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A numerical analysis of the large deflection of an elastoplastic cantilever

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Abstract. A simple numerical method is applied to calculate the large deflection of a cantilever beam under an elastic-plastic deformation by dividing the deformed axis into a number of small segments. Assuming that each segment can be approximated as a circular arc, the method allows large deflections and plastic deformation to be analyzed. The main interests are the load-deflection relationship, curvature distribution along the beam and the length of the plastic region. The method is proved to be easy and particularly versatile. Comparisons with other studies are given.

Key words: large deflection; elastic plastic deformation; cantilevers; curvature.

1. Introduction

The Elastica theory was first introduced by Euler to study the large deflection solutions of elastic beams. In the following two centuries, the research work in this field was mainly restricted to the beam made from linear elastic materials, i.e. merely geometrical nonlinearity was considered. The related results were summarized by Frisch-Fay (1962). Since then, the Elastica theory has been extended to cases in which both geometrical and material nonlinearities are involved. The effect of large deflection on the development of plastic deformation was studied by Reid and Reddy (1978) for a rigid, linear-hardening moment curvature relationship. They showed that a cantilever with a tip force acting in a fixed direction has a limited length for the plastically deformed region that extended from the root. Lo and Gupta (1978) investigated the case in which the stress-strain relationship takes the form of a logarithmic function when stress is beyond the elastic limit. Yu and Johnson (1982a) gave a closed form solution of a post-buckling behavior of an elastic-plastic Euler strut and called this type of problem "Plastica".

Changes in the distribution of curvature in a cantilever were considered by Wu and Yu (1986) for an elastic-perfectly plastic material model, and by Liu, Stronge and Yu (1991) for an elastoplastic strain hardening material. They successfully modeled the entire process of large deflection of a horizontal cantilever subjected to a concentrated force at its tip. Nevertheless, these approaches are of considerable complexity and numerical procedures such as the finite difference method

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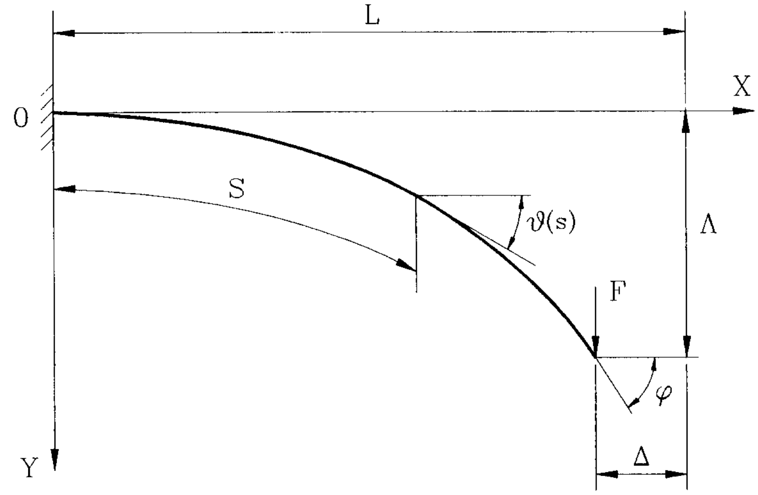


Fig. 1 A cantilever under a tip force load.

had to be applied to solve the equations. It is felt that a much simpler semi analytical, semi-numerical model can be established for such problems. Based on this approach, the new method presented in this paper may not only help to solve plastic, large deformation of a cantilever beam under tip loading, but also be able to deal with more complicated problems, such as distributed load, combined loading of bending and axial force, curved beams, etc.

2. Numerical model

The present paper studies the development of large deflection in an elastic-plastic bar. For easy understanding, a straight cantilever is chosen as an example, as shown in Fig. 1. The beam has a rectangular cross-section with depth H and width B; there is a force at the tip acting transversely to the undeformed cantilever and then it will remain fixed in its original direction during the deforming process. It is assumed that the cross-section of the beam will remain plane and axial stretching is negligible. The local curvature $K(S)$ at any section S is related to the rotation $\vartheta(S)$ of the cross-section,

$$K=\frac{d\vartheta}{dS} \tag{1}$$

A group of nondimensional variables are chosen: $m=M/M_e$, $f=FL/M_e$, $\delta=\Delta/L$, $\lambda=\Lambda/L$, $x=X/L$, $y=Y/L$, $s=S/L$ and $\beta=M_eL/EI=2\sigma_yL/EH$. Here forces and moments have been nondimensionalised with respect to the largest elastic moment $M_e=BH^2\sigma_y/6$, σ_y is the yield stress and E is the elastic modulus. β is a parameter that reflects the geometry and material properties of the beam and is equal to the nondimensional curvature $\kappa=LK=d\vartheta/ds$, pertaining to the maximum elastic bending moment.

2.1. Material modeling

In elastic deformation, the bending moment $m(s)$ is linearly related to the nondimensional curvature

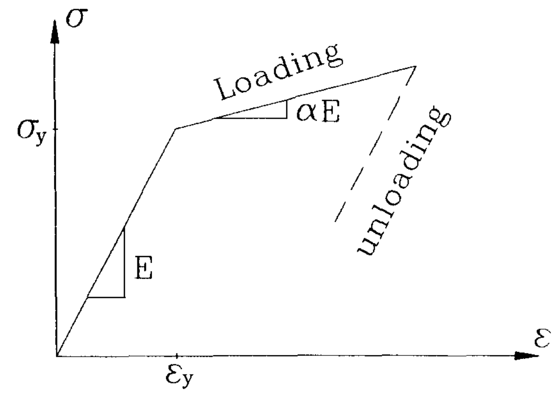


Fig. 2 Bi-linear material model.

$$m(s)=\kappa/\beta \tag{2}$$

The elastic response ends with the bending moment at the root reaching the elastic limit, $m=1$ and this elastic solution is only valid when $\kappa \leq \beta$ and $0<s<1$.

When the load f increases further, a plastic range initiates from the beam root and expands from root towards the tip. In the plastic region the stress distribution at a cross-section has an elastic core and is plastic near the top and bottom surfaces. For a rectangular section with elastic-perfectly plastic properties, the relation between the bending moment and the curvature of the neutral axis is

$$\rho^2=3-2m \tag{3}$$

where $\rho=\beta/\kappa$ is the ratio of the radius of the curvature to the radius of curvature at the elastic limit.

For a bilinear material property, as shown in Fig. 2, the above relation becomes

$$\rho^2-\alpha\left(\frac{2}{\rho}-3+\rho^2\right)=3-2m \tag{4}$$

α being the nondimensional plastic strain-hardening modulus.

2.2 Numerical modeling

In the present numerical modeling, the axis of the deformed beam will be replaced by a number of circular arcs tangent to each other at the points of intersection, as shown in Fig. 3, assuming the arc lengths are small enough that the bending moment for each arc can be approximated as constant and equal to the average bending moment. From the geometry of a general element of arc in Fig. 4, the following equations can be written for the ith arc,

$$s_i=\vartheta_i\rho_i \tag{5}$$

$$a_i=\rho_i\left(\sin\left(\phi-\sum_{j=1}^{i-1}\vartheta_j\right)-\sin\left(\phi-\sum_{j=1}^{i}\vartheta_j\right)\right) \tag{6}$$

$$b_i=\rho_i\left(\cos\left(\phi-\sum_{j=1}^{i-1}\vartheta_j\right)-\cos\left(\phi-\sum_{j=1}^{i}\vartheta_j\right)\right) \tag{7}$$

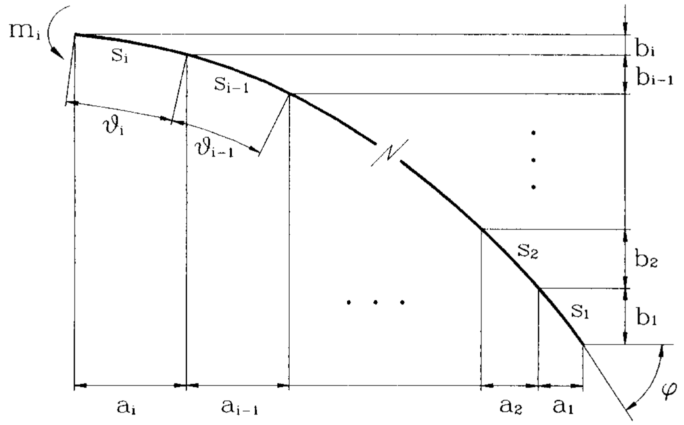


Fig. 3 Cantilever approximated by circular arcs.

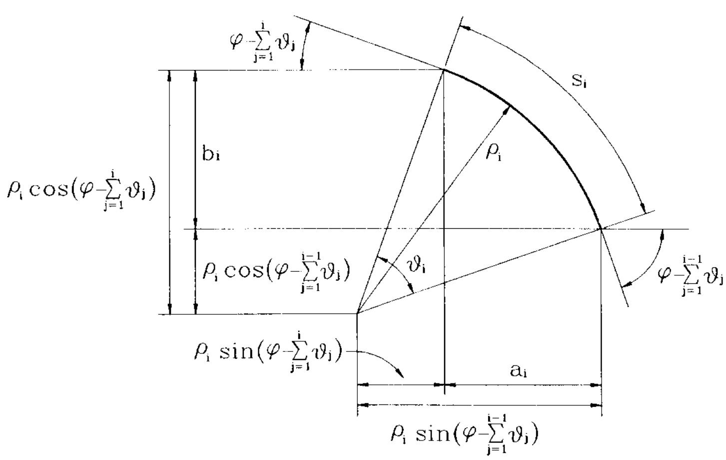


Fig. 4 Variables of the ith element.

where ρ_i is the nondimensionlised radius of the ith circular arc, ϑ_i is its corresponding angle, a_i and b_i are projections of the nondimensional arc length s_i in directions of X and Y axes respectively and

$$\phi=\sum_{j=1}^{N}\vartheta_j \tag{8}$$

is the sum of angles to all the arcs. N is the number of arcs into which the total length of the beam is divided. When s_i, a_i and b_i are known, the length, deflection and horizontal distance to the clamped end can be found,

$$L_i=L\sum_{j=1}^{i}s_j$$

$$X_i=L\sum_{j=1}^{i}a_j \tag{9}$$

$$Y_i = L\sum_{j=1}^{i} b_j$$

When the applied load is a concentrated tip force fixed in the direction perpendicular to the undeformed beam axis, the average bending moment for a general arc is

$$m_i = \left(\sum_{j=1}^{i-1} a_j + \frac{a_i}{2}\right)f \tag{10}$$

Thus if $a_j(j=1,\ i)$ are given, m_i can be calculated and with the assumed value of tip slope ϕ given, starting from the free end the nondimensional curvature of each arc can be solved using Eq. (2) for $m \leq 1$, (3) or (4) for $m > 1$, and all the rest of the above equations can then be solved in order.

When the value of a_i/ρ_i is greater than the corresponding value for

$$\sin\left(\phi - \sum_{j=1}^{i-1} \vartheta_j\right)$$

the last arc has been achieved, indicating too large a value of a_i was used. In this case, we have $N = i$ and for the last arc $\sin\left(\phi - \sum_{j=1}^{N} \vartheta_j\right)$ is equal to zero and

$$\frac{a_N}{\rho_N} = \sin\left(\phi - \sum_{j=1}^{N-1} \vartheta_j\right) \tag{11}$$

Combining Eq. (2) for $m \leq 1$, (3) or (4) for $m > 1$ with Eqs. (6), (10) and (11), the actual value of a_N of the last arc can be solved.

The correct result by this method requires a good estimate of ϕ at the beginning of the calculation. Iterations may need to be carried out if the deviation between $\sum_{j=1}^{N} s_j$ and 1 is too large, and ϕ has to be adjusted accordingly and the whole procedure repeated. Fig. 5 shows the flow chart of the algorithm.

3. Numerical results

Typical deflection profiles for an elastic-plastic cantilever with a vertical force at its tip are shown in Fig. 6; the material is a bilinear strain hardening model with $\alpha = 0.1$ and $\beta = 0.1$. In the calculation, the beam was divided into 100 segments, i.e. $N = 100$. It is clearly seen in Fig. 7 showing the bending moment distribution along the beam that the plastic range expands from the root towards the tip with increasing magnitude of the force. The plastic range increases rapidly with 63% of the total length entering the plastic state at $f = 3$, but after that the plastic length increases slowly to 76% at $f = 9$. Curvature changes along the beam are given in Fig. 8. Calculation was stopped after ϕ had reached 1.1.

Influences of α and β on beam deformation are given respectively in Figs. 9 and 10 with $f = 7$. It appears that for the same value of β, the higher α, the less κ/β, and with a constant α the higher β, the higher κ/β. A comparison between the numerical results and available experimental data of Reid and Reddy (1978) for mild steel cantilever is given in Fig.11, which shows reasonable agreements for $\alpha = 0.05$.

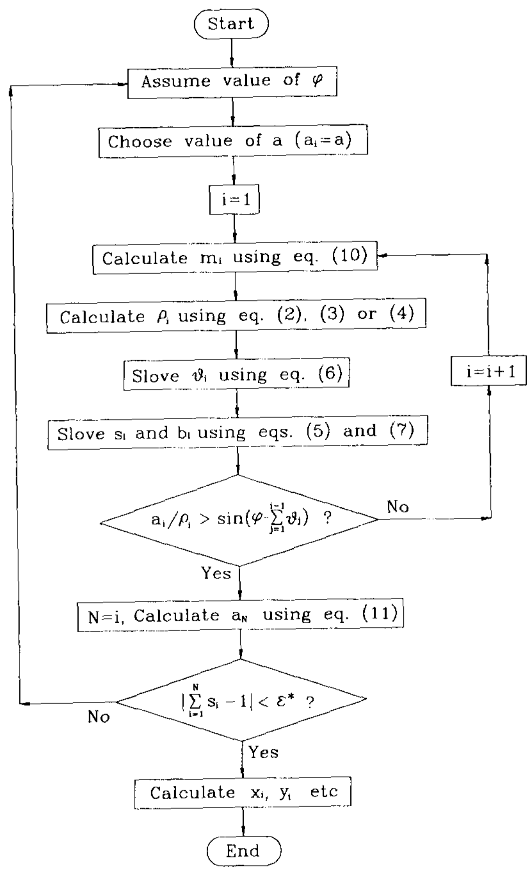


Fig. 5 Flow chart of the algorithm.

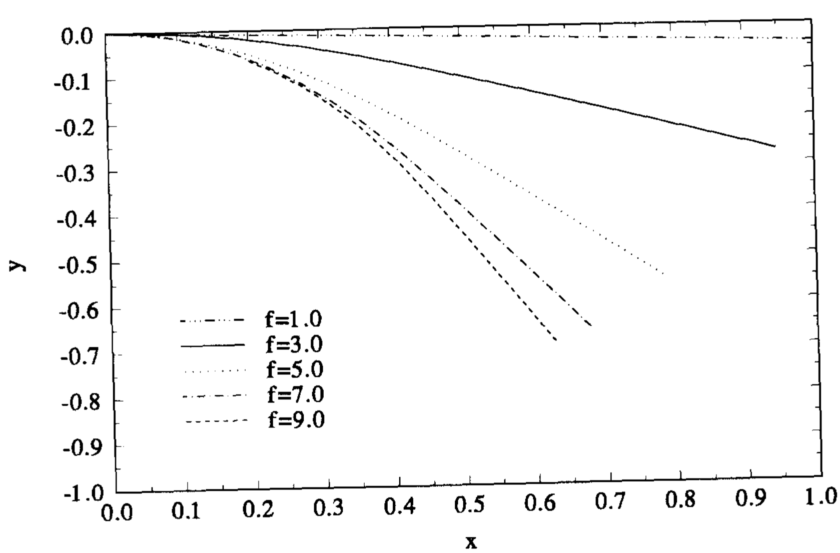


Fig. 6 Deformed configuration of a straight cantilever (α=0.1, β=0.1).

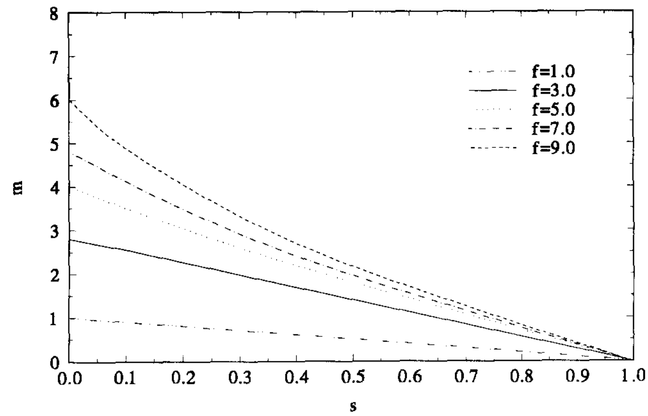


Fig. 7 Moment ratio, M/M_e against intrinsic coordinate, $s(\alpha=0.1,\ \beta=0.1)$.

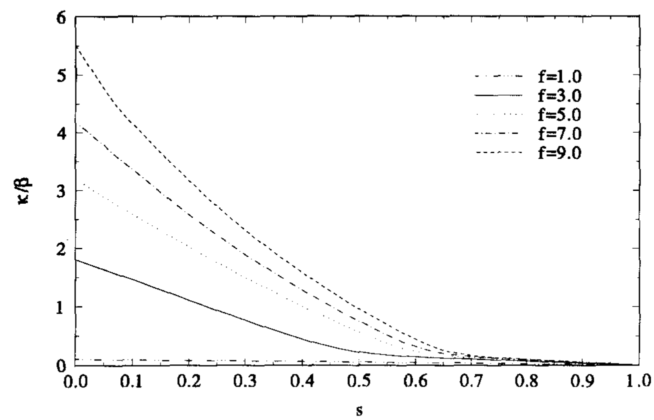
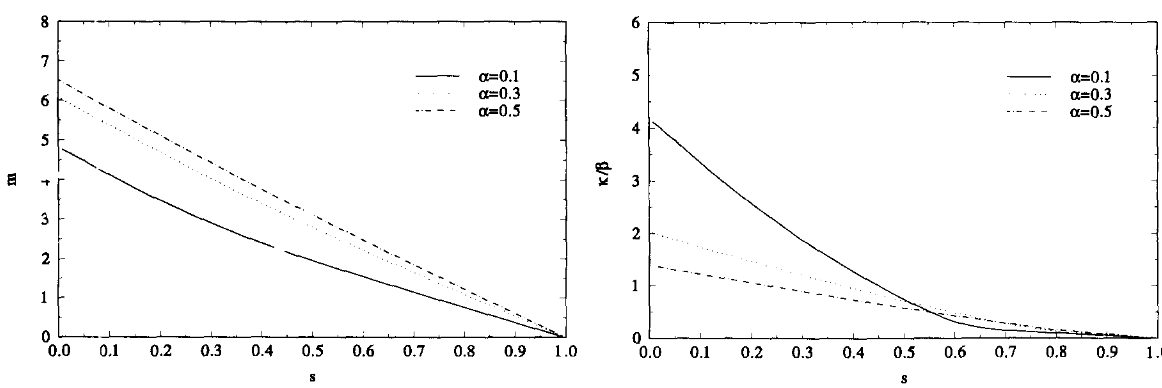


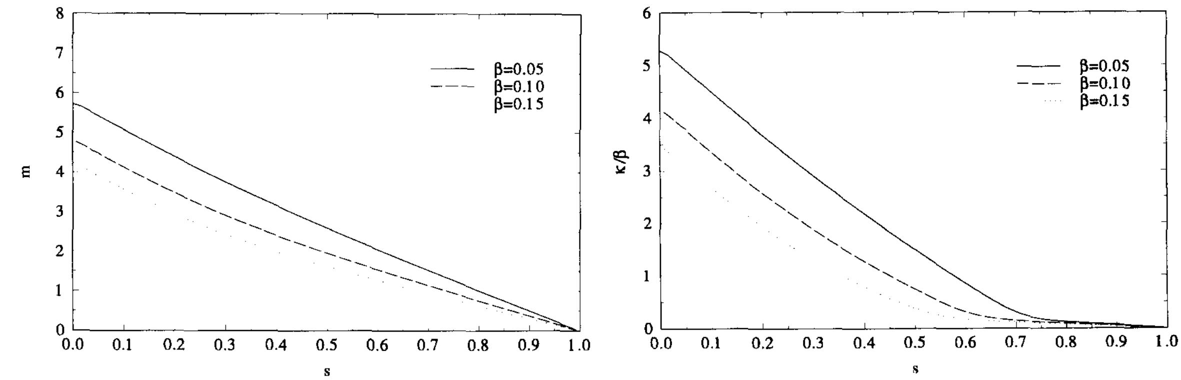
Fig. 8 Curvature ratio, κ/β against intrinsic coordinate, s $(\alpha=0.1,\ \beta=0.1)$.



a. Moment distribution

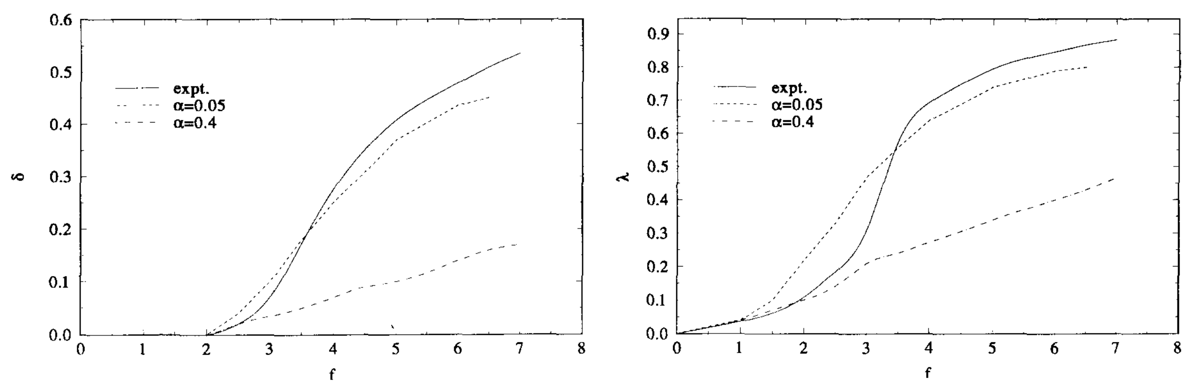
b. Curvature distribution

Fig. 9 Influence of α on bending moment and curvature distribution $(f=7.0,\ \beta=0.1)$.



a. Moment distribution b. Curvature distribution

Fig. 10 Influence of β on bending moment and curvature distribution (f=7.0, α=0.1).



a. Comparison with test results b. Comparison with test results

Fig. 11 Comparison between numerical results and experimental data (Reid and Reddy 1978), β=0.1475.

4. Discussions

The above example of a large deflection analysis on an elastoplastic straight cantilever demonstrates the effectiveness of this simple numerical approach. An advantage of this method is its significant versatility. Apart from a concentrated vertical force, the load can also be an inclined force or even a distributed force; only Eq. (10) needs to be changed accordingly as

$$m_i = f\left(\cos\gamma\left(\sum_{j=1}^{i-1} a_j + \frac{a_i}{2}\right) + \sin\gamma\left(\sum_{j=1}^{i-1} b_j + \frac{b_i}{2}\right)\right) \tag{10a}$$

and

$$m_i \approx \sum_{j=1}^{i-1} w_j s_j\left(\frac{a_j}{2} + \sum_{k=j}^{i-1} a_k + \frac{a_i}{2}\right) + \frac{w_i a_i^2}{8\cos\left(\phi - \sum_{j=1}^{i-1} \vartheta_j\right)} \tag{10b}$$

where γ is the inclined angle of the concentrated force to the undeformed beam axis, and Eqs.

(10a), (6) and (7) must be solved at the same time; w in Eq. (10b) is the intensity of distributed load per length, it is approximated here that over each arc element (not the whole beam), w is uniformly distributed.

Eqs. (2) and (3) or (2) and (4) represent respectively elastic-perfectly plastic and bilinear strain hardening materials with no axial force considered in the yield criterion. If the axial force is considered, for elastic-perfectly plastic beam under an inclined tip force, it gives

$$n_i=\frac{F}{2N_e}\left(\cos\left(\gamma+\phi-\sum_{j=1}^{i}\vartheta_j\right)+\cos\left(\gamma+\phi-\sum_{j=1}^{i-1}\vartheta_j\right)\right) \tag{10c}$$

and

$$\rho=\begin{cases}1/m & 0\leq m\leq 1-n\\ \left(3-\dfrac{m}{1-n}\right)^2/4(1-n) & 1-n\leq m\leq 1+n-2n^2\\ \sqrt{3(1-n^2)-2m} & 1+n-2n^2\leq m<3(1-n^2)/2\end{cases} \tag{12}$$

where $n=F/N_e$ and $N_e=BH\sigma_y$, as discussed by Yu and Johnson (1982b). If a work hardening relation of $\sigma=C\varepsilon^{\eta}$ is adopted, C and η being the material constant and working hardening exponent, respectively, the $\rho-m-n$ relation is given by El-Domiaty and Shabaik (1984). These relations can be easily adapted into the calculation.

This numerical method may also be applied to curved beams, such as a circular ring of radius r; then for elastic deformation, as an example, we have

$$\frac{1}{\rho}=m+\frac{1}{r}$$

and the coordinates of the beam tip become

$$X=r\left(\sin\left(\frac{L}{r}\right)-\sum_{i=1}^{N}a_i\right)$$

$$Y=r\left(\sum_{i=1}^{N}b_i-1+\cos\left(\frac{L}{r}\right)\right)$$

The method can be extended for any curved bar provided the geometry of the free shape allows to calculate pre-deformed r_i for each elemental arc.

Many authors (Reid and Reddy 1978, Wu and Yu 1986, Liu, *et al.* 1989) have noticed the so-called unloading phenomena. For a cantilever loaded with a tip load of fixed direction, the load magnitude increasing, the plastic range first generates from the beam root towards the free tip, then from a certain magnitude of f, the plastically deformed range begins to decrease. This is the result of large deflection which causes the bending moment in certain beam sections to reduce with increasing force. As shown in Fig. 2 the bi-linear material under this unloading condition behaves in the same way as in its elastic stage. Thus when using this numerical modeling the value of m_i should be recorded and compared with that produced by the next step of higher f. If unloading happens, the bending moment and curvature in the plastic unloading segment are reduced from the maximum values of m_u and its corresponding κ_u which occurs at a lower load. The relation between the bending moment and curvature in this segment is

$$m=m_u+\Delta\kappa/\beta \tag{13}$$

where the accumulated reduced curvature during unloading is $\Delta\kappa = \kappa - \kappa_u$.

The main difficulty in applying this numerical method is the estimate of the original value of ϕ, the assumed final tip slope. Experience shows that if a chosen value is too far from the real one, the algorithm may not be convergent to the required accuracy, then a new value of ϕ needs to be given. Therefore, a trial and error procedure is required in some cases.

Another problem is the accuracy of the curvature at beam segments where the curvature derivative is large, such as at the beam root. Due to the nature of the method, the beam element is assumed as a circular arc, the curvature is then in the sense of an average value. If a high accuracy is required, a finer discretization, viz. a smaller a_i should be used, particularly for the elements close to the beam root.

Finally, it is noted that if the analysis involves only elastic deformation, the above discussed method is in a way similar to the work done by Seames and Conway (1957).

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