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Structural reliability estimation using Monte Carlo simulation and Pearson's curves

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Abstract. At present Level 2 and importance sampling methods are the main tools used to estimate reliability of structural systems. But sometimes application of these techniques to realistic problems involves certain difficulties. In order to overcome the difficulties it is suggested to use Monte Carlo simulation in combination with two other techniques-extreme value and tail entropy approximations; an appropriate Pearson's curve is fit to represent simulation results. On the basis of this approach an algorithm and computer program for structural reliability estimation are developed. A number of specially chosen numerical examples are considered with the aim of checking the accuracy of the approach and comparing it with the Level 2 and importance sampling methods. The field of application of the approach is revealed.

Key words: structural reliability; Monte Carlo simulation; Pearson's curves; extreme value approximation; tail entropy approximation.

1. Introduction

It is well known (Bjerager 1990) that at present there exist two basic approaches to structural reliability analysis-Level 2 (FORM/SORM) methods and importance sampling technique.

The drawbacks of Level 2 methods (Hohenbichler, *et al.* 1987, Thoft-Christensen and Baker 1982) are well known and can be summarized as follows. The failure surface cannot be most commonly defined explicitly and therefore the derivatives essential in many techniques, used for reliability index calculation, can be determined only numerically. In the general case there is no one-to-one relationship between the reliability index and failure probability, so the value of failure probability is estimated only approximately. The failure surface can be of such a form that several local minima are present (multiple β-points). In this case if the process of calculation converges not to the global minimum structural reliability can be dangerously overestimated. To perform calculations non-normal variables should be transferred to normal variables (Rackwitz and Fiessler 1977, Rosenblatt 1952).

The application of importance sampling methods is sometimes troublesome as well, because it involves identification of the most likely failure region and proper selection of an importance sampling density function; some methods are sensitive to curvatures of the limit state function, to number of variables, to multiple β-points, etc. One of the most serious problems is that the majority of importance sampling methods use a gradient-based search algorithm to locate the important region. Therefore probability density functions and failure functions have to be conti-

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nuous (Engelund and Rackwitz 1993). Serious difficulties are experienced in application of importance sampling methods if more than one important region exist.

As was shown by Krakovski (1993), Monte Carlo simulation with subsequent approximation of the results by Pearson's curves (Elderton and Johnson 1969) can be used to solve practical problems of structural reliability estimation. The Pearson's curve represents a probability density function (pdf) of the limit state function. It should be noted that until recently this approach has not been recommended (see Thoft-Christensen and Baker 1982). A common objection against the approach is that simulation results are concentrated mostly around mean value, and therefore the approximation of pdf tails cannot be sufficiently accurate. In this study to represent the tails of distributions more accurately an improvement is introduced: it is suggested to use the approach in combination with two other techniques, namely, tail entropy (Lind and Hong 1991) and extreme value approximations.

The principal aim of this paper is to show that on the basis of the approach sufficient accuracy in reliability estimation can be achieved and the above drawbacks of Level 2 and importance sampling methods can be overcome. At the same time limitation on the field of application of the approach will be revealed.

With this in mind first the algorithm for structural reliability estimation is described. Thereafter a number of examples are considered where calculation results obtained on the basis of suggested and other approaches are compared with the aim of checking the accuracy of the suggested approach.

2. Algorithm for reliability estimation

2.1. General description of the algorithm

For reliability estimation of structural systems basic load and resistance variables are assumed to be statistically independent random variables with known probability density functions.

The pdf of the basic variables can be defined analytically by the equation of a curve or numerically in the form of a histogram. A deterministic method for structural analysis is assumed to be known. Structural failure is defined as violation of the requirements for the structure. The problem is to find the non-failure probability of the structure.

Algorithm to solve the problem is based on Monte Carlo simulation and consists of the following steps:

(1) According to specified pdf of basic variables obtain m sets of their random realizations.

(2) Carry out m deterministic analyses of the structure by the selected method. The analyses determine m values of the output parameter y_i ($i=1,\cdots,m$). Stresses, deflections, crack width, load-bearing capacity, etc. can be taken as the output parameter.

(3) Using the above values choose an appropriate pdf $z(y)$ out of the family of Pearson's curves.

(4) Find the ultimate value of the output parameter y^0, above or below which the failure occurs, such as the ultimate deflection, crack width, load-bearing capacity, etc.

(5) Determine reliability of the structure by numerical integration:

$$P=\int_{y^0}^{\infty} z(y)\,dy \tag{1}$$

or

$$P=\int_{-\infty}^{y^0} z(y)\,dy \tag{2}$$

Use Eq. (1) if y^0 is a lower bound, and Eq. (2) if y^0 is an upper bound. Failure probability is $P_f=1-P$.

The described algorithm was implemented in the form of a computer program. Two problems can be solved-direct and inverse. The direct problem is defined by Eqs. (1) and (2). The inverse problem is to find the value of y^0 so as to obtain a prescribed level of reliability P in Eqs. (1) and (2). In order to solve the inverse problem numerical integration in combination with one-dimensional search is performed.

Eqs. (1) and (2) represent a generalization of the well known approach with a limit state function f: $f>0$ and $f<0$ define failure and safe regions, respectively. Eq. (1) evaluates reliability if $f<y^0$ and $f>y^0$ define failure and safe regions, respectively, and Eq. (2) evaluates reliability if $f>y^0$ and $f<y^0$ define failure and safe regions, respectively. Of course, all these representations are equivalent, but they are convenient for solving inverse problems.

In the following the algorithm will be referred to as direct Monte Carlo simulation.

2.2. Generation of random realizations of basic variables

Let us consider the first step of the algorithm in more detail. In order to obtain sets of random realizations of basic variables the following procedure is proposed.

(1) Divide probability density functions of all basic variables into m equal-area parts.

(2) Find the centers of gravity of each part and the corresponding values of basic variables

$$\bar{x}_i^{(1)}<\bar{x}_i^{(2)}<\cdots<\bar{x}_i^{(m)} \tag{3}$$

All these values are equally probable.

(3) Using uniform distribution choose randomly a value of the first basic variable among m equally probable values. Remove this value from further calculations.

(4) Perform Step (3) for basic variables $2, 3, \cdots, n$. Obtain a set of values of basic variables.

(5) Put $m=m-1$. Go to Step (3).

(6) Perform Steps (3) to (5) until $m=0$.

Table 1 shows the results of numerical investigation of the accuracy of two algorithms-above described and conventional (in which random number generator is used to obtain realizations

Table 1 Comparison of accuracy of two algorithms generating random realizations of basic variables

Algorithm	Basic-moments	Exact values of μ_i	Calculated values of μ_i with sample size $m=$ 100	200	500	1000	2500	5000
Proposed	μ_1	0	8.490×10^{-4}	2.895×10^{-4}	2.524×10^{-4}	-2.288×10^{-4}	-1.892×10^{-4}	-1.417×10^{-4}
	μ_2	1	0.9858	0.9992	0.9996	0.9996	0.9998	0.9998
	μ_3	0	5.476×10^{-3}	-6.276×10^{-4}	-5.589×10^{-4}	-2.027×10^{-4}	-1.342×10^{-4}	-1.249×10^{-4}
	μ_4	3	2.7979	2.9424	2.9822	2.985	2.9882	2.9935
Conventional	μ_1	0	6.032×10^{-3}	7.257×10^{-3}	4.126×10^{-3}	2.992×10^{-3}	-2.207×10^{-3}	-2.756×10^{-3}
	μ_2	1	0.9738	0.9603	1.0153	0.9908	0.9911	1.012
	μ_3	0	0.1101	0.1499	0.1214	6.570×10^{-3}	3.245×10^{-3}	1.290×10^{-3}
	μ_4	3	3.0250	2.7912	2.8533	2.826	2.8064	2.7976

of each random variable). First four basic statistical moments μ_i ($i=1,\cdots,4$) are calculated for standard normal distribution with different sample sizes and compared with their exact values. As can be seen from Table 1, the described algorithm is much more accurate: even for rather small sample size $m=500$ the calculated μ_i are closer to their exact values than values computed using the conventional approach with sample size $m=5000$.

The higher accuracy of the results obtained by the proposed algorithm is attributable to the following reasons.

Assume that conventional Monte Carlo simulation with sample size m is performed t times for the ith basic variable, i.e., t samples of size m are obtained. The simulation results-random realizations $x_{ij}^{(1)}, x_{ij}^{(2)},\cdots,x_{ij}^{(m)}$-are ranged so that

$$x_{ij}^{(1)} \leq x_{ij}^{(2)} \leq \cdots \leq x_{ij}^{(m)},\ j=1,\cdots,t \tag{4}$$

and mean values are calculated

$$\tilde{x}_i^{(1)} = \frac{1}{t}\sum_{j=1}^{t} x_{ij}^{(1)};\cdots;\ \tilde{x}_i^{(m)} = \frac{1}{t}\sum_{j=1}^{t} x_{ij}^{(m)} \tag{5}$$

It is obvious that

$$\tilde{x}_i^{(1)} \to \bar{x}_i^{(1)},\cdots,\tilde{x}_i^{(m)} \to \bar{x}_i^{(m)} \tag{6}$$

when $t\to\infty$.

And it is also apparent that evenly distributed along the length of the curve mean values $\bar{x}_i^{(1)},\cdots,\bar{x}_i^{(m)}$ represent the original pdf of a basic variable much more accurately than randomly distributed along the length of the curve realizations $x_{ij}^{(1)},\cdots,x_{ij}^{(m)}$ used in conventional Monte Carlo simulation.

Thereupon to form a set of random realizations of basic variables the above algorithm is used.

2.3. Extreme value and tail entropy approximations

In order to represent the important tail regions more accurately two techniques can be used. The first is extreme value approximation. Consider the case when the argument is on the lower tail of the pdf. Calculations are carried out in the following way.

The results of simulation are divided into groups with k results in each group. Only one minimum result from each group is considered. All minimum results (extreme values) are approximated by one of the Pearson's curves, which is a pdf of extreme values. Then the exceedance probability P_k for any argument can be calculated by numerical integration of the pdf of extreme values.

The relationship between the exceedance probabilities in the parent distribution P and P_k is (Thoft-Christensen and Baker 1982):

$$P_k = P^k \tag{7}$$

Similar results can be obtained for the upper tail of a pdf.

The second technique, improving the results of direct Monte Carlo simulation, is tail entropy approximation (TEA) suggested by Lind and Hong 1991. In what follows the largest and the lowest observed values are denoted by x_m and x_1 respectively, and $F_0(x)$ denotes a reference distribution, represented in our case by one of the Pearson's curves. The end points of the domain of x are denoted by

$$x^{-}=F_0^{-1}(x);\ x^{+}=F_0^{-1}(1) \tag{8}$$

either or both may fall at infinity.

The lower tail entropy approximation of X is a random variable that has the distribution function $F_{-}(x)$:

$$F_{-}(x)=[1/(m+1)][F_0(x)/F_0(x_1)],\ x^{-}<x<x_1 \tag{9}$$

$$F_{-}(x)=\{1+m[F_0(x)-F_0(x_1)]/[1-F_0(x_1)]\}/(m+1),\ x_1<x<x^{+} \tag{10}$$

The upper tail entropy approximation of X is a random variable that has distribution $F_{+}(x)$ given by

$$F_{+}(x)=[m/(m+1)][F_0(x)/F_0(x_m)],\ x^{-}<x<x^{+} \tag{11}$$

$$F_{+}(x)=\{m+[F_0(x)-F_0(x_m)]/[1-F_0(x_m)]\}/(m+1),\ x_m<x<x^{+} \tag{12}$$

Eqs. (8) to (12) can be used to up-date the reliability calculated on the basis of Eqs. (1), (2) or (7).

2.4. *Verification of the approach*

In order to verify the suggested approach, below are considered 6 numerical examples. They are chosen from the following reasons.

Examples 1, 2 make it possible to compare the suggested approach with importance sampling technique. Example 1 deals with systems in which several failure modes occur. Recently a method for reliability estimation of such systems has been developed (Fu and Moses 1993) and it was of interest to compare their results with ours.

Example 2 is taken from the paper by Engelund and Rackwitz 1993, where six importance sampling methods are evaluated with respect to certain criteria. The example served to estimate the robustness of the methods with respect to multiple β-points. Only two methods out of six turned out to be not sensitive towards the existance of multiple β-points; two of them were sensitive, one failed to solve the problem and it was not possible to estimate the sensitivity of the last method. Therefore it was of interest to solve this difficult problem by the suggested approach.

The next two examples deal with Level 2 methods. Example 3 serves to compare results for an ordinary problem solved by FORM in Thoft-Christensen and Baker 1982. In Example 4 results are compared for both direct and inverse problems. Since it is difficult to solve the inverse problem on the basis of the conventional approach, a graphical representation is used.

The problem in Example 5 is chosen because, in author's opinion, it is very difficult, if not impossible, to solve this problem using importance sampling or Level 2 methods: the limit state function cannot be written analytically, only algorithm, checking whether or not failure occurs, is available; in addition, both direct and inverse problems are solved.

One further reason for choosing Examples 4 and 5 is their practical significance. Calculations similar to those in these examples were carried out for a revised version of the Russian Code 1985 in order to regulate reliability of RC structures. Inverse problems were solved and load-bearing capacities of structures M_c with an exceedance probability of 0.9986 were computed. Then a material property combination factor $k_c=M_c/M_0$ was used (M_0 is a load-bearing capacity determined from the Code using design strength of materials). Conventionally calculated load-bearing capacity of the structure is multiplied by k_c making all structures equally reliable with a reliability of 0.9986 (Krakovski 1993).

Example 6 shows the accuracy of the schemes where along with direct Monte Carlo simulation extreme value and tail entropy approximations are used.

3. Comparison of results: importance sampling methods and the suggested approach

3.1. Example 1

First let us consider an example given by Fu and Moses (1993). A system with two failure modes defined by failure functions

$$g_1(x)=R-S; \qquad g_2(x)=61-1.44R-S \tag{13}$$

is investigated. Here R and S are normally distributed random variables with mean values, 25, 10 and standard deviations, 2.5, 3.0, respectively. Exact value of failure probability is $P_f=7.462\times 10^{-4}$.

Monte Carlo simulation with sample size 1000 was performed. The result of each trial was min $[g_1(x), g_2(x)]$. These minima were approximated by a Pearson's curve. The curve turned out to be of type 4:

$$y(x)=0.2464\left[1+\left(\frac{x-18.497}{24.515}\right)^2\right]^{-28.342}\times \exp\left[-13.373\ \arctan\left(\frac{x-18.497}{24.515}\right)\right] \tag{14}$$

The least observed value was 0.5532. Then Eq. (9) was used:

$$F_0(x)=\int_{-\infty}^{0} y(x)dx=4.984\times 10^{-4};\ F_0(x_1)=\int_{-\infty}^{0.5532} y(x)dx=7.710\times 10^{-4}$$

$$P_f=F_-(x)=6.458\times 10^{-4} \tag{15}$$

The estimate $P_f=6.458\times 10^{-4}$ is not too far from the exact value and only slightly lower than the estimate $P_f=7.005\times 10^{-4}$ obtained by Fu and Moses with sample size $m=4000$.

3.2. Example 2

Consider the following example from Engelund and Rackwitz (1993). The limit state function is

$$g(x)=X_1X_2-PL \tag{16}$$

where P and L are deterministic parameters with values, 14.614 and 10.0, respectively; X_1 and X_2 are normally distributed with means, 78064.4 and 0.0104, and standard deviations, 11709.7 and 0.00156, respectively. As indicated above, the example was used to evaluate the robustness of different importance sampling methods with respect to multiple β-points. The exact value

Table 2 Results of calculations for Example 2

k	1	2	3	4	5	6
P_f	1.807×10^{-5}	6.620×10^{-6}	4.010×10^{-6}	2.453×10^{-6}	1.660×10^{-6}	9.188×10^{-6}

of failure probability is $P_f=1.451\times10^{-6}$.

The problem was solved using Monte Carlo simulation in combination with extreme value and tail entropy approximations. Calculations were carried out with sample size 5000. The results are given in Table 2 for different k from Eq. (7).

From Table 2 one can see how the accuracy improves with k if $k\leq5$. For $k=5$ the best result $P_f=1.660\times10^{-6}$ is obtained; it is rather close to the exact value. For $k=6$ the result is less accurate. The accuracy improves with k because, as k increases, the lower tail of the distribution is approximated more accurately. But if k is too large, then sample size m/k is small and the accuracy is impaired. The results also show that the proposed approach is not sensitive towards the existence of multiple β-points.

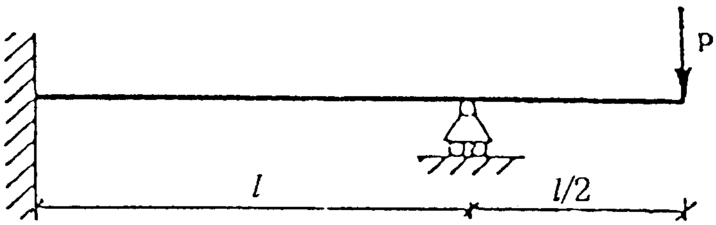

Fig. 1 Statically indeterminate beam.

4. Comparison of results: Level 2 methods and the suggested approach

4.1. Example 3

Consider the same statically indeterminate beam (Fig. 1) as in example 5.5 from Thoft-Christensen and Baker (1982). The following deflection failure criterion is used:

$$u_{max}=\frac{5}{48}\frac{pl^3}{ei}\geq\frac{1}{30} \tag{17}$$

where u_{max} is the maximum deflection, e the modulus of elasticity and i the relevent moment of inertia. The span of the beam $l=5$ m. Basic variables P, E and I are uncorrelated with the following mean values and standard deviations:

$$\begin{array}{ll} \bar{P}=4\ \text{kN} & \sigma_P=1\ \text{kN}, \\ \bar{E}=2\times10^7\ \text{kN/m}^2 & \sigma_E=0.5\times10^7\ \text{kN/m}^2 \\ \bar{I}=10^{-4}\ \text{m}^4 & \sigma_I=0.2\times10^{-4}\ \text{m}^4 \end{array} \tag{18}$$

In the normalized coordinate system the failure surface is given by

$$0.2z_1+0.25z_2+0.05z_1z_2-0.0391z_3+0.8438=0 \tag{19}$$

where normalized variables z_1, z_2, z_3 correspond to I, E, P, respectively.

The reliability index, determined by FORM, equals 3.29 (Thoft-Christensen and Baker 1982). Failure surface (19) is only slightly nonlinear. Therefore the failure probability is close to

$$P_f=\phi(-3.29)=5.009\times10^{-4} \tag{20}$$

Monte Carlo simulation with 5,000 trials has shown that the pdf of Eq. (19) can be approximated

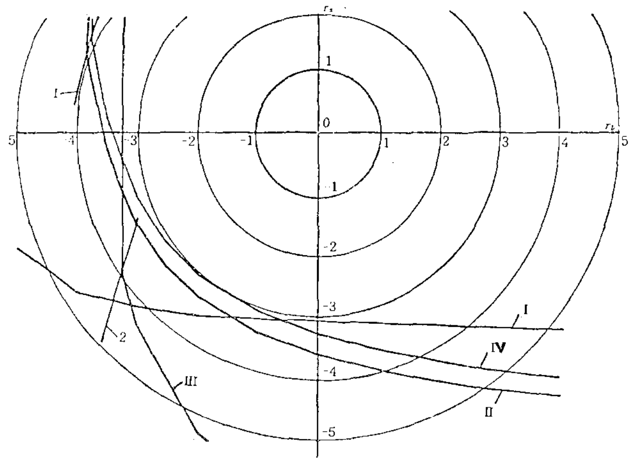

Fig. 2 Comparison of results obtained by the suggested approach and Level 2 method.

by Pearson's curve of type 6:

$$y(z)=10^{2439.836}=(z+0.5893)^{18.28294}(z+91.4649)^{-1243.036} \tag{21}$$

The failure probability computed by numerical integration of (21) is equal to 4.75×10^{-4} and very close to (20).

4.2. Example 4

Consider a reinforced concrete flexural member with rectangular cross section and singular reinforcement. The member is designed in accordance with the Russian Code 1985.

Two normally distributed random variables were considered, namely, strengths of concrete and reinforcement. Their mean values and standard deviations are

$$\bar{R}_b=28.5 \text{ MPa}, \ \sigma_b=4.56 \text{ MPa} \tag{22}$$

for concrete and

$$\bar{R}_s=435 \text{ MPa}, \ \sigma_s=20 \text{ MPa} \tag{23}$$

for steel. Design strengths of materials

$$R_b=14.5 \text{ MPa}, \ R_s=365 \text{ MPa} \tag{24}$$

have the exceedance probability of 0.9986.

The failure of the member is assumed to occur if

$$M<M_0 \tag{25}$$

where M is a load-bearing capacity (limit moment) computed for random realizations of material strengths, and M_0 is a load-bearing capacity computed for design strengths of materials.

The results of calculations on the basis of Level 2 approach are shown in Fig. 2. Normalized random variables

$$r_b=\frac{\tilde{R}_b-\bar{R}_b}{\sigma_b}; \ r_s=\frac{\tilde{R}_s-\bar{R}_s}{\sigma_s} \tag{26}$$

are plotted as abscissas and ordinates, respectively. Here $\tilde{R}_b$ and $\tilde{R}_s$ are random realizations of concrete and steel strengths, respectively.

The concentric circles represent the lines with equal reliability indices. Lines I, II, III represent failure surfaces, for which the ratios ξ/ξ_R are equal to 0.1, 0.6, 0.9 and relative moments $m=M/bh_0^2$ are equal to 810, 4390 and 5540 kNm/m^3, respectively. The moments m are determined by direct Monte Carlo simulation so that their exceedance probability equals 0.9986.

Here ξ is a relative depth of concrete compression zone and ξ_R is the maximum relative depth of concrete compression zone, for which limit state of the member is governed by stresses in reinforcement; b and h_0 are, respectively, width and effective depth of the member. The ratios ξ/ξ_R are defined for design strengths of materials (24).

To obtain a point at lines I, II, III first a value of r_b is fixed, and then the corresponding value of r_s is determined in accordance with the Russian Code 1985 for given values of ξ/ξ_R and m. Lines 1 and 2 correspond to the values of r_b and r_s, for which the condition $\xi=\xi_R$ is satisfied.

Moving along lines I, II, III one can see, that the values of r_s increase as the values of r_b decrease. But at the points of intersection of lines II and 1, III and 2 the value of r_b is minimal and corresponds to over-reinforcement. Thereupon lines II, III become parallel to the axis of ordinates, i.e., the load-bearing capacity of the member remains constant as steel strength increases and concrete strength is constant.

The reliability indices, obtained graphically from Fig. 1 as the shortest distances from the origin of coordinates to lines I, II, III are 3.05, 3.3 and 3.2, respectively.

If the failure surface had been linear, the non-failure probabilities for the above three cases could have been obtained from the normal distribution law; these probabilities would have been equal to 0.998817, 0.999517 and 0.999313, respectively. The non-failure probability, determined by the suggested approach, equals 0.9986 in all three cases. Therefore the results are not in conflict with a well known fact: the level 2 method, used to solve the problem under discussion, overestimates the non-failure probability for a convex safe region. And the more the failure surface differs from a linear surface, the more the disrepancy is between the non-failure probabilities determined by the suggested and level 2 methods.

Line IV was built by successive approximation with respect to the moment m so that the minimum distance from the origin of coordinates to the failure surface equals 3. The parameters of the member were identical with those used for line II. The moment m_0 with the exceedance probability 0.9986 turned out to be 4460 kNm/m^3, i.e. only slightly increased comparing to the moment 4390 kNm/m^3, used for line II. Thus, in spite of a rather significant disparity between the non-failure probabilities for lines II and IV (0.999517 and 0.9986) the limit moments are not too different.

5. Other examples

5.1. Example 5

The investigation was concerned with a reinforced concrete member with rectangular cross section under combined bending and compression. The member was designed according to the Russian Code (1985). Characteristics of the member:

—class of concrete compressive strength is B25;

—class of reinforcing steel is A-III;

- partial safety factor taking into account loading duration $\gamma_{b2}=1$;
- factor taking into account the effect of long-term loading on the deflection in the limit state $\phi_l=2$;
- $\mu'/\mu=1$, where μ and μ' are reinforcement ratios for compression and tension reinforcement, respectively;
- slenderness $\lambda=0$;
- $\xi/\xi_R=1.1$.

Strengths of concrete and reinforcement were assumed to be random normally distributed values. Mean values and standard deviations are

$$\bar{R}_b=24.84 \text{ MPa}, \ \sigma_b=3.35 \text{ MPa} \tag{27}$$

for concrete and

$$\bar{R}_s=426 \text{ MPa}, \ \sigma_s=17.04 \text{ MPa} \tag{28}$$

for steel.

The deterministic analysis carried out for the design values of concrete and reinforcement strengths has shown that for a fixed relative axial force

$$n=\frac{N}{bh_0}=9.628 \text{ MPa} \tag{29}$$

the limit relative bending moment is

$$m=\frac{M}{bh_0^2}=3.228 \text{ MPa} \tag{30}$$

The sample size was 250,000. In each trial for constant relative axial force $n=9.628$ MPa, a limit moment m_1 was determined. The failure of the member was assumed to occur if

$$m_1<m \tag{31}$$

The number of trials, in which condition (31) was satisfied, turned out to be 198. The failure probability P_{f1}, estimated as relative frequency, is

$$P_{f1}(m_1<m)=198/250{,}000=7.92\times10^{-4} \tag{32}$$

Type 4 Pearson's curve was used to approximate the results of the first 5,000 trials. The failure probability, determined from this curve is:

$$P_{f2}(m_1<3.288)=7.582\times10^{-4} \tag{33}$$

As can be seen P_{f2} coincides very closely with P_{f1}. Similar results were obtained for other values of concrete strength, λ, μ'/μ, ξ/ξ_R.

In order to investigate the effect of sample size on the results, a reinforced concrete member under combined bending and compression was considered once again. All characteristics of the member except $\lambda=10$ and $\xi/\xi_R=1$ were identical with those given above.

The direct Monte Carlo simulation was carried out. The total number of trials was equal to 55,000. In each trial the limit moment.

$$\alpha_m=\frac{m}{R_b} \tag{34}$$

for a fixed axial force

Table 3 Effect of sample size on the results of approximation by Pearson's curves

Way of calculation	Values of $a_{m0} \times 10^3$ for the number of trials $m \times 10^{-3}$										
	5	10	15	20	25	30	35	40	45	50	55
(i)	580	586	589	582	584	584	590	587	585	582	586
(ii)	580	582	585	582	583	583	585	586	586	586	585

Table 4 Comparison of exact and approximate values of P_f (direct Monte Carlo simulation)

x	$\beta=(\ln x-3)/0.2$	P_f, exact values	P_f, approximate values for sample size				
			500 (6)	1,000 (6)	5,000 (5)	10,000 (5)	25,000 (5)
10	-3.487	2.415×10^{-4}	1.417×10^{-4}	1.715×10^{-4}	3.350×10^{-4}	3.411×10^{-4}	3.362×10^{-4}
9.025	-4	3.167×10^{-5}	1.047×10^{-5}	1.557×10^{-5}	5.436×10^{-5}	5.585×10^{-5}	5.464×10^{-5}
8.166	-4.5	3.398×10^{-6}	4.214×10^{-7}	9.168×10^{-7}	7.961×10^{-6}	8.268×10^{-6}	8.018×10^{-6}
7.389	-5	2.868×10^{-7}	6.030×10^{-9}	2.806×10^{-8}	1.025×10^{-6}	1.079×10^{-6}	1.035×10^{-6}

$$a_n=\frac{n}{R_b}=0.5 \tag{35}$$

was determined.

In the investigation the inverse problem was solved. The moment a_{m0} with the exceedance probability 0.9986 was calculated after each 5,000 trials in two ways:

(i) using the previous 5,000 trials;

(ii) using all the trials performed.

The results are shown in Table 3. As can be seen from this table, the values of a_{m0} vary only slightly with the number of trials as well as from one series of 5,000 trials to another. Similar results were obtained for flexural reinforced concrete members and members under combined bending and tension. Therefore in these cases 5,000 trials permitted to obtain sufficiently accurate results.

5.2. Example 6

In this example the verification of the approach is carried out in the following way. Consider a log-normal distribution

$$f_X(x)=\frac{1}{\sigma\sqrt{2\pi}}\frac{1}{x}\exp\left[-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2\right] \tag{36}$$

The values of the parameters are

$$\mu=3,\ \sigma=0.2 \tag{37}$$

This is a distribution of concrete compressive strength (see example 2.12 in Thoft-Christensen and Baker 1982). Therefore the lower tail of the distribution is of interest.

Using pdf (36), (37) the direct Monte Carlo simulation is performed as described above and a suitable Pearson's curve is used to approximate (36), (37). It should be stressed that the log-normal curve is not among Pearson's curves. Therefore the values P_f of two probability distribution functions for the same x can be compared: the exact value (the log-normal distribution) and an approximate value (a Pearson's curve).

Table 5 Comparison of exact and approximate values of P_f (combination of extreme value and tail entropy approximations, sample size is 5,000)

x	$\beta=(\ln x-3)/0.2$	P_f, exact values	P_f, approximate values for $k=$ 1 (6)	2 (7)	3 (1)	4 (4)
10	-3.487	2.415×10^{-4}	1.620×10^{-4}	1.671×10^{-4}	1.637×10^{-4}	1.716×10^{-4}
9.025	-4	3.167×10^{-5}	1.583×10^{-5}	2.480×10^{-5}	1.903×10^{-5}	3.030×10^{-5}
8.166	-4.5	3.398×10^{-6}	1.058×10^{-6}	3.291×10^{-6}	1.798×10^{-6}	3.032×10^{-6}
7.389	-5	2.868×10^{-7}	4.074×10^{-8}	3.799×10^{-8}	1.298×10^{-7}	2.866×10^{-7}

The results of comparison for different sample sizes are shown in Table 4. Pearson's curve numbers used for approximation are given in parentheses next to the values of sample size. As can be seen from Table 4, the accuracy of the results decrease substantially as the exact value of failure probability decreases even for sufficiently large sample sizes.

In Table 5 are shown the results of comparison between exact and approximate values, when the last ones were calculated using extreme value and tail entropy approximations. Sample size 5,000 was used with $k=2$, 3, 4; $k=1$ denotes direct Monte Carlo simulation.

As can be seen from Table 5, the results with the use of the combination of extreme value and tail entropy approximations are considerably more accurate in comparison with those obtained by direct Monte Carlo simulation, especially for small values of P_f. An important point is that the accuracy of the results increases with k. For $k=4$ the approximate results are rather close to the exact ones.

6. Discussion of results

The following inferences can be made from the calculation results.

The suggested approach seems to be sufficiently accurate in the range of investigated failure probabilities (down to 10^{-7}). The key advantages of the approach over other methods are simplicity and versatility: it can be easily applied to different problems even when probability density functions and limit state functions are not continuous (Examples 4, 5). The above mentioned drawbacks of Level 2 and importance sampling methods can be overcome. In addition to direct problem, inverse problem can be solved. Extreme value and tail entropy approximations substantially improve results of direct Monte Carlo simulation (Examples 2, 6).

In order to achieve the best results in calculations a proper balance between general sample size m and parameter k regulating sample size in extreme value approximation should be established. The accuracy of approximation increases with k until sample size m/k used for extreme value approximation remains sufficiently large, but after a certain "threshold" in k the value of sample size m/k becomes too small and the accuracy of approximation is impaired (Example 2).

For relatively high failure probabilities of order 10^{-3}–10^{-4} (Examples 3, 4, 5) sufficiently accurate results can be obtained by direct Monte Carlo simulation without extreme value and tail entropy approximation; for lower failure probabilities of order 10^{-4}–10^{-7} (Examples 1, 2, 6) the application of these techniques is essential.

The main disadvantage of the suggested approach is that generally it fails to estimate very low failure probabilities (lower than 10^{-7}).

The approach is particularly advantageous when calculation for similar structures are carried out on a mass scale. In this situation first several typical cases are investigated. On the basis of obtained results it is decided whether or nor extreme value and tail entropy approximations are required and the optimum values of m and, if necessary, k are established. Thereafter all the rest of calculations are performed. For instance, such methodology was used for reliability regulation of RC flexural members and members under combined compression and bending in order to improve the Russian Code 1985: numerical values of material property combination factor k_c (see section 2.4 of this paper) were calculated for different combinations of member parameters.

7. Conclusions

It is suggested to use Monte Carlo simulation in combination with extreme value and tail entropy approximations. An appropriate Pearson's curve is fit to represent simulation results, reliability is estimated by numerical integration. Direct and inverse problems are considered. An algorithm and computer program for structural reliability estimation are developed. A number of specially chosen examples are used in order to verify the accuracy of the approach and compare numerical results with those obtained by the importance sampling and Level 2 methods. The suggested approach appears to be sufficiently accurate. The drawbacks of the importance sampling and Level 2 methods can be overcome: limit state function and probability density functions need not be differentiable and even continuous, the approach is insensitive towards existence of multiple β-points, non-normal basic variables need not be transferred to normal basic variables, etc. At the same time the approach usually fails to estimate failure probabilities lower than 10^{-7}. Therefore a proper field of application for the suggested approach exists and it can be used for practical purposes along with the Level 2 and importance sampling methods.

References

Bjerager, P. (1990), "On computation methods for structural reliability analysis", *Structural Safely*, **9**, 77-96.

Elderton, W. P. and Johnson, N. L. (1969), *Systems of frequency curves*, Cambridge University Press.

Engelund, S. and Rackwitz, R. (1993), "A benchmark study on importance sampling technique in structural reliability", *Structural Safely*, **12**, 255-276.

Fu, G. and Moses, F. (1993), "Multimodal simulation method for system reliability analysis", *J. Eng. Mech., ASCE*, **6**, 1173-1179.

Hohenbichler, M., Soltwitzer, S., Kruse, W. and Rackwitz, R. (1987), "New light on first- and second-order reliability methods", *Structural Safety*, **4**, 267-284.

Krakovski, M. B. (1993), "Improvement in Russian Codes for RC structures design on the basis of reliability theory", *Proceedings of IFIP WG7.5 Fifth Working Conference on Reliability and Optimization of Structural Systems*, Japan, March, North-Holland, 141-148.

Lind, N. C. and Hong, H. P. (1991), "Tail entropy approximations", *Structural Safety*, **4**, 297-306.

Rackwitz, R. and Fiessler, B. (1977), "An algorithm for calculation of structural reliability under combined loading", *Berichte zur Sicherheitstheorie der Bauwerke*, Lab. f. Konstr. Ingb., Munich.

Rosenblatt, M. (1952), "Remarks on a multivariate transformation", *The Annals of Mathematical Statistics*, **23**(1), 470-472.

Russian Code (1985), *Concrete and reinforced concrete structures, SNiP 2.03.01*, Stroyizdat Publishers, Moscow (in Russian).

Thoft-Christensen, P. and Baker, M. J. (1982), *Structural reliability theory and its applications*, Springer, New York.