Structural Engineering and Mechanics, Vol. 3, No. 1 (1995) 49-60
DOI: http://dx.doi.org/10.12989/sem.1995.3.1.049

Vibration and stability analyses of thick anisotropic composite plates by finite strip method

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Abstract. In the present study, a finite strip method for the vibration and stability analyses of anisotropic laminated composite plates is developed according to the higher-order shear deformation theory. This theory accounts for the parabolic distribution of the transverse shear strains through the thickness of the plate and for zero transverse shear stresses on the plate surfaces. In comparison with the finite strip method based on the first-order shear deformation theory, the present method gives improved results for very thick plates while using approximately the same number of degrees of freedom. It also eliminates the need for shear correction factors in calculating the transverse shear stiffness.

A number of numerical examples are presented to show the effect of aspect ratio, length-to-thickness ratio, number of plies, fibre orientation and stacking sequence on the natural frequencies and critical buckling loads of simply supported rectangular cross-ply and arbitrary angle-ply composite laminates.

Key words: composite plates; vibration; stability; higher order shear deformation; finite strip method.

1. Introduction

Laminated composite plates and beams are increasingly used in aerospace, automobile and other engineering constructions. Improving the accuracy and efficiency in vibration and stability analyses of such structures is attracting great attention from many engineers and researchers.

For the analysis of rectangular or sectorial plate structures, the finite strip method is a very efficient numerical method (Cheung 1976). This method uses a series of beam eigenfunctions to express the displacement variations in the longitudinal direction. Thus, the two dimensional analysis is transformed into one dimensional. The cost of analysis is reduced significantly not only due to the reduction in the required computer time and storage space, but also due to the substantial simplification of input data preparation. Furthermore, the accuracy of analysis is also improved, particularly for highly anisotropic laminates, because the numerical error attributed to material anisotropy is also cut down by the considerable decrease in the number of degrees of freedom involved in each analysis.

The finite strip method has been successfully employed for the vibration and stability analyses

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of composite laminates since 1976 (Hinton 1976, Craig and Dawe 1986, 1987, Dawe and Craig 1986, Akhras, *et al.* 1993, Cheung, *et al.* 1993). However, in all the existing finite strip analyses, the transverse shear deformation was taken into account according to the first order shear deformation theory, where any straight line originally normal to the plate middle surface is assumed to remain straight but not generally normal to the middle surface after deformation. This theory accounts for transverse shear deformation, but produces a constant distribution of transverse shear strain throughout the thickness. Although the existing finite strip method can be utilized to predict the natural frequencies and critical buckling loads of moderately thick composite plates, it fails to produce accurate results for very thick plates with length-to-thickness ratio lower than 10. Moreover, even for a moderately thick plate, the wave-length of a higher order mode shape could be only a fraction of the plate side length, and the plate may dynamically behave like a very thick plate under the higher order vibration. In addition, in all the existing analyses, a so-called shear correction factor must be introduced, and the evaluation of this factor is often tedious and inaccurate.

In order to overcome above problems, a higher-order shear deformation theory developed by J. N. Reddy (1984, 1985) is used in the present approach. This theory accounts for: (1) the transverse shear deformation; (2) a parabolic variation of the transverse shear strains throughout the thickness; and (3) the zero transverse shear stresses on the surfaces of the plates. Consequently, there is no need to use shear correction factors in computing the shear strain energy. Based on this theory, analytical solutions have been obtained for simply supported cross-ply laminates and anti-symmetric angle-ply laminates. This approach employs the same number of degrees of freedom as those of the first order shear deformation theory but gives more accurate results than the latter theory. However, if a displacement finite element model based on the higher-order shear deformation theory is to be developed, the interpolation function must guarantee interelement continuity for the deflection w and for its first derivatives $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$. Construction of such an element requires many degrees of freedom at each node and, therefore, much computer processing time. In an attempt to efficiently solve this problem, Putcha and Reddy (1986) developed a mixed formulation using this theory. The resulting finite element model consists of eleven degrees of freedom per node, namely three displacements, two rotations and six resultant moments. While this approach yields results in close agreement with the higher-order theory, it remains computationally intensive. The above difficulty can be readily solved by the finite strip simulation in which Hermitian cubic polynomials are used as the interpolation function of w in the x direction, and the beam eigenfunctions are utilized in the y direction so that the slopes $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are continuous across the nodal lines between the finite strips. As a result, the finite strip method could be extended successfully to the analysis of very thick plates and for higher order modes.

A number of numerical examples will be presented to validate the present method and to investigate the effect of structural parameters, such as aspect ratio, length-to-thickness ratio, number of plies, fibre orientation and stacking sequence, on the natural frequencies and critical buckling loads of simply supported cross-ply and arbitrary angle-ply laminated plates.

2. Finite strip simulation

In the present analysis, the rectangular composite laminate is modeled by a number of finite

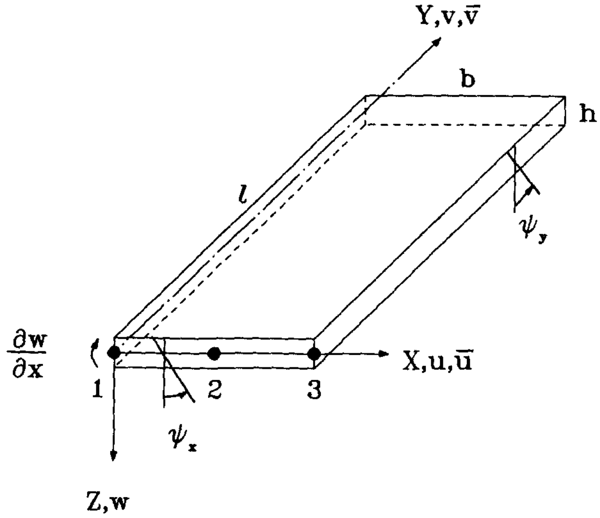

Fig. 1 A finite strip.

strips, each of which has 3 equally spaced nodal lines (Fig. 1). For the m-th harmonic, the displacement parameters of nodal line i are

$$\{\delta\}_{im}=\begin{cases}\left[u_{im},\ \bar{u}_{im},\ v_{im},\ \bar{v}_{im}\ w_{im},\ \left(\dfrac{\partial w}{\partial x}\right)_{im},\ \psi_{xim},\ \psi_{yim}\right]^T & \text{for } i=1,\ 3\\ \left[u_{im},\ \bar{u}_{im},\ v_{im},\ \bar{v}_{im},\ \psi_{xim},\ \psi_{yim}\right]^T & \text{for } i=2\end{cases} \tag{1}$$

For the laminates with two simply supported opposite sides, the boundary conditions at both ends of the strip are

$$w=\psi_x=0,\ M_y=0,\ N_y=N_{xy}=0 \ \text{ at } y=0 \text{ and } y=l \tag{2}$$

In this case, the midplane displacements and the normal rotations can be expressed in terms of the above displacement parameters as

$$\begin{aligned}
u_0&=\sum_{m=1}^{r}\sum_{i=1}^{3}\left[N_i(x)u_{im}\sin\frac{m\pi y}{l}+N_i(x)\bar{u}_{im}\cos\frac{m\pi y}{l}\right]\\
v_0&=\sum_{m=1}^{r}\sum_{i=1}^{3}\left[N_i(x)v_{im}\cos\frac{m\pi y}{l}+N_i(x)\bar{v}_{im}\sin\frac{m\pi y}{l}\right]\\
w&=\sum_{m=1}^{r}\sum_{i=1,3}\left[F_i(x)w_{im}+H_i(x)\left(\frac{\partial w}{\partial x}\right)_{im}\right]\sin\frac{m\pi y}{l}\\
\psi_x&=\sum_{m=1}^{r}\sum_{i=1}^{3}N_i(x)\,\psi_{xim}\sin\frac{m\pi y}{l}\\
\psi_y&=\sum_{m=1}^{r}\sum_{i=1}^{3}N_i(x)\,\psi_{yim}\cos\frac{m\pi y}{l}
\end{aligned} \tag{3}$$

where u_0 and v_0 are the displacements of the point $(x,\ y,\ 0)$ on the midplane, w is the deflection which is assumed to be constant in the thickness direction z, ψ_x and ψ_y are the rotations of the normal to the midplane about the y and x axes, respectively; r is the number of harmonics employed in the analysis; l is the length of the strip; $N_i(x)$ is the quadratic interpolation function for nodal line i ($i=1$ to 3) and has the following form:

$$N_i(x)=\prod_{\substack{j=1\\ j\neq i}}^{3}\frac{x-x_i}{x_i-x_j} \tag{4}$$

$F_i(x)$ and $H_i(x)$ ($i=1,3$) are the following Hermitian cubic polynomials:

$$\begin{aligned} F_1(x)&=1-3\left(\frac{x}{b}\right)^2+2\left(\frac{x}{b}\right)^3\\ F_3(x)&=3\left(\frac{x}{b}\right)^2-2\left(\frac{x}{b}\right)^3\\ H_1(x)&=x\left[1-2\frac{x}{b}+\left(\frac{x}{b}\right)^2\right]\\ H_3(x)&=x\left[\left(\frac{x}{b}\right)^2-\frac{x}{b}\right] \end{aligned} \tag{5}$$

in which b is the width of the strip.

According to the higher-order shear deformation theory, the displacements at any point (x, y, z) of a laminate are evaluated as

$$\begin{aligned} u&=u_0+z\psi_x-\frac{4}{3h^2}z^3\left(\psi_x+\frac{\partial w}{\partial x}\right)\\ v&=v_0+z\psi_y-\frac{4}{3h^2}z^3\left(\psi_y+\frac{\partial w}{\partial y}\right)\\ w&=w(x,\ y) \end{aligned} \tag{6}$$

Substituting Eq. (3) into Eq. (6), the following interpolation is obtained:

$$\{f\}=[u,\ v,\ w]^T=\sum_{m=1}^{r}\sum_{i=1}^{3}[N]_{im}\{\delta\}_{im} \tag{7}$$

where $[N]_{im}$ is the displacement matrix. For $i=1$ or 3, it is written as

$$[N]_{im}=\begin{bmatrix} N_iS_m & N_iC_m & 0 & 0 & -az^3F'_iS_m & -az^3H'_iS_m & (z-az^3)N_iS_m & 0\\ 0 & 0 & N_iC_m & N_iS_m & -az^3k_mF_iC_m & -az^3k_mH_iC_m & 0 & (z-az^3)N_iC_m\\ 0 & 0 & 0 & 0 & F_iS_m & H_iS_m & 0 & 0 \end{bmatrix} \tag{8}$$

where $S_m=\sin k_my$, $C_m=\cos k_my$, and $k_m=m\pi/l$; $(\ )'=\dfrac{d(\ )}{dx}$ and $(\ )''=\dfrac{d^2(\ )}{dx^2}$; $a=4/3h^2$, and h is the thickness of the plate.

For $i=2$, the above expression is applicable without the fifth and the sixth columns.

The following strain-displacement relationships are used in the analysis:

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{9}$$

By substituting Eq. (7) into Eq. (9), the strain vector can be expressed in terms of the displacement parameters as

$$\{\varepsilon\} = [\varepsilon_x,\ \varepsilon_y,\ \gamma_{xy},\ \gamma_{yz},\ \gamma_{zx}]^T = \sum_{m=1}^{r} \sum_{i=1}^{3} [B]_{im} \{\delta\}_{im} \tag{10}$$

where $[B]_{im}$ is the strain matrix. For $i=1$ or 3, $[B]_{im}$ has the following expression:

$$[B]_{im} = \left[\begin{matrix} N_i' S_m & N_i' C_m & 0 & 0 & -az^3 F_i'' S_m \\ 0 & 0 & -k_m N_i S_m & k_m N_i C_m & k_m^2 az^3 F_i S_m \\ k_m N_i C_m & -k_m N_i S_m & N_i' C_m & N_i' S_m & -2k_m az^3 F_i' C_m \\ 0 & 0 & 0 & 0 & k_m(1-3az^2) F_i C_m \\ 0 & 0 & 0 & 0 & (1-3az^2) F_i' S_m \end{matrix}\right.$$

$$\left.\begin{matrix} -az^3 H_i'' S_m & (z-az^3) N_i' S_m & 0 \\ k_m^2 az^3 H_i S_m & 0 & -k_m(z-az^3) \mathrm{N}_i S_m \\ -2k_m az^3 H_i' C_m & k_m(z-az^3) N_i C_m & (z-az^3) N_i' C_m \\ k_m(1-3az^2) H_i C_m & 0 & (1-3az^2) N_i C_m \\ (1-3az^2) H_i' S_m & (1-3az^2) N_i S_m & 0 \end{matrix}\right] \tag{11}$$

For $i=2$, the above expression is valid without the fifth and the sixth columns.

From Eq. (11) it can be observed that γ_{yz} and γ_{zx} vary proportionally to $(1-3az^2)$ throughout the thickness of the plate, and $\gamma_{yz} = \gamma_{zx} = 0$ at $z = \pm h/2$ which leads to $\tau_{yz} = \tau_{zx} = 0$ at the top and the bottom surfaces of the plate.

It is assumed that the laminate is manufactured from orthotropic layers (or plies) of preimpregnated unidirectional fibrous composite materials. Neglecting σ_z, for each layer, the stress-strain relations in the x-y-z coordinate system can be stated as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{26} & 0 & 0 \\ Q_{16} & Q_{26} & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{12}$$

or in a concise form:

$$\{\sigma\} = [Q]\ \{\varepsilon\} \tag{13}$$

Following the procedure commonly used in the finite strip vibration and stability analyses (Cheung 1976, Dawe and Roufaeil 1982) yields the stiffness matrix, the mass matrix and the geometrical stiffness matrix of the strip. The submatrices corresponding to nodal lines i and j can be evaluated as follows

$$[K]_{ijmn}=\int_h\int_l\int_b [B]_{im}^T[Q][B]_{jn}\,dx\,dy\,dz \tag{14}$$

$$[M]_{ijmn}=\int_h\int_l\int_b \rho[N]_{im}^T[N]_{jn}\,dx\,dy\,dz \tag{15}$$

$$[L]_{ijmn}=\int_h\int_l\int_b ([G_u]_{im}^T[\sigma^o][G_u]_{jn}+[G_v]_{im}^T[\sigma^o][G_v]_{jn}+[G_w]_{im}^T[\sigma^o][G_w]_{jn})\,dx\,dy\,dz \tag{16}$$

where h is the plate thickness, m and n denote the related series terms, and ρ is the mass density of the laminate.

$[G_u]_{im}$, $[G_v]_{im}$ and $[G_w]_{im}$ are defined as follows:

$$\begin{aligned}
\left[\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y}\right]^T &= \sum_{m=1}^{r}\sum_{i=1}^{3}[G_u]_{im}\{\delta\}_{im}\\
\left[\frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y}\right]^T &= \sum_{m=1}^{r}\sum_{i=1}^{3}[G_v]_{im}\{\delta\}_{im}\\
\left[\frac{\partial w}{\partial x},\ \frac{\partial w}{\partial y}\right]^T &= \sum_{m=1}^{r}\sum_{i=1}^{3}[G_w]_{im}\{\delta\}_{im}
\end{aligned} \tag{17}$$

$[\sigma^o]$ is the initial inplane stress matrix expressed as

$$[\sigma^o]=\begin{bmatrix}\sigma_x^o & \tau_{xy}^o\\ \tau_{xy}^o & \sigma_y^o\end{bmatrix} \tag{18}$$

The positive directions of the stresses are shown in Fig. 2.

The integrations in the above equations can be carried out analytically in the y and z directions, and the following expressions can be employed

$$\begin{aligned}
I_1&=\int_l \sin k_m y\ \sin k_n y\,dy=\begin{cases}\frac{1}{2} & \text{for } m=n\\ 0 & \text{for } m\neq n\end{cases}\\
I_2&=\int_l \cos k_m y\ \cos k_n y\,dy=\begin{cases}\frac{1}{2} & \text{for } m=n\\ 0 & \text{for } m\neq n\end{cases}\\
I_3&=\int_l \sin k_m y\ \cos k_n y\,dy=\begin{cases}\frac{2ml}{\pi(m^2-n^2)} & \text{for } m-n=2k+1\\ 0 & \text{for } m-n=2k\ \ (k=0,\ 1,\ 2,\ \cdots)\end{cases}
\end{aligned} \tag{19}$$

In order to eliminate the 'shear locking' of the thin plates, in calculating the transverse shear stiffness, the reduced integration technique (Cook, *et al.* 1989), i.e. the two point Gaussian quadrature, is used in the x direction.

Because the integral I_3 does not always vanish for $m\neq n$, different series terms are coupled in the analysis. However, for some laminates, e.g. cross-ply or antisymmetrical angle-ply laminates,

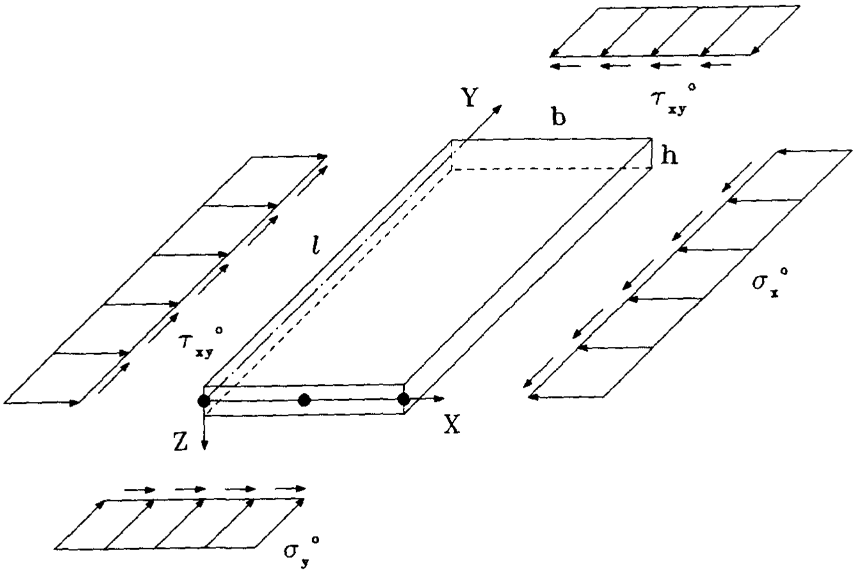

Fig. 2 Finite strip under inplane stresses.

all the stiffness coefficients associated with I_3 disappear. Consequently, the different series terms become uncoupled and the efficiency of the analysis is enhanced significantly.

After assembling the above strip matrices over the entire structure, the natural frequencies and critical load factors of the laminate can be obtained by solving the following matrix equations using standard computer subroutines:

$$[K]\{\delta\}=\omega^2[M]\{\delta\} \tag{20}$$

$$[K]\{\delta\}=\lambda[L]\{\delta\} \tag{21}$$

3. Numerical examples

3.1. Free vibration of (0°/90°) square laminate

A square laminates of side length a and thickness h is composed of equal thickness layers oriented at (0°/90°), and simply supported on all the edges. The lamina properties are assumed to be

$$E_1=40.0\ E_2,\ G_{12}=G_{13}=0.6\ E_2,\ G_{23}=0.5\ E_2 \text{ and } \nu_{12}=0.25,$$

Where 1 and 2 refer to the fiber and transverse to fiber inplane directions, respectively; and 3 denotes the direction normal to the plate midplane.

The fundamental free vibration of the plate is analyzed using the present method. By virtue of the symmetry of mode shape, half of the laminate is modeled by two proposed strips. And only one harmonic is required.

The resulting dimensionless fundamental frequency of the laminate is given in Table 1 as the function of the length-to-thickness ratio a/h. The analytical solutions based on the higher

Table 1 Fundamental frequencies of (0°/90°) square laminates

a/h	$\bar{\omega}=\omega a^2(\rho/E_2h^2)^{0.5}$			
	Present	HSDT	FSDT	CPT
2	5.717	5.699	5.191	8.499
4	8.355	8.294	7.975	10.292
10	10.570	10.449	10.355	11.011
20	11.108	10.968	10.941	11.125
100	11.303	11.156	11.155	11.163

Table 2 Fundamental frequencies of (45°/−45°) square laminates

a/h	$\bar{\omega}=\omega a^2(\rho/E_2h^2)^{0.5}$			
	Present	HSDT	FSDT	CPT
2	6.337	6.283	5.520	–
4	9.760	9.759	9.168	12.566
10	13.264	13.263	13.044	14.439
20	14.247	14.246	14.179	14.587
100	14.628	14.621	14.618	14.636

order shear deformation theory (HSDT) (Reddy and Phan 1985), the first order shear deformation theory (FSDT) and the classical plate theory with the rotary inertia included (CPT) are also listed in this table for comparison.

It can be seen that the present results are in good agreement with HSDT analytical solution while the CPT overestimates the frequency and the FSDT underestimates the frequency.

3.2. Free vibration of (45°/−45°) square laminate

A square laminate of side length a and thickness h consists of two equal thickness layers oriented at (45°/−45°). Its four edges are hinged and free in the tangential direction but immovable in the normal direction. The lamina properties are the same as for the previous example.

In the analysis, the entire plate is divided into four strips, and only one longitudinal harmonic is taken.

The resulting dimensionless fundamental frequency is shown in Table 2. The results are in an excellent agreement with the analytical solutions of the higher-order shear deformation theory (HSDT) (Reddy and Phan 1985).

3.3. Free vibration of symmetrical angle-ply square laminated plate

A square laminated plate is constructed from a number of equal thickness plies, which are oriented alternatively at θ and $-\theta$ and stacked symmetrically about the midplane of the plate. This means that for the 2-ply, 4-ply and 8-ply laminates the fiber orientations are (θ/θ), $(\theta/-\theta/-\theta/\theta)$ and $(\theta/-\theta/\theta/-\theta)_s$, respectively. It is well known that there is bending-twisting coupling but no coupling between bending and inplane deformation for such laminates.

The plate is simply supported on its four edges, and the lamina properties are identical to previous examples.

Table 3 Fundamental frequencies of square symmetrical angle-ply laminates $(\theta/-\theta/\cdots)_s$

a/h	No. of plies	$\bar{\omega}=\omega a^2(\rho/E_2h^2)^{0.5}$ $\theta=0°$	$\theta=15°$	$\theta=30°$	$\theta=45°$
2	2	5.263	5.412	5.754	5.968
	4	5.263	5.513	5.855	6.055
	8	5.263	5.684	6.038	6.198
	16	5.263	5.795	6.192	6.345
4	2	8.741	8.962	9.510	9.832
	4	8.741	9.284	10.120	10.475
	8	8.741	9.604	10.553	10.887
	16	8.741	9.729	10.726	11.055
10	2	14.720	14.757	14.983	15.340
	4	14.720	15.503	17.069	17.800
	8	14.720	16.118	18.276	19.133
	16	14.720	16.286	18.582	19.462
20	2	17.504	17.326	17.239	17.654
	4	17.504	18.312	20.182	21.170
	8	17.504	19.095	21.886	23.119
	16	17.504	19.287	22.274	23.553
100	2	18.829	18.528	18.343	18.954
	4	18.829	19.644	21.693	22.865
	8	18.829	20.511	23.642	25.100
	16	18.829	20.709	24.056	25.572

The free vibration of the laminate is analyzed by the present method. In each analysis, the plate is simulated by 8 strips with 8 longitudinal harmonics. Little improvement can be obtained if more strips or harmonics are employed.

The results are listed in Table 3, which shows the effect of fiber orientation θ, the number of plies and the length-to-thickness ratio a/h on the fundamental frequency ω of the plate.

3.4. Free vibration of (0°/45°/−45°/90°) laminate

A rectangular laminate of side lengths a and b is made up of four equal thickness layers stacked at (0°/45°/−45°/90°). Its four edges are hinged and free to move in both tangential and normal inplane directions but fixed at all the corners. The material properties are identical to example 1.

The free vibration of the plate is analyzed using four strips (parallel to the longer sides) with eight series terms. Little difference can be detected if more strips and terms are used. The resulting fundamental frequency is given in Table 4 as the function of the length-to-thickness ratio a/h and the aspect ratio b/a.

3.5. Stability of square cross-ply laminate under uniform compression

A square laminate of side length a and thickness h is composed of equal thickness layers oriented alternatively at 0° and 90°, and simply supported on all the edges. The lamina properties

Table 4 Fundamental frequencies of (0°/45°/−45°/90°) rectangular laminates

a/h	$\bar{\omega}=\omega a^2(\rho/E_2h^2)^{0.5}$ $b/a=1.0$	$b/a=1.5$	$b/a=2.0$
2	5.10	4.24	4.03
4	7.47	6.13	5.87
10	9.64	7.72	7.42
20	10.29	8.13	7.80
100	10.58	8.30	7.94

Table 5 $(\sigma_y)_{cr}$ of square cross-ply laminates

h	Method	$(\sigma_y)_{cr}a^2/E_2h^2$ (0°/90°)	(0°/90°/0°)
5	Present	8.368	10.674
	HSDT	8.628	11.008
	FSDT	8.142	10.525
10	Present	11.320	21.902
	HSDT	11.305	22.160
	FSDT	11.099	21.643
20	Present	12.502	30.978
	HSDT	12.268	30.922
	FSDT	12.208	30.664
100	Present	12.945	35.935
	HSDT	12.614	35.602
	FSDT	12.611	35.589
	CPT	12.628	35.831

are the same as example 1. The stability of the plate under uniform compressive stress σ_y is analyzed using the present method. By virtue of symmetry of the buckling mode shape, only half of the laminate is modeled by two proposed strips. And only one harmonic is required.

The resulting dimensionless critical stress is exhibited in Table 5 as the function of the length-to-thickness ratio and layering. The analytical solutions based on the higher order shear deformation theory (HSDT) (Reddy and Phan 1985), the first order shear deformation theory (FSDT) and the classical plate theory (CPT) are also given in this table for comparison. It can be seen that the present method yields an acceptable accuracy as compared to the HSDT solutions.

3.6. Stability of square angle-ply laminated plates under uniaxial compression

The square laminated plates of side length a and thickness h are stacked from equal thickness layers, which are oriented at (45°/−45°), (45°/−45°/45°/−45°) and (45°/−45°/−45°/45°) respectively. The material properties of each layer are as follows

$$E_1=40.0\ E_2,\ G_{12}=G_{23}=G_{31}=0.5\ E_2,\ \nu_{12}=0.25.$$

Table 6 $(\sigma_y)_{cr}$ of square angle-ply laminates

a/h	Method	$(\sigma_y)_{cr}a^2/E_2h^2$		
		(45°/−45°)	(45°/−45°/45°/−45°)	(45°/−45°/−45°/45°)
5	Present	11.455	14.895	9.404
	FS-FSDT	10.365	15.117	9.715
10	Present	17.554	32.770	24.863
	FS-FSDT	16.942	33.321	26.125
20	Present	20.477	47.545	41.732
	FS-FSDT	20.278	47.886	42.402
100	Present	21.670	55.719	49.775
	FS-FSDT	21.669	55.759	49.729
	CPT	21.709	56.088	−

For the first two laminates, all the edges are hinged and free to move in the tangential direction but immovable in the normal inplane direction, and the analysis is carried out using four strips with one series term. For the third laminate, the four sides are simply supported, and the plate is modeled by eight strips with eight series terms. Further increasing strips and series terms produces little improvement.

The resulting critical stresses $(\sigma_y)_{cr}$ are given in Table 6 in comparison with the CPT solutions (Jones, *et al.* 1973) and the finite strip solutions based on the first order shear deformation theory (FS-FSDT). The results reveal that the first order solution underestimates the critical stress for the (45°/−45°) laminate and overestimates the critical stress for others in comparison with higher order solutions.

4. Conclusions

In the present study, a finite strip method for vibration and stability analysis of anisotropic laminated composite plates is developed according to the higher-order shear deformation theory. This theory accounts for the parabolic distribution of the transverse shear strains through the thickness of the plate and for zero transverse shear stresses on the plate surfaces. Therefore, the present method yielded more accurate results than the finite strip method based on the first-order shear deformation theory and eliminates the need to use shear correction factors in calculating the transverse shear stiffness.

Acknowledgements

The financial supports from the Department of National Defence and the Natural Sciences and Engineering Research Council of Canada are gratefully acknowledged.

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