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Stress analyses of solids with rectangular holes by 3-D special hybrid stress elements

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Abstract. Two kinds of special 3-dimensional 12-node finite elements-each one contains a traction-free planar surface-have been developed based on Hellinger-Reissner principle by assumed stress hybrid method. Example solutions have demonstrated the advantage of using these special elements for analyzing plates and solids with rectangular holes.

Key words: rectangular hole; special hybrid stress element.

1. Introduction

For stress analyses of solids with traction-free cylindrical surfaces by the finite element method, it has been shown that the special hybrid stress elements which contain cylindrical surfaces not only provide much more accurate results than those obtained by using conventional assumed displacement elements and ordinary assumed stress hybrid elements, but also possess high computational efficiency (Pian and Tian 1986, Tian 1990a, 1990b, Tian and Tian 1990, Tian, *et al.* 1991).

In the paper two kinds of 3-dimensional 12-node special elements with traction-free planar surfaces are derived by hybrid stress method. The combination of the new special elements with above special hybrid stress elements which contain traction-free circular surfaces can be conveniently used for stress analyses around rectangular holes with rounded corners.

2. Hybrid formulation of two 12-node special solid elements

The special elements are shown Fig. 1(a) and (b). The plane 1584 is a traction-free surface. The element stiffness matrix is formulated by Hellinger-Reissner principle (Pian and Chen 1982). The energy functional for an individual element is given by

$$\Pi_R=\int_{V_n}\left[-\frac{1}{2}\sigma^T\boldsymbol{S}\sigma+\sigma^T(\boldsymbol{D}\boldsymbol{u})-\overline{\boldsymbol{F}}^T\boldsymbol{u}\right]dV-\int_{S\sigma_n}\overline{\boldsymbol{T}}^T\boldsymbol{u}\,dS-\int_{Su_n}\boldsymbol{T}^T(\boldsymbol{u}-\overline{\boldsymbol{u}})dS \tag{1}$$

$$=stationary$$

where

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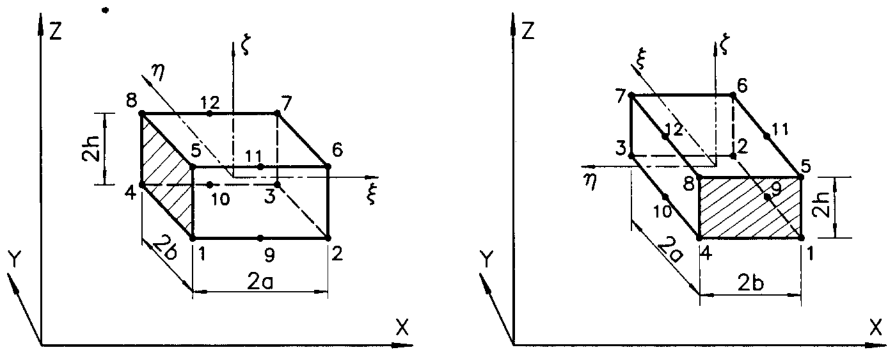

(a) (b)

Fig. 1 Geometry of special 12-node elements with traction-free planar surfaces.

V_n : volume of individual element
σ : stresses
$\boldsymbol{S}$: elastic compliance
$\boldsymbol{u}$: displacements
$\boldsymbol{Du}$: strains
$\bar{\boldsymbol{F}}$: prescribed body forces
S_{σ_n} : part of element boundary on which the surface tractions are prescribed
$\bar{\boldsymbol{T}}$: prescribed surface tractions
S_{u_n} : part of element boundary on which the displacements are prescribed
$\boldsymbol{T}$: surface tractions
$\bar{\boldsymbol{u}}$: prescribed surface displacements

The stress field can be expressed in terms of stress parameters β:

$$\sigma=\boldsymbol{P}\beta \tag{2}$$

The displacements $\boldsymbol{u}$ are interpolated in terms of nodal displacements $\boldsymbol{q}$.

$$\boldsymbol{u}_q=\boldsymbol{Nq} \tag{3}$$

From the variation of Π_R the element stiffness matrix $\boldsymbol{K}$ is then given by

$$\boldsymbol{K}=\boldsymbol{G}^T\boldsymbol{H}^{-1}\boldsymbol{G} \tag{4}$$

where

$$\boldsymbol{H}=\int_{Vn}\boldsymbol{P}^{*T}\boldsymbol{S}\,\boldsymbol{P}^{*}\,dV \qquad \boldsymbol{G}=\int_{Vn}\boldsymbol{P}^{*T}(\boldsymbol{DN})\,dV \tag{5}$$

The stress fields are then chosen to be uncoupled and complete in quadratic terms of ξ. By the use of the equilibrium equations and the boundary conditions of the traction-free surface, the following stress assumption $\sigma^*(=\boldsymbol{P}^*\beta)$ can be obtained.

$$\sigma_x{}^a=\sigma_y{}^b=(1+\xi)^2\beta_1$$
$$\sigma_y{}^a=\sigma_x{}^b=\beta_2+\xi\beta_3+\eta\beta_4+\zeta\beta_5+\xi^2\beta_6+\xi\eta\beta_7+\xi\zeta\beta_8+\frac{2d}{c}\xi\eta\zeta\beta_{16}$$
$$+(\eta\zeta+2\xi\eta\zeta)\beta_{25}+\xi^2\eta\beta_{27}+\xi^2\zeta\beta_{28}$$

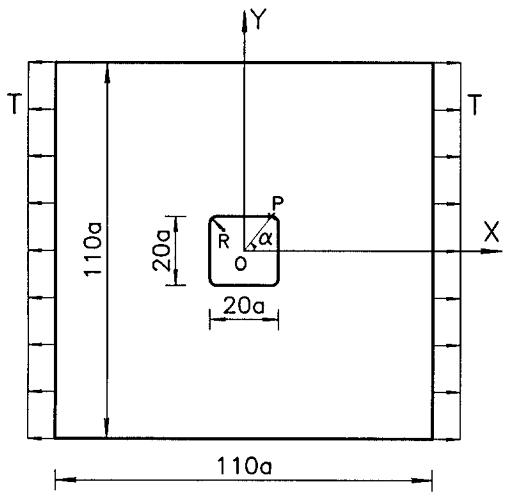

Fig. 2 Thin square plate with a center square hole of rounded corners.

$$
\begin{aligned}
\sigma_z{}^a = \sigma_z{}^b = {} & \beta_9 + \xi\beta_{10} + \eta\beta_{11} + \xi^2\beta_{12} + \xi\eta\beta_{13} - c\zeta\beta_{19} - c\xi\zeta\beta_{21} - d\zeta\beta_{22} \\
& + 2d\xi\eta\zeta\beta_{23} - (d\zeta + 2d\xi\zeta)\beta_{24} + (\eta\zeta + 2\xi\eta\zeta)\beta_{26} + \xi^2\eta\beta_{29} + \xi^2\zeta\beta_{30} \\
\tau_{xy}{}^a = \tau_{xy}{}^b = {} & -\frac{d}{c}(\eta + \xi\eta)\beta_1 + (1+\xi)\beta_{14} + (\xi + \xi^2)\beta_{15} + (\zeta - \xi^2\zeta)\beta_{16} \\
& - \frac{c}{d}(\xi\zeta + \xi^2\zeta)\beta_{25} \\
\tau_{yz}{}^a = \tau_{zx}{}^b = {} & \frac{d^2}{c}\eta\zeta\beta_1 - c\zeta\beta_4 - c\xi\zeta\beta_7 - d\zeta\beta_{14} - (d\zeta + 2d\xi\zeta)\beta_{15} + \beta_{17} \\
& + \xi\beta_{18} + \eta\beta_{19} + \xi^2\beta_{20} + \xi\eta\beta_{21} - c\xi^2\zeta\beta_{27} - \frac{1}{c}\xi^2\eta\beta_{30} \\
\tau_{zx}{}^a = \tau_{yz}{}^b = {} & -(d\zeta + d\xi\zeta)\beta_1 + (1+\xi)\beta_{22} + (\eta + \xi^2\eta)\beta_{23} + (\xi + \xi^2)\beta_{24} \\
& - \frac{1}{d}(\xi\eta + \xi^2\eta)\beta_{26}
\end{aligned}
\tag{6}
$$

where σ^a : stress assumption of special element "a"
σ^b : stress assumption of special element "b"
$c = h/b$ (for element "a"); $-h/b$ (for element "b")
$d = h/a$ (for element "a" and element "b")

The 30 independent β parameters are the minimum numbers required for the suppression of kinematic deformation modes.

3. Numerical results

3.1. Square plate with a square hole of rounded corners

A thin square plate with a centre rectangular hole is acted upon by uniform tension T (Fig.

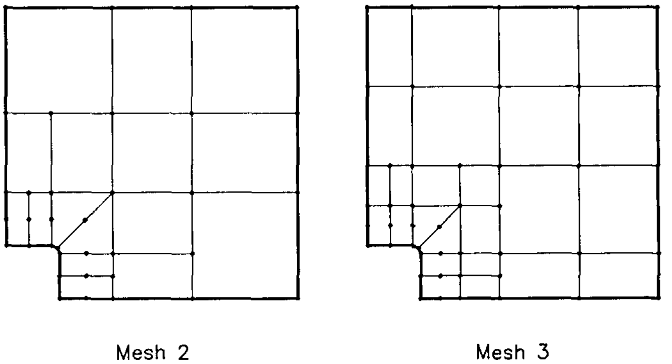

Fig. 3 Meshes for 1/4 plate with square hole of rounded corners.

2). The problem is analyzed by using only one layer of 3-D elements in three different meshes for 1/4 of the plate. The mesh 2 and mesh 3 with 16 and 25 elements are shown in Fig. 3 while the coarse mesh 1 consists of only 7 elements. Fig. 3 also shows the combination of following different elements which are used to analyze the stress around the rectangular hole.

(1) Two kinds of present special 12-node assumed stress hybrid element with a traction-free planar surface on the straight sides of the hole
(2) The special 12-node assumed stress hybrid element with a traction-free cylindrical surface at the corners of the hole (Tian 1990a)
(3) The ordinary assumed displacement isoparametric element in the rest area of the plate

The computed results are given in Table 1. For comparison of element performance the results obtained by using the ordinary assumed displacement solid elements everywhere are also included in the Table. The reference solutions are given by Shawen (1965) by using angle-preserving mapping method. Case A and Case B in the Table are corresponding to using 3 and 4 terms of the mapping function given by Shawen. The values R are the desirable radii at the corner to imitate the mapping function with the rectangular hole of rounded corners.

It can be seen that the stress concentration factors provided by the present special elements

Table 1 Computed stress concentration factor (SCF) (2-D problems; Thin square plate with a rectangular hole of rounded corners under tension)

Type of elements	Mesh 1		Mesh 2		Mesh 3	
	SCF	error %	SCF	error %	SCF	error %
Case A: Reference solution SCF=5.156; $R=0.616a$						
Present special elements and previous special elements and ordinary displacement elements	5.509	6.85	5.465	5.99	5.151	−0.10
Ordinary displacement elements	6.294	22.07	5.686	10.28	5.464	5.97
Case B: Reference solution SCF=6.464; $R=0.347a$						
Present special elements and previous special elements and ordinary displacement elements	5.760	−10.89	6.066	−6.16	6.088	−5.82
Ordinary displacement elements	9.387	46.58	7.981	24.63	7.608	18.80

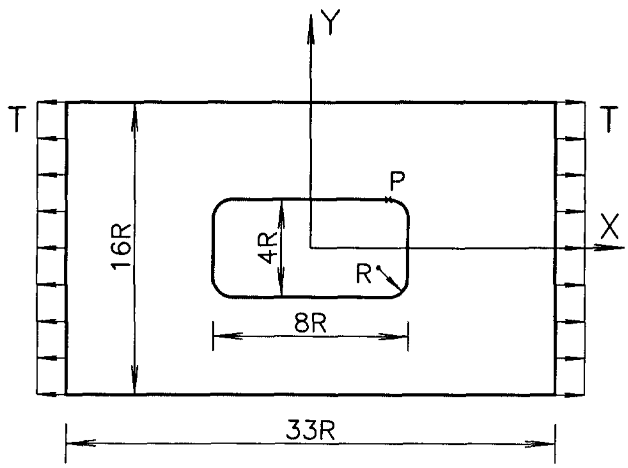

Fig. 4 Thin rectangular plate with a rectangular hole of rounded corners.

are much closer to the reference solutions in comparison with those by using the ordinary assumed displacement elements. When the largest circumferential stress σ_θ is acting on point P, letting α be the angle between x-axis and radial line OP, the values of α obtained by the special elements and the ordinary assumed displacement elements are the same. i.e. α equals to 46.82° for Case A and 46.01° for Case B. They are different from 45.67° (Case A) and 45.32° (Case B) given by Shawen's formula. For both of Case A and Case B the location of point P obtained by the present finite element method is very close to the intersection point of the straight side and the circular arc and is near by the y-axis. The result is proved by photoelastic method (Nisida 1982).

3.2. Rectangular plate with a rectangular hole of rounded corners

A thin rectangular plate with a centre rectangular hole of rounded corners under uniform tension is shown in Fig. 4. The problem is also analyzed by only one layer of 16 elements for 1/4 of the plate.

The distributions of the circumferential stresses σ_θ at the rim of the hole obtained by the present elements are shown in Fig. 5. It is seen that it is close to the analytical solution (Nisida 1982). But the stress concentration factor will be reduced to 2.25 by using fine meshes.

3.3. Square block with a square hole of rounded corners

A square block with a centre hole is acted upon by uniform tension over two opposite faces. The Poisson's ratio ν for the material is taken as 0.25. The problem is analyzed by using five different meshes with 16, 32, 64, 128 and 256 elements for 1/8 of the block. The mesh 3 is shown in Fig. 6. The top view of Fig. 6 is the same as that shown in Fig. 3. The results of the stress concentration factors at the point P and point Q are given in Table 2.

It shows that the stress concentration factors are equal to 6.1 in the middle plane and 6.4 on the face when a thick square block with a centre square hole of rounded corners under tension over two opposite faces.

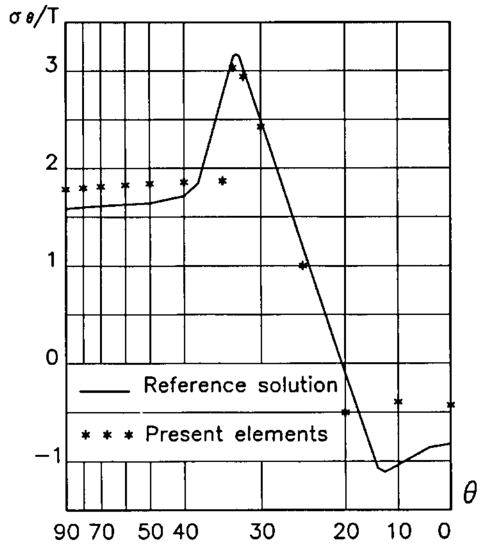

Fig. 5 Variations of σ_θ/T at the rim of the hole.

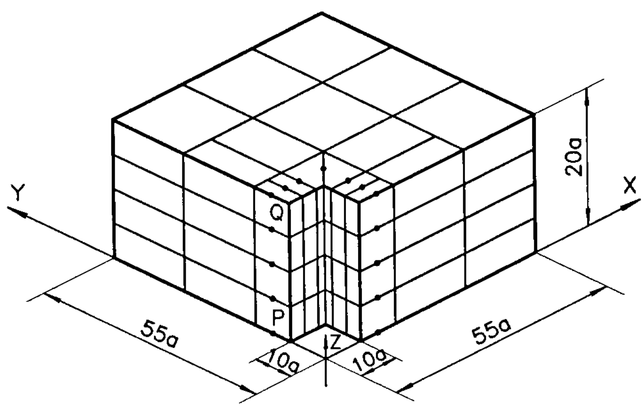

Fig. 6 Thick block with square hole of rounded corners under uniform tension (Mesh for one-eighth of block)

Table 2 Computed stress concentration factors (ν=0.25; 3-D problems; Thick square block with a square hole of rounded corners)

Meshes	1	2	3	4	5
Degrees of freedom	155	250	440	820	1580
Stress concentration factors of middle plane	5.980	6.052	6.070	6.068	6.064
Stress concentration factors of face	6.020	6.192	6.291	6.357	6.386

4. Conclusions

Two kinds of special 12-node solid assumed stress hybrid elements with traction-free planar surfaces are developed by hybrid stress approach. These special elements together with ordinary solid assumed displacement elements can be efficiently used for stress analyses of rectangular holes in a solid or in a plate.

Acknowledgements

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