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A discussion on simple third-order theories and elasticity approaches for flexure of laminated plates

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Abstract. It is well known that two-dimensional simplified third-order theories satisfy the layer interface continuity of transverse shear strains, thus these theories violate the continuity of transverse shear stresses when two consecutive layers differ either in fibre orientation or material. The third-order theories considered herein involve four/or five dependent unknowns in the displacement field and satisfy the condition of vanishing of transverse shear stresses at the bounding planes of the plate. The objective of this investigation is to examine (i) the flexural response prediction accuracy of these third-order theories compared to exact elasticity solution (ii) the effect of layer interface continuity conditions on the flexural response. To investigate the effect of layer interface continuity conditions, three-dimensional elasticity solutions are developed by enforcing the continuity of different combinations of transverse stresses and/or strains at the layer interfaces. Three dimensional twenty node solid finite element (having three translational displacements as degrees of freedom) without the imposition of any of the conditions on the transverse stresses and strains is also employed for the flexural analysis of the laminated plates for the purposes of comparison with the above theories. These shear deformation theories and elasticity approaches in terms of accuracy, adequacy and applicability are examined through extensive numerical examples.

Key words: third-order theories; elasticity; flexure; composite; laminated; layer interface; continuity; degrees of freedom; analytical; finite element method; symmetric lay-up; antisymmetric lay-up; cross-ply; angle-ply.

1. Introduction

It is well known that the classical plate theory based on Kirchoff's hypothesis of inextensional straight normals under predicts deflections and over predicts frequencies and buckling loads due to the neglect of transverse shear flexibility. Further, since the advanced composites in use to date have low ratio of transverse shear modulus to the in-plane moduli, transverse shear deformations play even more significant role in reducing the flexural stiffness than in metals. To account for the transverse shear flexibility and other non-classical factors such as transverse normal strains, several approaches have been proposed over the years. Most of these approaches have been critically reviewed in the state of the art papers by Reissner (1985), Noor and Burton (1989) and Reddy (1990). While Reissner (1985) provided survey of several two dimensional theories

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and variational statements of 3-dimensional elasticity problems. Reddy (1990) reviewed all the third-order theories that satisfy vanishing of transverse shear stresses at the bounding planes of the plate. Both these papers do not include numerical results. Noor and Burton (1989) assessed the effect of thickness ratio, modular ratio, number of layers and stacking sequence on the response prediction accuracy of several two dimensional theories.

Bhimaraddi and Stevens (1984), Reddy (1984) have proposed simple third-order shear deformation theories involving five dependent unknowns as in classical shear deformation theory due to Mindlin (1951). The displacement field proposed by Bhimaraddi and Stevens (1984) includes first-order and classical plate theories as its subset, while Reddy's (1984) theory degenerates to first-order theory only. These two approaches (Bhimaraddi and Stevens 1984 and Reddy 1984) are basically the same and can be obtained one from the other by proper substitution of the variables. Bhimaraddi and Stevens (1984) investigated the problem of free vibrations of laminated plates only, while Reddy's (1984) theory has been employed for deflection, free-vibration and buckling analysis of laminated plates (refer Kant and Pandya 1988, Putcha and Reddy 1986 and Reddy and Phan 1985). The displacement fields employed by Reddy (1984) can be obtained by neglecting transverse normal strain and imposing zero transverse shear stress at the top and bottom surfaces of the plate in the displacement model proposed by Lo, *et al.* (1977a, 1977b). Lim, *et al.* (1988) further simplified Reddy's (1984) theory by incorporating the assumption that in-plane rotation tensor is constant through the thickness and investigated the flexural response of isotropic plates. The resulting theory has only four dependent unknowns in the displacement field. Based on the same assumption, Bhimaraddi and Stevens' (1984) theory can also be degenerated to a displacement field having four dependent unknowns as that of Lim, *et al.* (1988). The simplified theories discussed herein allow parabolic variation of transverse shear strain and do not require shear correction factors. These simplified theories impose the continuity of displacements and transverse shear strains at the layer interfaces. All these investigators have compared their results with the analytical elasticity solution due to Pagano (1970), Pagano and Hatifield (1972) and Srinivas and Rao (1970), wherein the continuity of transverse normal and shear stresses along with displacements at the layer interfaces is imposed. Therefore, the effect of layer interface continuity condition on the flexural response prediction accuracy of simplified third-order theories need to be studied.

The objective of the present paper is to critically evaluate these simplified third-order theories involving five variables (Bhimaraddi and Stevens 1984 and Reddy 1984) and four variables (Lim, *et al.* 1988) for flexural response of laminated composite plates. Another simple third-order theory involving four variables degenerated from the displacement field of Bhimaraddi and Stevens (1984) is also considered for the purpose. The method of solution proposed by Pagano (1970) is employed for the following layer interface continuity conditions.

(1) Transverse normal and shear stresses are continuous at the layer interface.
(2) Transverse normal stress and transverse shear strains are continuous at the layer interface.
(3) Transverse normal and shear strains are continuous at the layer interface.

In addition representative problems are solved using a twenty node three-dimensional solid element available in general purpose finite element program MSC NASTRAN and the results are compared with analytical elasticity solution and those of simplified higher-order theories. Since, the element has only three translations as degrees of freedom, the continuity of displacements and in-plane strains at the layer interface alone is assured, unlike in Pagano's (1970) analytical elasticity solution. Further, the stress free conditions at the bounding planes are not imposed in this idealization. The comparison of plate centre deflections obtained from these modelling approaches have brought forth many features regarding the adequacy and accuracy of these formulations. The present study is limited to plate centre deflections alone as (i) such results are readily available in the open literature and (ii) it is well known that transverse normal and shear stresses computed by utilizing three-dimensional elasticity equilibrium equations rather than constitutive relations are fairly accurate.

2. Third order theories-formulations

The displacement field, strain-displacement relationship and constitutive equations of the simplified third-order theories considered in this study can be expressed as:

$$u(x, y, z)=u_o(x, y)+z\,\phi_1(x, y)+\frac{4z^3}{3h^2}\psi_1(x, y)$$

$$v(x, y, z)=v_o(x, y)+z\,\phi_2(x, y)+\frac{4z^3}{3h^2}\psi_2(x, y)$$

$$w(x, y, z)=w_b(x, y)+w_s(x, y) \tag{1}$$

where u, v, w are the components of displacement anywhere in the plate in x, y and z direction respectively. u_o and v_o are the components of mid-plane displacements in x and y directions respectively. The transverse displacement is expressed interms of two components w_b and w_s. The displacement field employed in two of the third-order theories considered herein is such that the derivative of the component w_b is numerically equal to the rotation of the cross-section $(-\nabla w_b)$ and w_s is the displacement due to shear deformation of the section. ϕ's and ψ's are additional variables, required to represent rotation of the cross-sectioin. h is the thickness of laminated plate and x, y and z are the cartesian coordinates.

On substituting displacement field Eq. (1) in the strain-displacemnt relationship, the strains in terms of dependent unknowns can be written as:

$$\varepsilon_x=\varepsilon_{xo}+z\,\kappa_{xo}+\frac{4z^3}{3h^2}\kappa_{x1};\quad \varepsilon_y=\varepsilon_{yo}+z\,\kappa_{yo}+\frac{4z^3}{3h^2}\kappa_{y1}$$

$$\gamma_{xy}=\gamma_{xyo}+z\,\kappa_{xyo}+\frac{4z^3}{3h^2}\kappa_{xy1};\quad \gamma_{xz}=\left(1-\frac{4z^2}{h^2}\right)\gamma_1$$

$$\text{and}\quad \gamma_{yz}=\left(1-\frac{4z^2}{h^2}\right)\gamma_2 \tag{2}$$

with

$$\varepsilon_{xo}=u_{o,x};\quad \varepsilon_{yo}=v_{o,y};\quad \gamma_{xyo}=u_{o,y}+v_{o,x};\quad \kappa_{xo}=\phi_{1,x}$$

$$\kappa_{yo}=\phi_{2,y};\quad \kappa_{xy}=\phi_{1,y}+\phi_{2,x};\quad \kappa_{x1}=\psi_{1,x};\quad \kappa_{y1}=\psi_{2,y}$$

$$\kappa_{xy1}=\psi_{1,y}+\psi_{2,x};\quad \gamma_1=\phi_1+w_{b,x}+w_{s,x}\quad \text{and}\quad \gamma_2=\phi_2+w_{b,y}+w_{s,y}$$

The constitutive relations for a laminated plate corresponding to this model can be written as:

$$\begin{Bmatrix} N_i \\ M_i \\ P_i \\ Q_i \end{Bmatrix}=\begin{bmatrix} A_{ij} & B_{ij} & E_{ij} & 0 \\ B_{ij} & D_{ij} & F_{ij} & 0 \\ E_{ij} & F_{ij} & H_{ij} & 0 \\ 0 & 0 & 0 & X_{lm} \end{bmatrix}\begin{Bmatrix} \varepsilon_j \\ \kappa_{oj} \\ \bar{\kappa}_{1j} \\ \phi_m \end{Bmatrix}\qquad \begin{matrix}(i, j=1,\ 2,\ 6)\\(l, m=4,\ 5)\end{matrix}$$

where,

$$N_i=[N_x\ N_y\ N_{xy}];\ M_i=[M_x\ M_y\ M_{xy}]$$
$$P_i=[P_x\ P_y\ P_{xy}];\ Q_i=[Q_x\ Q_y]$$
$$\varepsilon_j=[\varepsilon_{xo}\ \varepsilon_{yo}\ \varepsilon_{xyo}];\ \kappa_j=[\kappa_{xo}\ \kappa_{yo}\ \kappa_{xyo}]$$
$$\bar{\kappa}_{1j}=\frac{4}{3h^2}[\kappa_{x1}\ \kappa_{y1}\ \kappa_{xy1}] \text{ and } \phi_m=[\gamma_1\ \gamma_2]$$

A_{ij}, B_{ij} and D_{ij} are the usual extensional, bending-extensional and bending stiffness coefficients. And E_{ij}, F_{ij} and H_{ij} are the higher-order stiffness coefficients given as:

$$E_{ij}=\frac{1}{4}\sum_{k=1}^{L}\bar{Q}_{ij}^k(h_{k+1}^4-h_k^4)$$

$$F_{ij}=\frac{1}{5}\sum_{k=1}^{L}\bar{Q}_{ij}^k(h_{k+1}^5-h_k^5)$$

$$H_{ij}=\frac{1}{7}\sum_{k=1}^{L}\bar{Q}_{ij}^k(h_{k+1}^7-h_k^7)$$

where $\bar{Q}_{ij}$ are the reduced transformed stiffnesses with respect to the material axes of k^{th} layer.
The simplified theories due to Bhimaraddi and Stevens (1984), Reddy (1984) and Lim, *et al.* (1988) can be deduced from Eqs. (1)-(3) by appropriately specifying $\phi's$ and $\psi's$ and $w's$ e.g.:

Model A

Third-order theory due to Bhimaraddi and Stevens (1984) can be obtained by substituting

$$\phi_1=\phi_x-w_{o,x};\ \phi_2=\phi_y-w_{o,y};\ \psi_1=-\phi_x$$
$$\psi_2=-\phi_y;\ w_b=w_o \text{ and } w_s=0 \tag{4}$$

in Eqs. (1)-(3)
The equilibrium equations of the theory consistent with the assumed displacement-field (strain-displacement relationship) and appropriate constitutive Eqs. (3) are:

$$N_{x,x}+N_{xy,y}=0 \tag{5a}$$

$$N_{xy,x}+N_{y,y}=0 \tag{5b}$$

$$M_{x,xx}+2M_{xy,xy}+M_{y,yy}+q=0 \tag{5c}$$

$$M_{x,x}+M_{xy,y}-\frac{4}{3h^2}[P_{x,x}+P_{xy,y}]-Q_x=0 \tag{5d}$$

$$M_{xy,x}+M_{y,y}-\frac{4}{3h^2}[P_{xy,x}+P_{y,y}]-Q_y=0 \tag{5e}$$

Model B

Third-order theory due to Reddy (1984) can be obtained by substituting:

$$\phi_1=\psi_x;\ \phi_2=\psi_y;\ \psi_1=w_{o,x}-\psi_x$$
$$\psi_2=w_{o,y}-\psi_y;\ w_b=w_o \text{ and } w_s=0 \tag{6}$$

in Eqs. (1)-(3).
The equilibrium equations of the theory consistent with the assumed displacement-field (strain-

displacement relationship) and appropriate constitutive Eqs. (3) can be written as:

$$N_{x,x}+N_{xy,y}=0 \tag{7a}$$

$$N_{xy,x}+N_{y,y}=0 \tag{7b}$$

$$\frac{4}{3h^2}[P_{x,xx}+2P_{xy,xy}+P_{y,yy}]-Q_{x,x}-Q_{y,y}+q=0 \tag{7c}$$

$$M_{x,x}+M_{xy,y}-\frac{4}{3h^2}[P_{x,x}+P_{xy,y}]-Q_x=0 \tag{7d}$$

$$M_{xy,x}+M_{y,y}-\frac{4}{3h^2}[P_{xy,x}+P_{y,y}]-Q_y=0 \tag{7e}$$

The two theories (models A & B) are basically same and one can be derived from the other, by substituting $\phi_x=\psi_x+w_{o,x}$ and $\phi_y=\psi_y+w_{o,y}$ in Eqs. (4) or (6). Further, the deletion of higher-order terms from model B results in first-order shear deformation theory only while model A can be degenerated to first-order and classical plate theories.

Model C

Third-order theory due to Lim, *et al.* (1988) can be obtained by substituting:

$$\phi_1=-w_{b,x};\ \phi_2=-w_{b,y};\ \psi_1=-w_{s,x};\ \psi_2=-w_{s,y} \tag{8}$$

in Eqs. (1)-(3).

The displacement model C can be obtained directly from model B by assuming $\psi_x+w_{o,x}=w_{s,x}$, $\psi_y+w_{o,y}=w_{s,y}$ and $w=w_b+w_s$. This is equivalent to assuming that the in-plane rotation tensor is constant through the thickness. The resulting theory will, thus involve only four dependent unknowns.

The equilibrium equations for this model can be written as:

$$N_{x,x}+N_{xy,y}=0 \tag{9a}$$

$$N_{xy,x}+N_{y,y}=0 \tag{9b}$$

$$M_{x,xx}+2M_{xy,xy}+M_{y,yy}+q=0 \tag{9c}$$

$$\frac{4}{3h^2}[P_{x,xx}+2P_{xy,xy}+P_{y,yy}]+Q_{x,x}+Q_{y,y}=0 \tag{9d}$$

In the present study, this model is extended to investigate the flexural response of laminated plates.

Model D

Third-order theory involving four dependent unknowns can also be obtained by substituting:

$$\phi_1=-w_{o,x}+w_{s,x};\ \phi_2=-w_{o,y}+w_{s,y};\ \psi_1=-w_{s,x};\ \psi_2=-w_{s,y} \tag{10}$$

in Eqs. (1)-(3).

The equilibrium equations for this model can be written as:

$$N_{x,x}+N_{xy,y}=0 \tag{11a}$$

$$N_{xy,x}+N_{y,y}=0 \tag{11b}$$

$$M_{x,xx}+2M_{xy,xy}+M_{y,yy}+q=0 \tag{11c}$$

$$M_{x,xx}+2M_{xy,xy}+M_{y,yy}-\frac{4}{3h^2}[P_{x,xx}+2P_{xy,xy}+P_{y,yy}]+Q_{x,x}+Q_{y,y}=0 \tag{11d}$$

All the models considered above allow parabolic variation of transverse shear strain through the thickness and satisfy the condition of zero transverse shear strain and stresses at the top and bottom surface of the plate. Hence, the shear correction factors required in classical shear deformation theory are not needed. Models C and D have the advantage of easily allowing the development of C^1 continuous plate bending finite element over the models A and B. This implies that no numerical gimmicks such as reduced/or selective integrations are required if finite element is based on models C and D.

3. Elasticity solutions

Three dimensional elasticity solution proposed by Pagano (1970) is employed in this section for various layer interface continuity conditions. Pagano (1970) solved elasticity equilibrium equations expressed in terms of displacements, in each orthotropic layer for a simply-supported plate. The simply-supported edge conditions assumed are such that in-plane normal displacements are allowed and tangential displacements are constrained. The boundary value problem so formed is solved for the following boundary conditions:

$$\sigma_z\left(x, y, \frac{h}{2}\right)=q_o\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}; \quad \sigma_z\left(x, y, \frac{-h}{2}\right)=0$$
$$\tau_{xz}\left(x, y, \frac{h}{2}\right)=\tau_{xz}\left(x, y, \frac{-h}{2}\right)=0; \ \tau_{yz}\left(x, y, \frac{h}{2}\right)=\tau_{yz}\left(x, y, \frac{-h}{2}\right)=0 \tag{12}$$

along with simply-supported edge conditions as:

$$\text{At } x=0,\ a:\ \sigma_x=v=w=0$$
$$\text{At } y=0,\ b:\ \sigma_y=u=w=0 \tag{13}$$

For the sake of simplicity, the notations adopted by Pagano (1970) are followed.

The flexural response of laminated plates is defined by the solution of boundary value problem satisfying elasticity equilibrium equations in each layer, boundary conditions expressed by Eqs. (12) and (13) and layer interface continuity conditions. The analytical solutions is developed for the following layer interface continuity conditions:

Model E: u, v, w, σ_z, τ_{xz} and τ_{yz} are equated at the adjacent layers at each interface.

Model F: u, v, w, σ_z, γ_{xz} and γ_{yz} are equated at the adjacent layers at each interface.

Model G: u, v, w, ε_z, γ_{xz} and γ_{yz} are equated in the adjacent layers at each interface.

For the sake of brevity, details of the steps involved in the analysis are omitted, the interested reader may refer Pagano (1970).

4. Numerical results and discussions

Among the two dimensional higher-order shear deformation theories discussed in the preceding section, model B, has been employed by several investigators and as a result, numerical results for certain configurations of laminated plates subjected to transverse loading are available. But, with regard to models A, C and D, no results have been reported for the flexural response of laminated plates. Hence, analytical solutions for simply-supported, symmetrically laminated cross-ply plates subjected to sinusoidal transverse loading, based on models A-D are developed. For this purpose, governing equations resulting from the equilibrium Eqs. (5), (7), (9) and (11) and strain-displacement relations and constitutive equations are solved to numerically evaluate the flexural response of symmetrically laminated, simply-supported cross-ply plates. In the case of symmetrically laminated cross-ply plates, one term approximations for various variables defined as:

$$(w,\ w_b,\ w_s)=(w_o,\ w_{bo},\ w_{so})\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$(\phi_x,\ \psi_x)=(\phi_{xo},\ \psi_{xo})\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$(\phi_y,\ \psi_y)=(\phi_{yo},\ \psi_{yo})\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}$$

(with a and b being the length and width of the plate)

give exact solution, when plate is subjected to sinusoidal transverse loading. During the course of investigation, it was realized that shear deformation theories involving five unknowns, that is models A and B yield identical plate centre deflections. Same is true for models C and D. Because some results for antisymmetrically laminated plates, based on model B are available in the literature, a finite element based on the displacement field of model C alone is developed herein. As the displacement field of this model allows C^1 continuous plate bending element, hence Bogner, *et al.* (1966) approach of Hermite interpolation formulae ensuring interelement compatibility is utilized. The four node rectangular finite element developed herein has fourteen degrees of freedom per node such as u_i, $u_{i,x}$, $u_{i,y}$, v_i, $v_{i,x}$, $v_{i,y}$, w_{bi}, $w_{bi,x}$, $w_{bi,y}$, $w_{bi,xy}$, w_{si}, $w_{si,x}$, $w_{si,y}$, $w_{si,xy}$ resulting in 56×56 element stiffness matrix and 56×1 load vector. Further, this element does not require any reduced/or selective integration scheme and is free from shear locking phenomenon. The convergence of the plate centre deflection, for simply-supported, antisymmetrically laminated plates ia attained by employing a 4×4 finite element mesh over the whole plate. The elasticity solutions based on models E-G, allowing different interface continuity conditions as explained in the preceding section, for symmetrically and unsymmetrically laminated cross-ply plates are also developed herein.

Following material properties are used for numerical investigation, unless otherwise stated:

$$E_L/E_T=25;\ G_{LT}=G_{LZ}=0.5E_T;\ V_{LT}=0.25;\ G_{TZ}=0.2E_T$$

The transverse plate centre deflection presented throughout this section is normalized as $\bar{w}=100\,wE_Th^3/(q_oa^4)$, where h is the total thickness of the plate.

The effect of side to thickness ratio (a/h) and modulus ratio (E_L/E_T) on the transverse plate centre deflection of orthotropic plates, subjected to sinusoidal transverse pressure, using various models described in the preceding section is presented in Table 1. It may be observed that the simplified theories due to Bhimaraddi and Stevens (1984) [model A] and Reddy (1984) [model B] involving five unknowns result in exactly the same transverse deflection. Same is true with, the theories involving four dependent unknowns that is models C and D. It may be mentioned here that models C and D slightly over estimate the flexural stiffness when compared to model E, as modular and thickness to side ratio increases, while models A and

Table 1 Normalized plate centre deflection ($\bar{w}$) of orthotropic plates subjected to sinusoidal transverse pressure. ($G_{LT}=G_{LZ}=0.5\ E_T$; $\nu_{LT}=0.25$; $G_{TZ}=0.2\ E_T$)

Source	a/h	E_L/E_T			
		40	25	15	10
Model A	5	1.0840	1.2150	1.4162	1.6323
Model B		1.0840	1.2150	1.4162	1.6323
Model C		0.9578	1.1129	1.3496	1.5962
Model D		0.9578	1.1129	1.3496	1.5962
Model E		1.0768	1.2022	1.3986	1.6106
NASTRAN		1.0939	1.2134	1.3996	1.6007
Model A	10	0.4970	0.6371	0.8576	1.0921
Model B		0.4970	0.6371	0.8576	1.0921
Model C		0.4522	0.6041	0.8366	1.0810
Model D		0.4522	0.6041	0.8366	1.0810
Model E		0.4954	0.6353	0.8543	1.0879
Model A	20	0.3375	0.4835	0.7124	0.9540
Model B		0.3375	0.4835	0.7124	0.9540
Model C		0.3261	0.4746	0.7068	0.9511
Model D		0.3261	0.4746	0.7068	0.9511
Model E		0.3371	0.4830	0.7117	0.9531
Model A	50	0.2916	0.4396	0.6713	0.9151
Model B		0.2916	0.4396	0.6713	0.9151
Model C		0.2897	0.4382	0.6704	0.9146
Model D		0.2897	0.4382	0.6704	0.9146
Model E		0.2915	0.4396	0.6712	0.9150

B are found to marginally under estimate it. Though, the models A and B yield results which are closer to the one's obtained using model E, but this contradicts the common notion because two dimensional theories should result in lower deflection compared to the three dimensional elasticity solution due to the neglect of transverse normal strain. The results presented by Librescu and Khdeir (1988) also indicate this trend. The over estimation of flexural stiffness using models C and D may be due to the neglect of transverse normal strains and inherent assumption of in-plane rotation tensor being constant through the thickness. The MSC NASTRAN results included in this paper are obtained, using a twenty node three dimensional solid element with 8×8×3 mesh, over the whole plate which is decided based on the convergence study. It is interesting to note that the three dimensional finite element solution over predicts the plate centre deflection as modulus ratio and thickness to side ratio increases compared to analytical elasticity solution model E. To verify the three dimensional finite element results obtained using MSC NASTRAN, an orthotropic plate made of Argonite crystals, having side to thickness ratio 10 and subjected to uniform pressure is solved. The normalized transverse plate centre deflection so obtained ($\bar{w}=0.137659$) compares excellently with the available elasticity solution ($\bar{w}=0.137714$) due to Srinivas and Rao (1970).

Tables 2 and 3 give, the normalized plate centre deflections of symmetric cross-ply plates, subjected to sinusoidal transverse pressure. This study attempts to focus the attention on the layer interface continuity conditions. As mentioned in the previous sectioin, models A-D are based on the transverse shear strain (γ_{xz}, γ_{yz}) continuity at the layer interface, hence, the compari-

Table 2 Effect of layer interface continuity conditions on the normalized plate centre deflection ($\bar{w}$) of square symmetric cross-ply $[0°/90°]_s$ plate subjected to sinusoidal transverse pressure

a/h	Model A or B	Model C or D	Model E	Model F	Model G	NASTRAN
5	1.4237	1.1129	1.4685	1.3360	1.3762	1.0547
10	0.7147	0.6045	0.7370	0.7753	0.8114	0.5795
20	0.5060	0.4749	0.5129	0.6232	0.6569	0.4624
50	0.4434	0.4382	0.4446	0.5792	0.6120	0.4371

Table 3 Effect of layer interface continuity conditions on the normalized plate centre deflection ($\bar{w}$) of a square symmetric cross-ply $[(0°/90°)_4]_s$ plates subjected to sinusoidal transverse pressure ($G_{LT}=G_{LZ}=0.5\ E_T$; $\nu_{LT}=0.25$; $G_{TZ}=0.2\ E_T$)

Source	a/h	E_L/E_T			
		40	25	15	10
Model A	4	1.3389	1.5038	1.7467	1.9954
Model B		1.3389	1.5038	1.7467	1.9954
Model C		1.3209	1.4852	1.7283	1.9780
Model D		1.3209	1.4852	1.7283	1.9780
Model E		1.5660	1.7229	1.9579	2.1993
Model F		1.4217	1.5966	1.8412	2.0693
Model G		1.4315	1.6072	1.8513	2.0774
Model A	10	0.4599	0.6088	0.8410	1.0848
Model B		0.4599	0.6088	0.8410	1.0848
Model C		0.4522	0.6041	0.8366	1.0810
Model D		0.4522	0.6041	0.8366	1.0810
Model E		0.4984	0.6647	0.8777	1.1203
Model F		0.5492	0.7242	0.9732	1.2073
Model G		0.5553	0.7315	0.9807	1.2137
Model A	100	0.2846	0.4330	0.6652	0.9094
Model B		0.2846	0.4330	0.6652	0.9094
Model C		0.2845	0.4330	0.6652	0.9094
Model D		0.2845	0.4330	0.6652	0.9094
Model E		0.2850	0.4337	0.6658	0.9101
Model F		0.3777	0.5540	0.8049	1.0409
Model G		0.3829	0.5605	0.8119	1.0469

son with analytical elasticity solution (model E) which is based on the layer interface continuity of transverse stresses (σ_z, τ_{xz}, τ_{xz}) is not really appropriate. Models F and G ensuring the layer interface continuity of transverse normal stress and transverse shear strains/and transverse normal and shear strains respectively are found to yield erroneous results especially in the thin plate region. Based on the results presented in Tables 2 and 3, it can be concluded that the interface continuity conditions considered in models F and G are not realistic. In that case how the two dimensional theories discussed herein give results comparable to three dimensional elasticity solution model E? Are these comparisons just a coincidence? This question remains unanswered. The three dimensional finite element results (MSC NASTRAN) for which no layer interface continuity conditions on transverse stresses/or strains are imposed, but equilibrium in the overall sense is satisfied are also not comparable with those of model E (see Table 2). It may be observed that the deflections obtained using models C and D compare well with those of three dimensional finite element results. Again the layer interface continuity conditions are different. On comparing plate centre deflections given in Tables 1-3 following may be noticed:

- Models A, B and E reveal that symmetric cross-ply square plates are more flexible compared to orthotropic square plates.
- Models C and D reveal that the transverse deflection of symmetric cross-ply and orthotropic, square plates is identically same.
- Three dimensional finite element model shows that cross-ply square plates are stiffer than the square orthotropic plates.

This shows that there is considerable ambiguity regarding the correctness of flexural response predictions obtained using above mentioned approaches.

The effect of number of layers on transverse plate centre deflection of symmetrically laminated square and rectangular ($b/a=3$), cross-ply plates is presented in Tables 4 and 5. Following observations are made.

- The transverse plate centre deflection obtained using models C and D for square plates is invariant with number of layers and for rectangular plates it decreases with increase in number of layers.
- The transverse plate centre deflections obtained using models A-D are comparable with each other when the number of layer is large.
- The transverse plate centre deflections for a four layered (0°/90°) plate obtained using models A and B compare well with elasticity solution (model E). As the number of layers increase, especially in thick plates ($a/h=4$), the comparison is no good.

The effect of side to thickness ratio (a/h) and number of layers on the transverse plate centre deflections of antisymmetrically laminated cross-ply plates, subjected to sinusoidal transverse

Table 4 Effect of number of layers on the normalized plate centre deflection ($\bar{w}$) of square symmetric cross-ply plates subjected to sinusoidal transverse pressure ($G_{LT}=G_{LZ}=0.5\ E_T$; $\nu_{LT}=0.25$; $G_{TZ}=0.2\ E_T$)

Source	a/h	$[0°/90°]_s$	$[(0°/90°)_2]_s$*	$[(0°/90°)_4]_s$*	$[(0°/90°)_5]_s$*
Model A		1.8937	1.5643	1.5038	1.4970
Model B		1.8937	1.5643	1.5038	1.4970
Model C	4	1.4865	1.4865	1.4865	1.4865
Model D		1.4865	1.4865	1.4865	1.4865
Model E		1.9367	1.7839	1.7229	1.7147
Model A		0.7147	0.6245	0.6088	0.6071
Model B		0.7147	0.6245	0.6088	0.6071
Model C	10	0.6045	0.6045	0.6045	0.6045
Model D		0.6045	0.6045	0.6045	0.6045
Model E		0.7370	0.6647	0.6467	0.6445
Model A		0.4343	0.4332	0.4330	0.4330
Model B		0.4343	0.4332	0.4330	0.4330
Model C	100	0.4332	0.4332	0.4332	0.4332
Model D		0.4332	0.4332	0.4332	0.4332
Model E		0.4346	0.4337	0.4335	0.4335

$[(0°/90°)_{1st}/(0°/90°)_{2nd}/\cdots/(0°/90°)_{nth}]_s$

Table 5 Effect of number of layers on the normalized plate centre deflection ($\bar{w}$) of rectangular ($b/a=3$) symmetric cross-ply plates subjected to sinusoidal transverse pressure ($G_{LT}=G_{LZ}=0.5\ E_T$; $\nu_{LT}=0.25$; $G_{TZ}=0.2\ E_T$)

Source	a/h	$[0°/90°]_s$	$[(0°/90°)_2]_s$	$[(0°/90°)_4]_s$	$[(0°/90°)_5]_s$
Model A		3.1089	2.7689	2.7879	2.8046
Model B		3.1089	2.7689	2.7879	2.8046
Model C	4	2.7970	2.6096	2.6682	2.6916
Model D		2.7970	2.6096	2.6682	2.6916
Model E		3.2337	3.1867	3.2149	3.2295
Model A		0.9796	1.0361	1.1196	1.1411
Model B		0.9796	1.0361	1.1196	1.1411
Model C	10	0.9209	1.0043	1.0950	1.1177
Model D		0.9209	1.0043	1.0950	1.1177
Model E		1.0183	1.1091	1.1926	1.2137
Model A		0.5547	0.6907	0.7881	0.8110
Model B		0.5547	0.6907	0.7881	0.8110
Model C	100	0.5544	0.6907	0.7882	0.8112
Model D		0.5544	0.6907	0.7882	0.8112
Model E		0.5552	0.6916	0.7890	0.8119

Table 6 Effect of number of layers on the normalized plate centre deflection ($\bar{w}$) of square antisymmetric cross-ply plates subjected to sinusoidal transverse pressure

Source	a/h	$[0°/90°]$	$[0°/90°]_2$	$[0°/90°]_4$	$[0°/90°]_5$
Model B*		1.9563	–	–	–
Model C⁺	4	1.9987	1.6107	1.5181	1.4944
Model E		2.0680	1.9581	1.7903	1.7245
Model B*		1.2128	–	–	–
Model C⁺	10	1.2155	0.6870	0.6233	0.6091
Model E		1.2275	0.7624	0.6698	0.6479
Model B*		1.0656	–	–	–
Model C⁺	100	1.0644	0.5086	0.4499	0.4373
Model E		1.0657	0.5093	0.4502	0.4375

*Results taken from Kant and Pandya (1988).
⁺Results obtained from the finite element developed herein.

pressure loading is presented in Table 6. Elasticity solution (model E), and finite element results based on models B and C are included. It is interesting to note that for two layered thin/thick plates ($a/h=4$ to 100) the two dimensional higher-order theories based on the layer interface continuity of transverse shear strains yield deflections comparable with elasticity solution. However, the results for thick plates ($a/h=4$) differ significantly as the number of layers increase. Though, the results based on model B for large number of layers (4, 8, 16) are not available in the literature, it is expected that these values will be close to the one's obtained using models C. This argument is based on the study presented in Table 4. It is because, for regular antisymmetric cross-ply plate and symmetric cross-ply plates with large number of layers, the flexural stiffness in both the directions is same ($D_{11} \approx D_{22}$).

Table 7 Normalized plate centre deflection ($\bar{w}$) of eight layer antisymmetric angle-ply $[45°/-45°]_4$ plates subjected to sinusoidal transverse pressure

a/h	Source		
	Model B*	Model C**	Model B$^+$
4	1.28	1.2835	1.2792
10	0.42	0.4167	0.4193
50	0.26	0.2500	0.2522
100	—	0.2447	0.2469

*Results read from graph (Putcha and Reddy 1986)
$^+$Results taken from Kant and Pandya (1988)
**Results obtained from the finite element developed herein.

Tables 7 gives transverse plate centre deflections of a square antisymmetrically laminated, eight layered angle-ply plates subjected to sinusoidal transverse pressure loading. Elasticity solution for this type of lay-up are not available. Hence, the finite element results based on models B and C alone are presented herein. It may be observed that the deflection obtained from both the models are in good agreement. It may be due to the fact that the flexural stiffness in both the directions is same ($D_{11}=D_{22}$). Therefore, for antisymmetrically laminated angle-ply plates with fibre angle other than -45, models B and C may result in different deflections.

5. Conclusions

Various two dimensional, simple third-order shear deformation theories involving four and five dependent unknowns in the displacement field and three dimensional elasticity approaches are examined for the flexural response of laminated plates. Following conclusions can be drawn based on present investigations:

- The plate centre deflections of orthotropic plates obtained using theories involving only four dependent variables are slightly lower for large modulus ratio and small side to thickness ratio. Whereas, theories involving five dependent variables consistently yield results which are comparable with analytical elasticity solution. However, for small modulus ratio ($E_L/E_T<$ 15) and/or large side to thickness ratio ($a/h>10$), all the two dimensional theories discussed herein yield nearly same deflections.
- The strain-displacement relationship of models C and D have the advantage of allowing the development of a C^1 continuous plate bending element. Therefore, these models might turn out to be computationally more economic and will not have the problem of shear locking in thin plates.
- Three dimensional analytical solution shows that orthotropic plates are stiffer compared to symmetric cross-ply plates, while three dimensional finite element solution show the reverse trend.
- Analytical elasticity solutions based on the continuity of transverse normal stress and transverse shear strains/and transverse normal and shear strains do not yield even the classical lamination theory results. Hence, these layer interface continuity conditions may not be realistic.
- Transverse deflections of symmetric cross-ply plates, obtained using third-order theories

involving five dependent unknowns are comparable with analytical elasticity solutions, while those obtained using the theories involving four unknowns are comparable with the three dimensional finite element results.

– The performance of all these simple higher-order theories deteriorate compared to analytical elasticity solution, as the number of layers increases. For plates made of large number of layers, all these two dimensional theories yield nearly same deflection, which of course is far away especially for thick plates (a/h<4.0) from the analytical elasticity solution.

In brief the authors feel that the two dimensional shear deformation theories discussed herein need to be refined further for better flexural response predictions, especially when number of layers, modulus ratio and thickness to side ratios are large. The effects of layer interface continuity on the flexural response predictions need to be studied more deeply. The applicability of three dimensional finite element idealization for layered plates needs to be examined.

References

Bhimaraddi, A. and Stevens, L. K. (1984), "A higher order theory for vibration of orthotropic homogeneous and laminated rectangular plates", *J. Appl. Mech.,* **51**, pp. 107-113.

Bogner, F. K., Fox, R. L. and Schmit, L. A. (1966), "The generation of interelement compatible stiffness and mass matrices by the use of interpolation formulas", *Proc. Conf. Matrix Methods in Structural Mech. AFFDL-TR-66-80. Wright-Patterson A.F.B.,* Ohio (October, 1966).

Kant, T. and Pandya, B. N. (1988), "A simple finite element formulation of a higher order theory for unsymmetrically laminated plates", *Composite Structures,* **9**, pp. 215-246.

Librescu, L. and Khdeir, A. A. (1988), "Analysis of symmetric cross-ply laminated elastic plates using a higher order theory-Part I", 7, pp. 189-213.

Lim, S. P., Lee, K. H., Chow, S. T. and Santhilnathan, N. R. (1988), "Linear and nonlinear bending of shear deformable plates", *Computers and Structures,* **30**, pp. 945-952.

Lo, K. H., Chirstensen, R. M. and Wu, E. M. (1977a), "A higher order theory of plate deformation, Part I: Homogeneous plates", *J. Appl. Mech.,* **44**, pp. 663-668.

Lo, K. H., Chirstensen, R. M. and Wu, E. M. (1977b), "A higher order theory of plate deformation, Part II: Laminated plates", *J. Appl. Mech.,* **44**, pp. 669-676.

Mindlin, R. D. (1951), "Influence of rotatory inertia and shear on the flexural motion of isotropic, elastic plates", *J. Appl. Mech.,* **18** (TRANS ASME 73), A31.

Noor, A. K. and Burton, W. S. (1989), "Assessment of shear deformation theories for multilayered composite plates", *Appl. Mech. Rev.,* **42**(1), pp. 1-12.

Pagano, N. J. (1970), "Exact solutions for rectangular bidirectional composites and sandwich plates", *J. Composite Materials,* **4**, pp. 20-34.

Pagano, N. J. and Hatifield, S. J. (1972), "Elastic behavior of multilayered bidirectional composites", *AIAA Journal,* **10**(7), pp. 931-933.

Putcha, N. S. and Reddy, J. N. (1986), "A refined mixed shear flexible finite element for the nonlinear analysis of laminated plates", *Computers and Structures,* **22**, pp. 529-538.

Reddy, J. N. (1984), "Refined higher-order theory for laminated composite plates", *J. Appl. Mech.,* **51**, pp. 745-752.

Reddy, J. N. (1990), "A general non-linear third order theory of plates with moderate thickness", *Int. J. Nonlinear Mech.,* **25**, pp. 677-686.

Reddy, J. N. and Phan, N. D. (1985), "Stability and Vibration of isotropic, orthotropic and laminated plates according to a higher-order shear deformation theory", *J. Sound and Vibration,* **98**, pp. 157-170.

Reissner, E. (1985), "Reflection on the theory of elastic plates", *Appl. Mech. Rev.,* **38**(4), pp. 1453-1464.

Srinivas, S. and Rao, A. K. (1970), "Bending vibration and buckling of simply-supported thick orthotropic rectangular plates", *Int. J. Solids and Structures,* **6**, pp. 1463-1481.