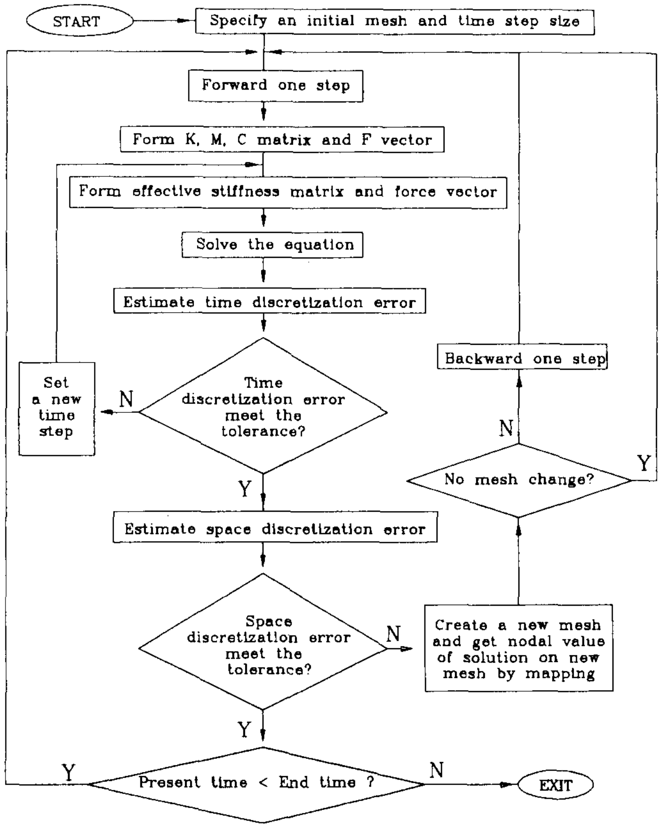


Fig. 3 Adaptive procedure.

4.1. Control of temporal discretization error

If error estimate does not satisfy the given error tolerance, the time step size is updated according to the local refinement index until the required accuracy is achieved. As many other studies (Zienkiewicz and Xie 1991) in which the error per step is used for error control, a lower error limit and upper error limit are introduced by

$$\gamma_1 \delta \eta \leq \frac{E(e_T, \dot{e}_T)^{1/2}}{E(u, \dot{u})^{1/2}} \leq \gamma_2 \delta \eta \tag{22}$$

where $0 \leq \gamma_1 \leq 1$ and $\gamma_2 \geq 1$ are two parameters. Whenever Eq. (22) is satisfied, the solution is accepted and the time integration is proceeded to the next step without changing the time step size. However, if the upper limit is violated, reject the solution, update the time step size and perform a re-calculation for the current time.

For the Newmark integration, the rate of convergence of the local error should achieve $O(\Delta t^3)$. Therefore when Eq. (22) is not satisfied a new time step size $\Delta t'$ may be predicted as

$$\Delta t' = \left(\frac{\delta\, \eta E(u, \dot{u})^{1/2}}{E(e_T, \dot{e}_T)^{1/2}} \right)^{1/3} \Delta t \tag{23}$$

where Δt is the current time step size.

4.2. Control of spatial discretization error

For each time station, the estimation of spatial discretization error as described in the preceding section is made first. If the given error tolerance is not satisfied, the mesh is updated according to the local refinement parameter until the required accuracy is achieved. In order to perform the computation more efficiently and economically for a given tolerance, a lower error limit and an upper error limit are introduced as

$$\gamma_1 \varepsilon \eta \leq \frac{E(e_S)^{1/2}}{E(u,\ \dot{u})^{1/2}} \leq \gamma_2 \varepsilon \eta \tag{24}$$

If this condition is not satisfied, the spatial mesh needs to be updated and a re-calculation for the current time should be performed. Based on the optimal mesh hypothesis, in order that the energy norm of the error for an element should be within a prescribed error bound, the critical error is defined as

$$\|e_S^c\| = \frac{\varepsilon \eta E(u,\ \dot{u})}{\sqrt{n}} \tag{25}$$

This critical error measure provides a refinement criterion, that is, any element for which the energy norm of the error calculated from Eq. (18) is greater than the critical error norm should be refined. A new element size required over the domain of each present element can be predicted by the use of the well-known fact that the convergence rate of the error is $O(h^p)$ for optimal mesh. Thus in each subdomain the predicted mesh size h_k^a is required as

$$h_k^a = \left(\frac{\|e_S^c\|}{\|e_S^i\|} \right)^{1/p} h_k \tag{26}$$

where h_k is existing mesh size.

Because of the mesh modification by adaptive methods, the value of all history dependent variables must be remapped to the location of new nodal points. The variables defined at nodal points, such as accelerations and displacements, can be easily remapped using the shape functions of an element such that

$$x(a) = N_i(a) x_i \tag{27}$$

where $x(a)$ is the value at the location of new nodal points, x_i is the value at the ith node (in local numbering) of the element to which the new point belongs and a is the position of the new point in the parametric coordinate system. If the problems without damping subjected to a shock-like external load and the corresponding meshes are modified too frequently or abruptly, the remapping error becomes considerably large.

There are generally two approaches for the mesh modification in an adaptive analysis. The one is the element refinement/coarsening approach and the other is remeshing approach. With the element refinement/coarsening approach, the element that violates the allowable error range is subdivided into smaller elements or merged into a bigger element. The remeshing approach involves completely regenerating a new mesh, either in regions of high error only, or over the entire domain. Unlike the element refinement/coarsening approach, the remeshing approach changes the location of the whole nodes. Therefore, from the view point of minimizing the remapping error, the element refinement/coarsening approach is more efficient than the reme-

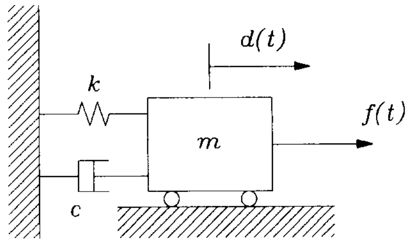
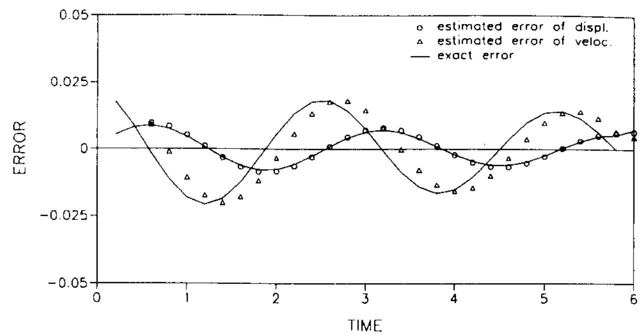
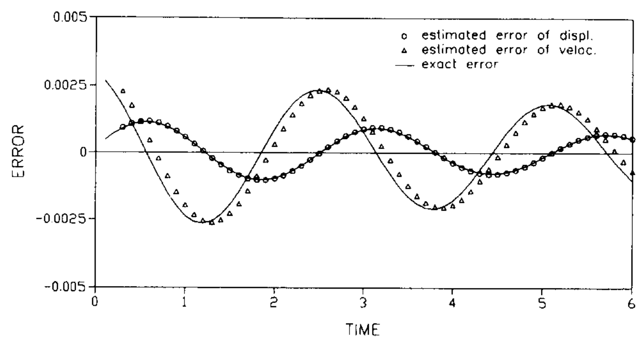


Fig. 4 A single-degree-of-freedom model.



(a)



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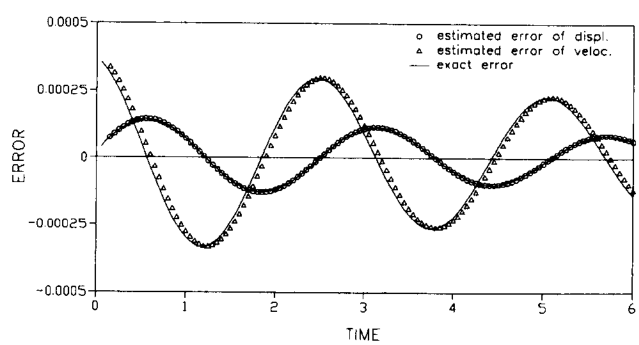
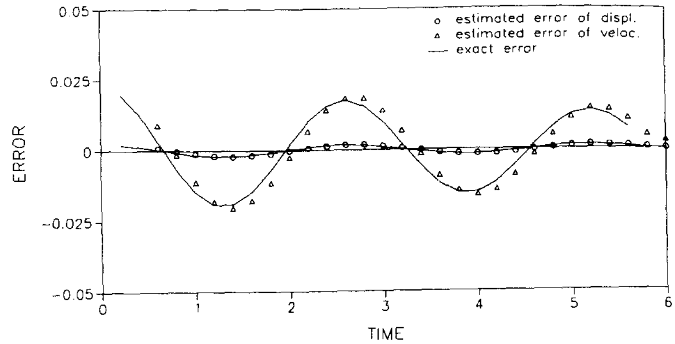
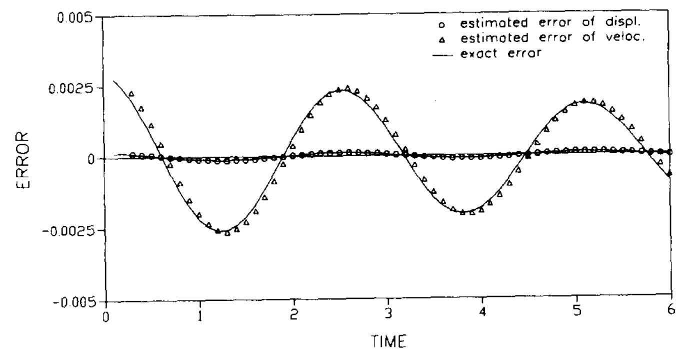


(c)

Fig. 5 Exact and estimated local error for constant average acceleration method: (a) $\Delta t=0.2$; (b) $\Delta t=0.1$; (c) $\Delta t=0.05$.



(a)



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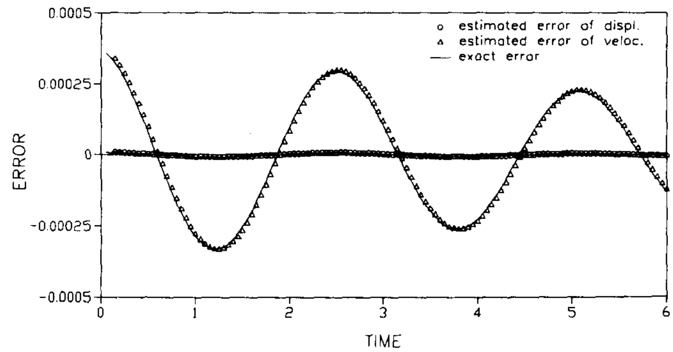

(c)

. Fig. 6 Exact and estimated local error for linear acceleration method: (a) $\Delta t=0.2$; (b) $\Delta t=0.1$; (c) $\Delta t=0.05$.

shing approach. For this reason, the element refinement/coarsening approach is used in this study.

5. Numerical examples

A single-degree-of-freedom problem was solved to evaluate the accuracy of the proposed time discretization error estimate. And to demonstrate the performance of the proposed adaptive

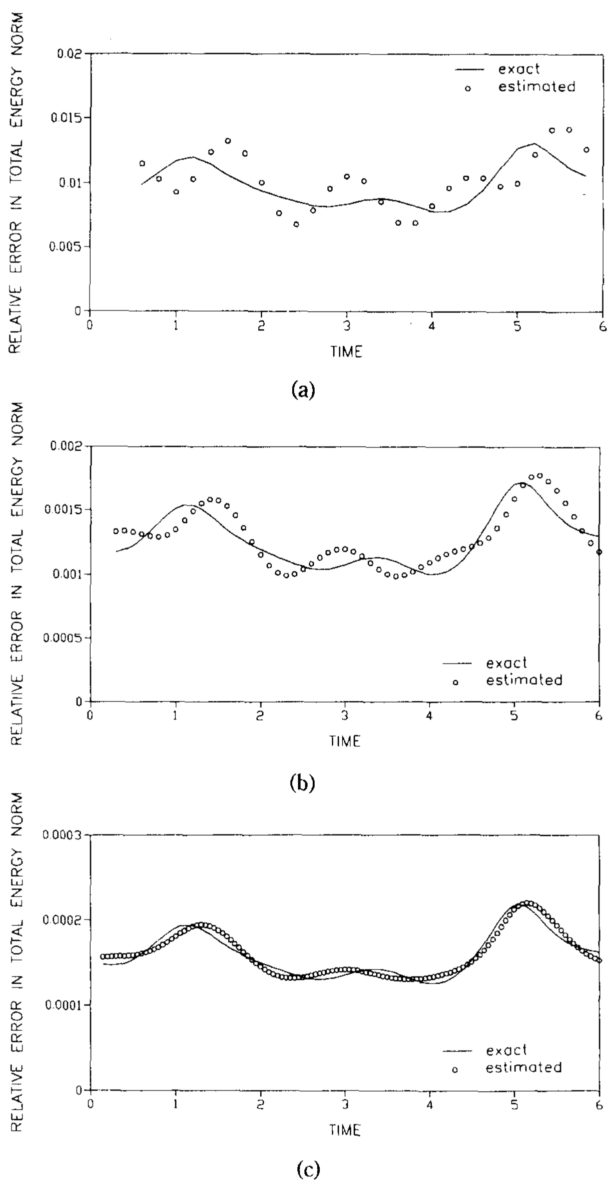

Fig. 7 Relative error of energy norm for constant average acceleration method: (a) $\Delta t=0.2$; (b) $\Delta t=0.1$; (c) $\Delta t=0.05$.

procedure, uni-dimensional tests were carried out since the results can be verified intuitively with ease. Several test problems with different error tolerances η, and parameters ε and δ were analyzed to obtain the reasonable value of ε or δ.

5.1. Estimate of time discretization error

To investigate the performance of the proposed time discretization error estimate, a SDOF model (Fig. 4) for which an exact solution can be obtained analytically is considered. This

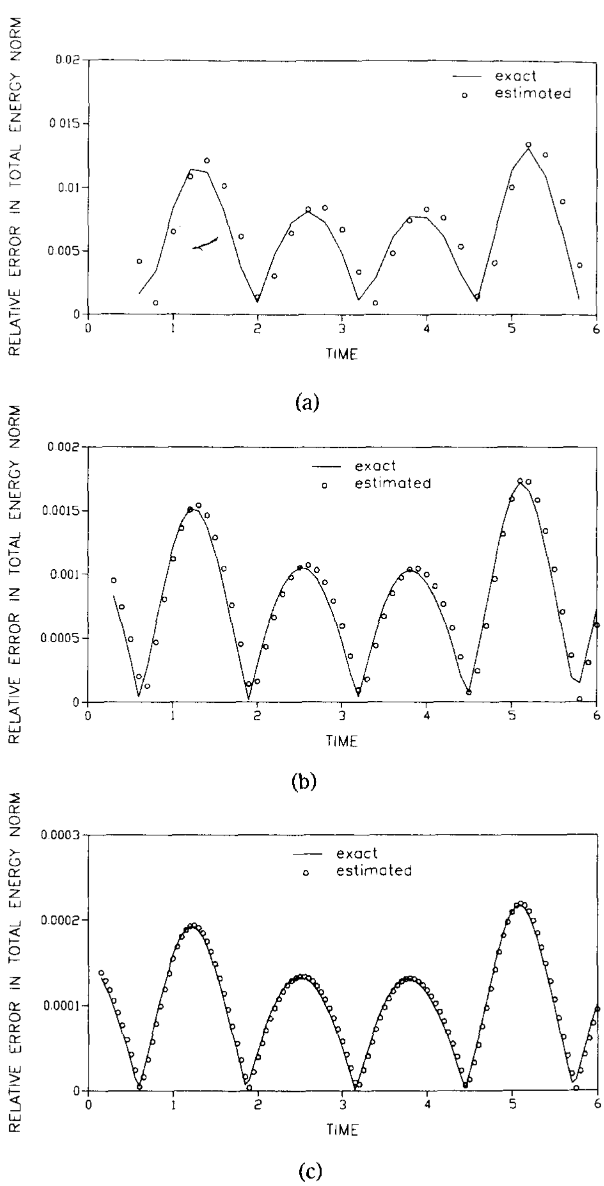

Fig. 8 Relative error of energy norm for linear acceleration method: (a) $\Delta t=0.2$; (b) $\Delta t=0.1$; (c) $\Delta t=0.05$.

dynamic system subjected to a sinusoidal loading can be presented by the following second order differential equation:

$$\ddot{d}+0.19596\dot{d}+6d=\sin t \tag{28}$$

When the Newmark integration step forwards from a time station t_n to t_{n+1}, the exact local error of displacement and velocity for this time step is obtained respectively by

$$e_{ex}=d_{n+1}^{N}-d(t_{n+1}) \tag{29a}$$

$$\dot{e}_{ex}=\dot{d}_{n+1}^{N}-\dot{d}(t_{n+1}) \tag{29b}$$

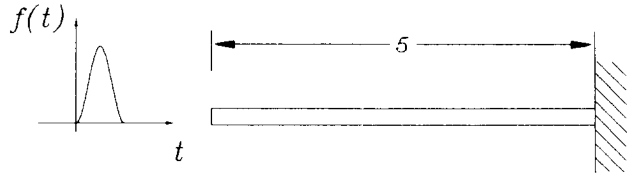

Fig. 9 One-dimensional model (Material properties are: $E=10.0$, $A=10.0$, $\rho=5.0$)

Table 1 Conventional analysis

	NDOF	Time step size	Computing time (sec)
Case A	300	0.01	592.09
Case B	30	0.03	24.16

where d_{n+1}^N and $\dot{d}_{n+1}^N$ are respectively the displacement and velocity given by Newmark's scheme at time t_{n+1}, and $d(t_{n+1})$ and $\dot{d}(t_{n+1})$ are obtained by integrating Eq. (28) analytically from t_n to t_{n+1} taking the values d_n^N and $\dot{d}_n^N$ as initial conditions (Zeng, *et al.* 1992). The exact and estimated local errors of displacements and velocities using the three different time step size $\Delta t=0.2$, 0.1 and 0.05 are shown in Fig. 5 for the constant average acceleration method and in Fig. 6 for the linear acceleration method. From Figs. 5 and 6 it is seen that the local error estimates for displacement perform very well while the local error estimate for velocities is not so good as for displacements. Nevertheless, the local error estimate for velocities is asymptotically convergent in most stages. The relative error distributions for the energy norm with different time step sizes and integration schemes are shown in Figs. 7 and 8. Noting that the proposed estimate gives an accurate measure of the relative energy norm of exact error induced by time discretization, this scheme can be used in the following example.

5.2. Adaptive analysis of one-dimensional problem

An one-dimensional elastic bar (Fig. 9) subjected to a sinusoidal pulse is analyzed to demonstrate the effectiveness of the proposed adaptive analysis proçedure. The sinusoidal pulse is given by

$$f(t)=\begin{cases}\sin^2 8\pi t & \text{if } 0\leq t\leq 0.125\\ 0 & \text{if } t>0.125\end{cases} \tag{30}$$

The Rayleigh damping is assumed and Rayleigh damping coefficients are set to 0.02.

Two uniform meshes and constant time step sizes are used for the conventional analysis. The first one which consists of 300 degrees of freedom is analyzed with a constant time step size of $\Delta t=0.01$. In the second, a mesh consisting of 30 degrees of freedom is used and the time step size is set to 0.03 (Table 1). And as an adaptive analysis, the dynamic responses in a time interval of [0.0, 10.0] are to be computed with parameters $\gamma_1=0.5$, $\gamma_2=1.5$ and $\varepsilon=0.5$ ($\delta=0.5$) and two different relative error tolerance $\eta=0.01$ and 0.02. Cases E and F are partial adaptive analyses with the control of time discretization error only and space discretization error only, respectively. Detailed conditions and computing times of each cases are shown

Table 2 Adaptive analysis

	η	ε	δ	Initial NDOF	Initial time step size	Computing time (sec)
Case C	0.02	0.5	0.5	100	0.01	19.11
Case D	0.01	0.5	0.5	20	0.01	42.51
Case E	0.01	–	1.0	70	0.01	21.91
Case F	0.01	1.0	–	20	0.03	21.10

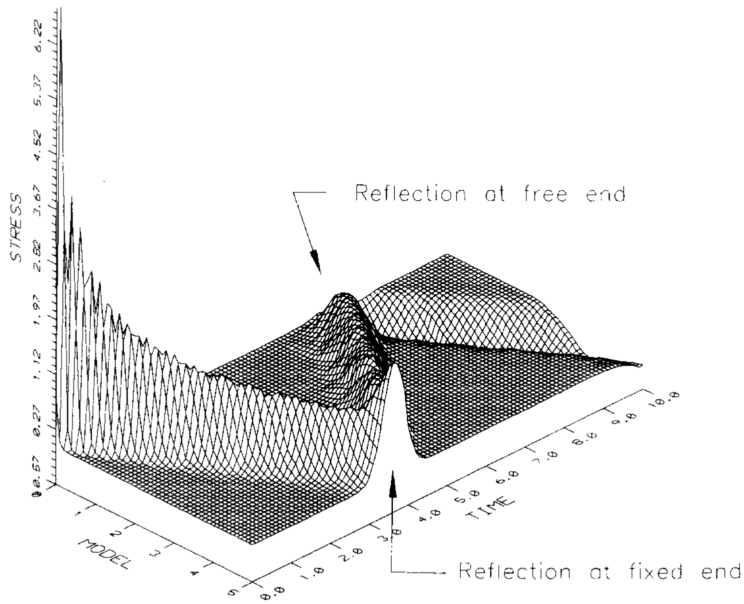

Fig. 10 Stress wave propagation.

in Tables 1 and 2. It should be noted that the optimal values of ε and δ are not known yet. In Fig. 10, it can be observed that the shape of stress wave varies from a sharp peak to a rather smoothed shape as the energy is being damped out. Also observed are the reflections of stress wave at the free and fixed ends. In Figs. 11 and 12, the variation of time step sizes and the mesh adaptation for two different cases are plotted. The element refinement/coarsening algorithm used in this study is not affected by the initial meshes which the user defined (Cases C and D) and both cases show a virtually the same variation of meshes after a first few iterations. The relative errors adaptively controlled within the allowable range are shown in Figs. 13 and 14.

The comparison between the solutions of conventional and adaptive analysis is shown in Figs. 15 and 16. It is noticed that the adaptive analysis gives more accurate results than conventional analysis with same computing time. The Newmark's average acceleration method is non-dissipative and unconditionally stable, but its accuracy in the transient analysis of wave propagation problems depends not only on the spatial discretization but also on the temporal discretization (Wang, Murti and Valliappan 1992). Figs. 15 and 16 show that the solutions

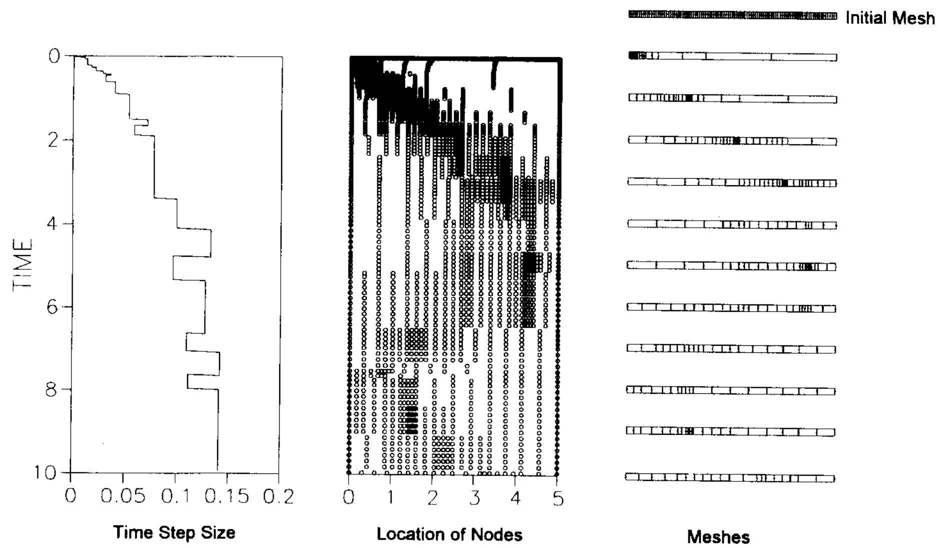

Fig. 11 Variation of time step size and mesh evolution in the bar (Case C).

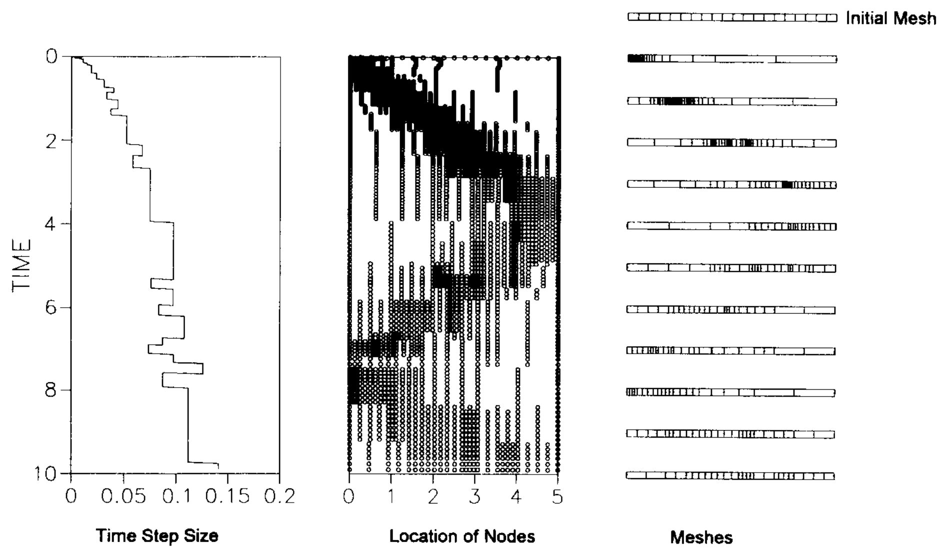

Fig. 12 Variation of time step size and mesh evolution in the bar (Case D).

of Case C (simultaneous adaptive control of spatial and temporal discretization errors) are more accurate than the solutions with either of Case E (adaptive control of temporal discretization error only) and Case F (adaptive control of spatial discretization error only). Therefore the simultaneous control of spatial and temporal discretization errors is more practical than the adaptive control of only one of those errors.

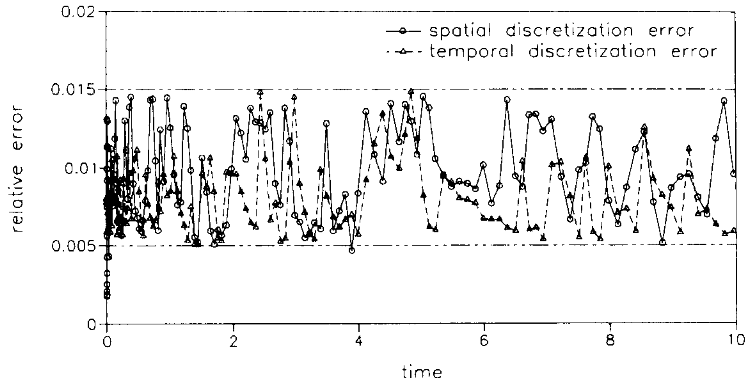

Fig. 13 Relative error achieved in the adaptive computation (Case C).

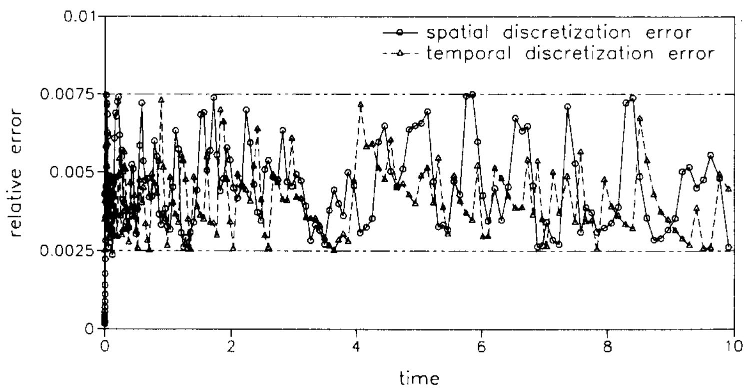

Fig. 14 Relative error achieved in the adaptive computation (Case D).

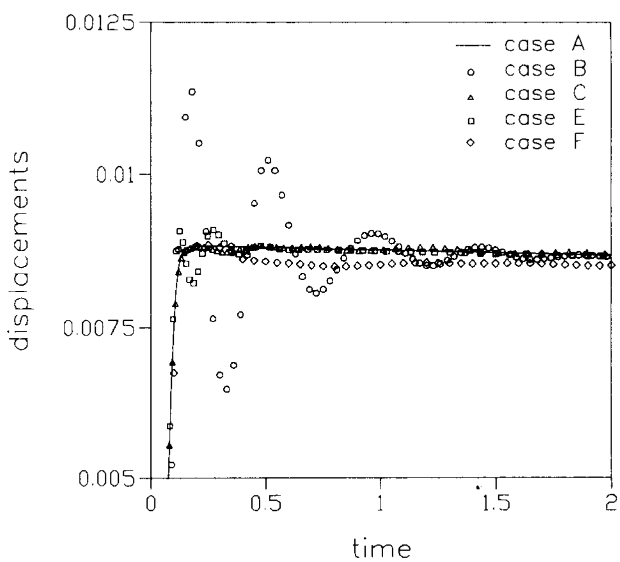

Fig. 15 Comparison of displacements at the free end, $x=0.0$ in time.

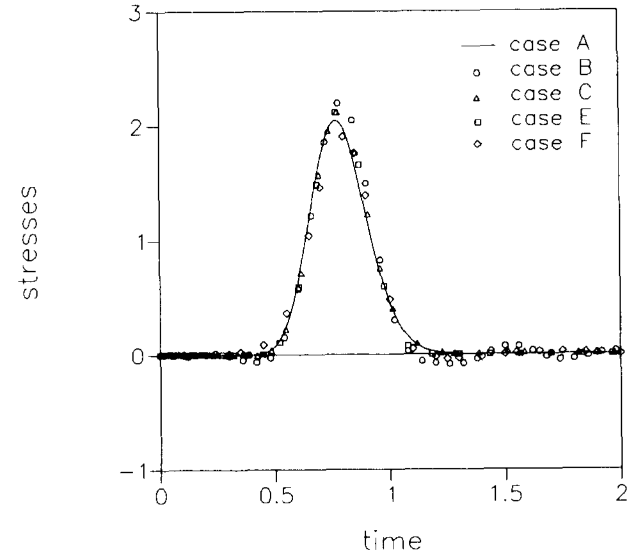

Fig. 16 Comparison of stresses at $x=1.0$ in time.

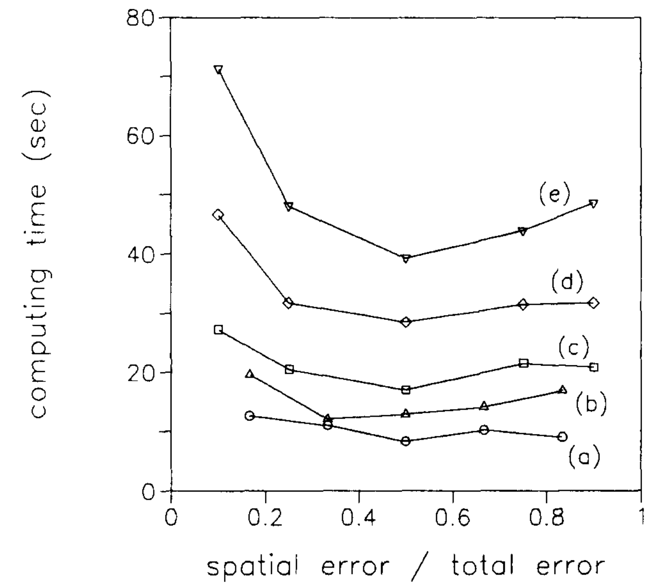

Fig. 17 Comparison of computing time: (a) $\eta=0.12$; (b) $\eta=0.06$; (c) $\eta=0.04$; (d) $\eta=0.02$; (e) $\eta=0.01$.

5.3. The optimal value of the parameter, ε or δ

In order to find the optimal value of ε or δ, a parametric study on the problem used in section 5.2 is carried out with various ratio of spatial and time discretization error tolerance. In Fig. 17 the relationship between the total computing time and ε or δ is shown. It is seen that when a proper value of ε or δ is selected, the total computing time can be reduced greatly at the same error tolerance. In this study it can be observed that the reasonable value of ε or δ is about 0.5 in the aspect of computing time which is also the value used in the preceding example. For two and three dimensional problems, however, it would not be the case because, unlike the control of the temporal discretization error, the control of the spatial discretization error is more expensive and time consuming than the one dimensional case.

6. Conclusions

In this study the spatial discretization error and the temporal discretization error were estimated consistently and an effective algorithm which controlls the errors automatically and simultaneously by adaptive modification of the mesh distribution and the time step size is proposed. In such a way, the best performance attainable by the finite element analysis of dynamic problems can be obtained.

The temporal discretization error can be estimated simply by comparing the solutions obtained by the Newmark method with solutions obtained by the locally exact quadratic function. This error estimate converge to the exact error as the size of time step is decreased. Since the error estimate by quadratic function is not affected by the specific time integration method used, the temporal discretization error estimate proposed in this study can be applied to the various single step method.

The remapping error occurred in control procedure of spatial discretization error becomes considerably large for the problems without damping. Therefore more extended studies are needed on the remapping technique.

Based on the parametric studies a reasonable ratio of spatial and time discretization errors, which should be specified by the user in an adaptive analysis of dynamic problem is also proposed.

The adaptive procedure in this study can be extended to two dimensional and three dimensional problems directly. For this, however, the use of an effective automatic mesh generator is recommended.

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