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Vibration frequencies for elliptical and semi-elliptical Mindlin plates

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Abstract. This paper presents new frequency results for elliptical and semi-elliptical Mindlin plates of various aspect ratios, thicknesses and boundary conditions. The results were obtained using the recently developed computerized Rayleigh-Ritz method for thick plate analysis. For simply supported elliptical plates, it is proposed that the penalty function method be used to enforce the condition of zero rotation of the midplane normal in the tangent plane to the plate boundary.

Key words: elliptical plate; semi-elliptical plate; Mindlin theory; Rayleigh-Ritz method; vibration.

1. Introduction

The vibration of elliptical thin (Kirchhoff) plates has been extensively investigated, and Pavlik (1937) was probably the first researcher to analyse the vibration problem. Using the Rayleigh-Ritz method, he computed the natural frequencies of the first seven normal vibrations for a free-elliptical plate of aspect ratio 1.29 and compared the results with experimental observations. Further experiments conducted by Waller (1950) yielded many useful frequency results. Later studies on vibrating elliptical plates with completely free edges were made by Beres (1974) and Narita (1985) using the Rayleigh-Ritz method. Sato (1973) derived exact solutions in terms of Mathieu functions for such unconstrained vibrating elliptical plates. Frequency values for the first five modes were presented and they were in good agreement with those obtained experimentally. Sato pointed out that Pavlik's frequency results are higher than his by about 4% and 10% for the first and second modes of vibration. Sato (1976) also extended the study to elastically constrained elliptical plates. Vibration of unsupported elliptical plates of lenticular section, whose middle surfaces may be flat or uniformly curved, was thoroughly studied by Harris and Mansfield (1967).

Shibaoka (1956) examined the vibration of clamped elliptical plates and presented the fundamental frequencies for different aspect ratios. DeCapua and Sun (1972) used the Rayleigh-Ritz method to solve the same problem and presented vibration results for the first six modes. McNitt (1962) employed Galerkin's method to obtain approximate results for the first two vibration

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frequencies of a damped elliptical clamped plate. Nayfeh, et al. (1976) proposed a perturbation procedure which was used in conjunction with a transfer of boundary conditions to solve the vibration of nearly annular and circular plates. They illustrated the method by considering two clamped elliptical plates of aspect ratios $b/a=0.9$ and $b/a=0.8$ and obtained the natural frequencies and mode shapes. The method, however, may not furnish accurate results when the elliptical plates are longish.

Vibration of simply supported elliptical plates was treated by Leissa (1967). He determined the fundamental frequencies for various values of Poisson ratios and aspect ratios. Sato (1971) gave exact solutions for simply supported plates in terms of Mathieu functions. Numerical results were given for the first five natural frequencies for a complete range of eccentricities.

More recently, Wang, et al. (1994) used the Rayleigh-Ritz method to analyse the free flexural vibration problem of super-elliptical plates.

In the case of thick elliptical plates (which requires the consideration of transverse shear deformation and rotary inertia), it is rather surprising that so little work has been done. An exceptional paper is that of Callahan (1964) who used the Mindlin theory to derive the governing frequency equations for eight boundary conditions. The equation for each boundary case is an infinite determinant and each element in it is an infinite series of Mathieu functions containing an unknown frequency. Although a method for calculating the roots (normal modes of vibration) of the infinite determinant was proposed by Callahan, no results were presented.

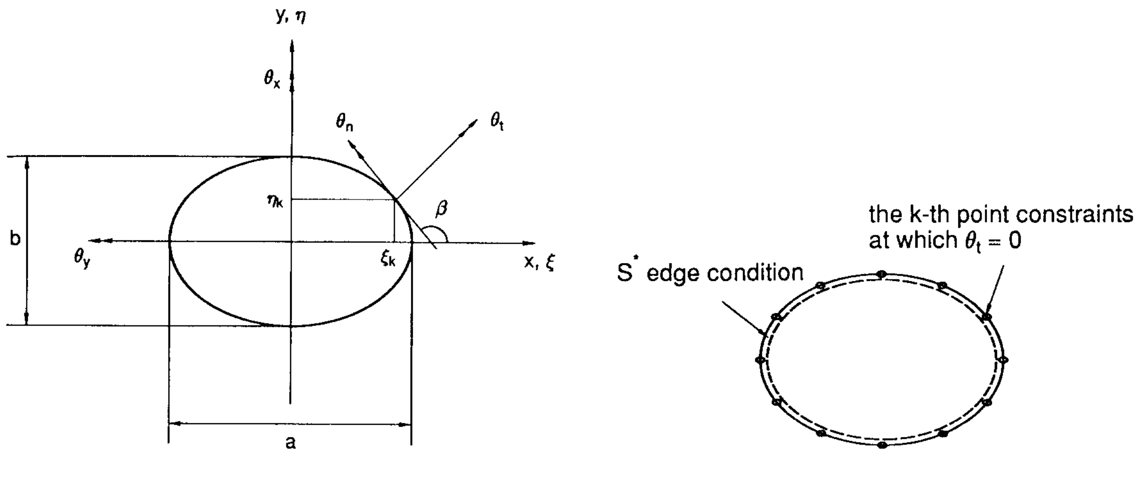
In view of the dearth of Mindlin vibration results for elliptical plates, the authors were prompted to use their recently developed computerized Rayleigh-Ritz method to analyze this vibration problem (Wang, et al. 1993, Xiang, et al. 1993, Kitipornchai, et al. 1993). The key feature of the method lies in the definition of the Ritz function which consists of the product of a mathematically complete two-dimensional polynomial function and the equations of the boundaries raised to appropriate powers. The method should yield accurate vibration solutions for elliptical plates since there is no discretization of the curved plate edges. A sufficient degree of polynomial function is necessary, however, to ensure converged results.

Using this computerized Rayleigh-Ritz method, extensive Mindlin vibration results have been computed and presented for elliptical plates and semi-elliptical plates, the latter plate shape not studied hitherto. These new vibration results should be useful to engineering designers who are dealing with elliptical plates and panels.

A particular feature of this paper is the treatment of the (S) simply supported edge condition in the Rayleigh-Ritz method. This kind of S-supporting edge requires zero transverse deflection, $w=0$ and zero rotation normal to the edge, $\theta_t=0$. For Mindlin plates whose S-edges are either parallel to the Cartesian axes or where boundaries can be expressed simply in polar coordinates (such as the cases of circular and annular sectorial plates), the Ritz functions involving the product of the boundary equations raised to appropriate powers can automatically satisfy the geometric boundary conditions of this kind of S support. The same Ritz function cannot be used, however, when the S-edge is inclined (for instance in polygonal plates) or curved (as in elliptical plates). Note that for skew plates, one can still use the Ritz function if skew coordinates are adopted. To handle the aforementioned S-supporting edge, a penalty function method is proposed and demonstrated on elliptical and semi-elliptical plates with S-edges.

2. Energy functional, boundary conditions and Ritz functions for Mindlin plates

Consider a flat, isotropic elliptical plate of uniform thickness h, major axis length a, minor axis length b, Young's modulus E, shear modulus $G=E/[2(1+\nu)]$ and Poisson's ratio ν (see



(a) Geometry and coordinate systems

(b) Using point constraints to simulate S boundary condition

Fig. 1 Elliptical Mindlin plate: (a) geometry and coordinate systems (b) simulating S boundary conditions by point constraints.

Fig. 1(a)).

By adopting the following nondimensionalization,

$$\xi=\frac{x}{a};\ \eta=\frac{y}{b};\ \chi=\frac{a}{b};\ w=\frac{\bar{w}}{b};\ \lambda^2=\omega^2\left(\frac{b}{2}\right)^4\frac{\rho h}{D};\ \alpha=\frac{h}{b};\ \phi=\frac{6\kappa(1-\nu)}{\alpha^2} \tag{1}$$

the total energy functional F of the vibrating Mindlin plate may be written as

$$F=\int_A\left\{\left(\frac{1}{\chi}\frac{\partial\theta_x}{\partial\xi}+\frac{\partial\theta_y}{\partial\eta}\right)^2-2(1-\nu)\left[\frac{1}{\chi}\frac{\partial\theta_x}{\partial\xi}\frac{\partial\theta_y}{\partial\eta}-\frac{1}{4}\left(\frac{\partial\theta_x}{\partial\eta}+\frac{1}{\chi}\frac{\partial\theta_y}{\partial\xi}\right)^2\right]\right.$$
$$\left.+\phi\left[\left(\theta_x+\frac{1}{\chi}\frac{\partial w}{\partial\xi}\right)^2+\left(\theta_y+\frac{\partial w}{\partial\eta}\right)^2\right]-16\lambda^2\left(w^2+\frac{\alpha^2}{12}(\theta_x^2+\theta_y^2)\right)\right\}dA \tag{2}$$

in which D=flexural rigidity of the plate=$Eh^3/[12(1-\nu^2)]$, a and b are the plate maximum dimensions in the x- and y-directions, respectively, κ=shear correction factor=5/6 (Reissner 1945), $\bar{w}(\xi,\eta)$=transverse displacement; $\theta_x(\xi,\eta)$ and $\theta_y(\xi,\eta)$ the normal rotations in the xz- and yz--planes, respectively; $w(\xi,\eta)$=nondimensionalized transverse displacement; A=nondimensionalized area of the plate and $dA=d\xi d\eta$.

The boundary conditions for Mindlin plates are (Huang 1989):

–Free edge (F): $Q_n=0,\ M_n=0,\ M_{nt}=0$ (3a)

in which Q_n=the shearing force, M_n=bending moment and M_{nt}=twisting moment.

–Simply supported edge (S): $w=0,\ M_n=0$ and $\theta_t=0$ (3b)

in which θ_t is the rotation of the midplane normal in the tangent plane tz to the plate boundary (see Fig. 1(a)).

–Simply supported edge (S*): $w=0,\ M_n=0,\ M_{nt}=0$ (3c)

–Clamped edge (C): $w=0,\ \theta_n=0,\ \theta_t=0$ (3d) in which θ_n is the rotation of the midplane normal to the clamped edge (see Fig. 1(a)).

The Ritz functions for approximating the transverse displacement and the rotations in the x- and y-directions are taken as

$$\left(w,\ \theta_x,\ \theta_y\right)=\sum_{q=0}^{p}\sum_{i=1}^{q}\left(c_r\Phi_r,\ d_r\Psi_{xr},\ e_r\Psi_{yr}\right) \tag{4}$$

in which p is the degree of the mathematically complete two dimensional polynomial function, c_r, d_r and e_r the unknown coefficients to be varied with the subscript r given by

$$r=\frac{(q+1)(q+2)}{2}-i \tag{5}$$

and

$$(\Phi_r,\ \Psi_{xr},\ \Psi_{yr})=(\xi^i\ \eta^{q-i})(\Phi_1,\ \Psi_{x1},\ \Psi_{y1}) \tag{6}$$

The basic functions, Φ_1, Ψ_{x1}, Ψ_{y1} must satisfy the geometric boundary conditions given in Eq. (3). For the plate shapes considered and the adopted Cartesian coordinate system shown in Fig. 1, they take on:

$$\Phi_1,\ \Psi_{x1},\ \Psi_{y1}=\begin{cases}\left(\xi^2+\eta^2-\dfrac{1}{4}\right)^{\Omega_1} & \text{for elliptical plates}\\[2ex] (\eta)^{\Omega_2}\left(\xi^2+\eta^2-\dfrac{1}{4}\right)^{\Omega_1} & \text{for semi-elliptical plates}\end{cases} \tag{7}$$

For Φ_1, the power Ω assumes values of

$\Omega=0$ if the edge is free (F) (8a)

$\Omega=1$ if the edge is simply supported (S and S^*) or clamped (C) (8b)

For Ψ_{x1}, the power Ω assumes values of

$\Omega=0$ if the edge is free (F) or simply supported (S^*) or simply supported (S) parallel to the y-direction (8c)

$\Omega=1$ if the edge is simply supported (S) parallel to the x-direction or clamped (C) (8d)

For Ψ_{y1}, the power Ω assumes values of

$\Omega=0$ if the edge is free (F) or simply supported (S^*) or simply supported (S) parallel to the x-direction (8e)

$\Omega=1$ if the edge is simply supported (S) parallel to the y-direction or clamped (C) (8f)

3. Treatment of simply supported edge (S)

The Ritz functions, Ψ_{x1} and Ψ_{y1} cannot be used to satisfy the S type support condition for the elliptical plate and the curved edge of the semi-elliptical plate since the condition requires $\theta_t=0$. In order to handle this kind of support condition, it is proposed that in addition to the S^* edge condition a series of n point constraints, as shown in Fig. 1(b), be introduced to ensure that at these points

$$\theta_{tk}=\theta_{xk}\cos(\pi-\beta_k)-\theta_{yk}\sin(\pi-\beta_k)=0,\ k=1,\ 2,\ \cdots,\ n \tag{9}$$

where β_k is the angle the tangent line (to the k-th point on the plate curved edge) makes with the x-axis. For simplicity, the positions of these points at the elliptical edge are generated using equi-spaced angles.

Using the penalty function method, the energy functional in Eq. (2) may be augmented to

$$\begin{aligned} F^* &= F+\mu\sum_{k=1}^{n}\theta_{tk}^2 \\ &= F+\mu\sum_{k=1}^{n}(\theta_{xk}\cos\beta_k+\theta_{yk}\sin\beta_k)^2 \end{aligned} \tag{10}$$

where μ is the penalty multiplier.

4. Eigenvalue equation for natural frequencies

On the basis of the foregoing energy functional and Ritz functions, the application of the Rayleigh-Ritz method furnishes the following eigenvalue equation:

$$([K]-\lambda^2[M])\begin{Bmatrix}\{c\}\\ \{d\}\\ \{e\}\end{Bmatrix}=\{0\} \tag{11}$$

in which the Ritz coefficients

$$\{c\}=\begin{Bmatrix}c_1\\ c_2\\ \vdots\\ c_m\end{Bmatrix};\ \{d\}=\begin{Bmatrix}d_1\\ d_2\\ \vdots\\ d_m\end{Bmatrix};\ \{e\}=\begin{Bmatrix}e_1\\ e_2\\ \vdots\\ e_m\end{Bmatrix} \tag{12}$$

the stiffness matrix

$$[K]=\begin{bmatrix}[K_{cc}] & [K_{cd}] & [K_{ce}]\\ & [K_{dd}] & [K_{de}]\\ \text{symmetric} & & [K_{ee}]\end{bmatrix} \tag{13}$$

and the mass matrix

$$[M]=\begin{bmatrix}[M_{cc}] & [M_{cd}] & [M_{ce}]\\ & [M_{dd}] & [M_{de}]\\ \text{symmetric} & & [M_{ee}]\end{bmatrix} \tag{14}$$

The elements of $[K]$ and $[M]$ are given by

$$K_{ccij}=\frac{\Phi}{\chi}\int_A\frac{\partial\Phi_i}{\partial\xi}\frac{\partial\Phi_j}{\partial\xi}dA+\chi\phi\int_A\frac{\partial\Phi_i}{\partial\eta}\frac{\partial\Phi_j}{\partial\eta}dA; \tag{15a}$$

$$K_{cdij}=\phi\int_A\frac{\partial\Phi_i}{\partial\xi}\psi_{xj}dA; \tag{15b}$$

$$K_{ceij}=\chi\phi\int_A \frac{\partial\Phi_i}{\partial\eta}\psi_{yj}dA; \tag{15c}$$

$$K_{ddij}=\frac{1}{\chi}\int_A \frac{\partial\psi_{xi}}{\partial\xi}\frac{\partial\psi_{xj}}{\partial\xi}dA+\chi\frac{(1-\nu)}{2}\int_A \frac{\partial\psi_{xi}}{\partial\eta}\frac{\partial\psi_{xj}}{\partial\eta}dA+\chi\phi\int_A \psi_{xi}\psi_{xj}dA$$
$$+\sum_{k=1}^{n}\mu\cos^2\beta_k\ \psi_{xi}(\xi_k,\eta_k)\ \psi_{xj}(\xi_k,\eta_k) \tag{15d}$$

$$K_{deij}=\nu\int_A \frac{\partial\psi_{xi}}{\partial\xi}\frac{\partial\psi_{yj}}{\partial\eta}dA+\frac{(1-\nu)}{2}\int_A \frac{\partial\psi_{xi}}{\partial\eta}\frac{\partial\psi_{yj}}{\partial\xi}dA$$
$$+\sum_{k=1}^{n}\mu\cos\beta_k\ \sin\beta_k\,\psi_{xi}(\xi_k,\eta_k)\ \psi_{yj}(\xi_k,\eta_k) \tag{15e}$$

$$K_{eeij}=\chi\int_A \frac{\partial\psi_{yi}}{\partial\eta}\frac{\partial\psi_{yj}}{\partial\eta}dA+\frac{(1-\nu)}{2x}\int_A \frac{\partial\psi_{yi}}{\partial\xi}\frac{\partial\psi_{yj}}{\partial\xi}dA+\chi\phi\int_A \psi_{yi}\psi_{yj}dA$$
$$+\sum_{k=1}^{n}\mu\sin^2\beta_k\ \psi_{yi}(\xi_k,\eta_k)\ \psi_{yj}(\xi_k,\eta_k) \tag{15f}$$

$$M_{ccij}=\frac{16}{\chi}\int_A \Phi_i\Phi_j dA; \tag{16a}$$

$$M_{cdij}=M_{ceij}=M_{deij}=0; \tag{16b}$$

$$M_{ddij}=\frac{4\chi\alpha^2}{3}\int_A \psi_{xi}\psi_{xj}dA; \tag{16c}$$

$$M_{eeij}=\frac{4\chi\alpha^2}{3}\int_A \psi_{yi}\psi_{yj}dA; \tag{16d}$$

where $i, j=1, 2, \cdots, m$; $m=(p+1)(p+2)/2$. Note that the terms involving the penalty multiplier μ are only taken into consideration when the edge is S-supported.

The natural frequency parameter λ may be determined by solving the eigenvalue problem given by Eqs. (11)-(16) using the EISPACK subroutines. Integrations have been performed exactly using the software MATHEMATICA (Wolfram 1991).

5. Numerical results

5.1. Convergence and comparison studies

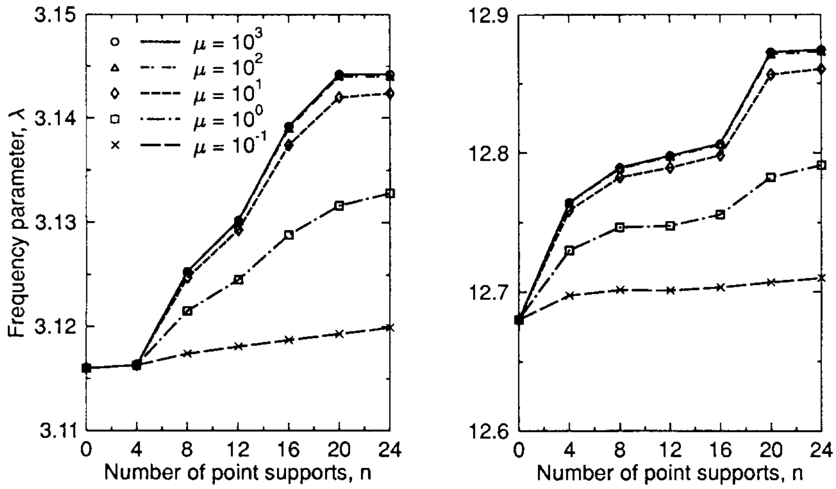
Convergence studies have been conducted to establish the degree of polynomial required for accurate solutions. Table 1 presents a typical case study on elliptical plates with aspect ratios, $\chi=2, 3$ and thickness-minor axis length ratio, $h/b=0.001, 0.15$. It can be observed from the tabulated results that $p=12$ is sufficient for converged results, and this degree has been used to generate all further results for elliptical plates reported herein. Note that $p=10$ has been used to calculate the results for semi-elliptical plates since it is sufficient to generate converged solutions.

Checks were also made using thin plate solutions available in the literature. It can be seen from Table 1 that the results corresponding to $h/b=0.001$ are in very close agreement with those obtained by previous researchers such as Shibaoka (1956), Sato (1971, 1973), Narita (1985), Leissa

Table 1 Convergence and comparison studies of frequency parameters, $(\omega b^2/4)\sqrt{\rho h/D}$, for elliptical plates with different boundary conditions

χ	Mode	p	Free		Simply supported S^*		Clamped	
			h/b		h/b		h/b	
			0.001	0.15	0.001	0.15	0.001	0.15
2	1	6	1.668	1.618	3.304	3.119	6.845	5.833
		9	1.667	1.617	3.303	3.116	6.844	5.833
		12	1.667	1.617	3.303	3.116	6.844	5.832
		13	1.667	1.617	3.300	3.116	6.844	5.832
			1.667[a]		3.306[a]		6.869[b]	
			1.667[b]		3.304[b]		6.875[e]	
			1.667[c]		3.303[c]		6.845[f]	
					3.293[d]			
	3	6	4.303	3.977	9.841	8.466	14.08	11.01
		9	4.230	3.941	9.581	8.267	14.00	10.97
		12	4.230	3.940	9.581	8.266	13.99	10.97
		13	4.230	3.940	9.580	8.266	13.99	10.97
			4.231[b]		9.582[b]		13.99[b]	
			4.230[c]				13.99[f]	
	6	6	8.174	7.237	16.16	13.00	22.17	15.67
		9	7.885	6.988	15.69	12.69	22.02	15.62
		12	7.878	6.986	15.69	12.68	22.01	15.62
		13	7.878	6.986	15.69	12.68	22.01	15.62
			7.881[b]		15.69[b]		22.01[b]	
			7.878[c]				22.01[f]	
3	1	6	0.751	0.740	3.010	2.849	6.312	5.400
		9	0.751	0.739	3.009	2.846	6.311	5.399
		12	0.751	0.739	3.009	2.846	6.311	5.399
		13	0.750	0.739	3.009	2.846	6.310	5.399
			0.750[a]		3.009[b]		6.322[b]	
			0.750[b]				6.322[e]	
			0.750[c]					
	3	6	1.958	1.881	6.638	5.904	10.23	8.269
		9	1.923	1.857	6.323	5.636	10.04	8.167
		12	1.923	1.856	6.322	5.634	10.02	8.163
		13	1.923	1.856	6.321	5.634	10.02	8.163
			1.916[b]		6.323[b]		10.03[b]	
			1.923[c]					
	6	6	5.694	5.117	14.20	11.67	19.78	14.20
		9	5.452	4.887	11.59	9.722	16.68	12.40
		12	5.441	4.882	11.44	9.619	16.67	12.40
		13	5.441	4.882	11.44	9.619	16.67	12.40
			5.466[b]		11.62[b]		16.68[b]	
			5.441[c]					

a-Sato (1973); *b*-Liew (1990); *c*-Narita (1958); *d*-Leissa (1967); *e*-Shibaoka (1956) and *f*-DeCapua and Sun (1972).



(a) First mode (b) Sixth mode

Fig. 2 Variation of frequency parameter λ versus the number of point constraints n for an elliptical plate having $\chi=2$ and $h/b=0.15$ with different values of penalty multiplier μ.

(1967) and Liew (1990). The good agreement of results lends support to the correctness of the numerical algorithm.

For the case of the S-supported elliptical plate, it is necessary also to determine the number of point constraints n and the value of the penalty multiplier μ for converged results. Taking an elliptical plate of aspect ratio $\chi=2$ and $h/b=0.15$, Fig. 2 shows the variation of the frequencies of the first and sixth mode with respect to n and μ. It can be observed that $n=24$ points and $\mu=1000$ are sufficient to simulate the S-support condition. Note that when $n=0$, the solutions correspond to those belonging to elliptical plates with S^* support.

5.2. Elliptical plates

Four different edge conditions F, S^*, S and C have been considered for the elliptical Mindlin plate. The first six natural frequencies have been computed and are presented in Tables 2-5 for $\chi=1.0$, 1.5, 2.0, 2.5, 3.0 and $h/b=0.001$, 0.05, 0.10, 0.15.

The following observations are made:

- The results corresponding to $\chi=a/b=1.0$ are those for circular plates and they agree with those obtained by Irie, et al. (1980).
- For elliptical plates with free edges, the first three natural frequencies are zero as they are associated with rigid body modes.
- The frequency parameters decrease with increasing aspect ratio $\chi=a/b$ particularly in the higher modes.
- The frequency parameters decrease with increasing plate thickness due to increasing effects of shear deformation and rotary inertia, especially for plates with greater restraints such as clamped plates.
- The frequencies are a few percent higher for S-type supported plates than for the corresponding S^*-type supported plates. The maximum differences occur when the plate thickness is relatively large and when considering higher vibration modes. These differences are not

Table 2 Frequency parameters $(\omega b^2/4)\sqrt{\rho h/D}$, for free ($F$) elliptical plates

a/b	h/b	Mode sequences					
		4	5	6	7	8	9
	0.001	5.358	5.358	9.003	12.43	12.43	20.47
1	0.05	5.278	5.278	8.868	12.06	12.06	19.71
	0.10	5.115	5.115	8.508	11.32	11.32	17.99
	0.15	4.895	4.895	8.013	10.41	10.41	16.00
	0.001	2.878	3.549	7.142	7.337	7.799	12.89
1.5	0.05	2.855	3.501	7.016	7.232	7.635	12.54
	0.10	2.806	3.415	6.745	6.972	7.304	11.76
	0.15	2.736	3.300	6.383	6.615	6.876	10.80
	0.001	1.667	2.636	4.230	5.503	6.941	7.878
2	0.05	1.659	2.598	4.184	5.403	6.840	7.734
	0.10	1.642	2.537	4.083	5.214	6.601	7.413
	0.15	1.617	2.459	3.940	4.967	6.272	6.986
	0.001	1.077	2.093	2.751	4.222	5.161	6.710
2.5	0.05	1.073	2.060	2.730	4.148	5.097	6.613
	0.10	1.066	2.013	2.685	4.019	4.951	6.387
	0.15	1.055	1.953	2.621	3.852	4.747	6.077
	0.001	0.750	1.734	1.922	3.419	3.618	5.441
3	0.05	0.748	1.706	1.912	3.358	3.585	5.331
	0.10	0.745	1.667	1.889	3.261	3.510	5.135
	0.15	0.739	1.618	1.856	3.136	3.403	4.882

Table 3 Frequency parameters $(\omega b^2/4)\sqrt{\rho h/D}$, for simply supported ($S^*$) elliptical plates

a/b	h/b	Mode sequences					
		1	2	3	4	5	6
	0.001	4.936	13.89	13.89	25.61	25.61	29.71
1	0.05	4.894	13.51	13.51	24.32	24.32	28.25
	0.10	4.778	12.63	12.63	21.71	21.71	25.03
	0.15	4.606	11.53	11.53	18.94	18.94	21.64
	0.001	3.678	7.926	12.19	14.12	18.39	22.28
1.5	0.05	3.652	7.779	11.90	13.70	17.69	21.32
	0.10	3.579	7.450	11.20	12.77	16.20	19.27
	0.15	3.471	7.014	10.31	11.64	14.48	17.01
	0.001	3.300	5.910	9.580	11.53	14.37	15.69
2	0.05	3.274	5.812	9.357	11.27	13.92	15.17
	0.10	3.209	5.608	8.877	10.64	12.95	14.03
	0.15	3.116	5.335	8.266	9.826	11.78	12.68
	0.001	3.122	4.983	7.479	10.66	11.17	14.29
2.5	0.05	3.094	4.905	7.324	10.38	10.93	13.86
	0.10	3.034	4.748	7.003	9.797	10.33	12.90
	0.15	2.948	4.539	6.589	9.062	9.561	11.73
	0.001	3.009	4.456	6.321	8.642	10.94	11.44
3	0.05	2.983	4.390	6.198	8.437	10.71	11.12
	0.10	2.926	4.259	5.952	8.017	10.14	10.44
	0.15	2.846	4.084	5.634	7.487	9.393	9.619

Table 4 Frequency parameters $(\omega b^2/4)\sqrt{\rho h/D}$, for simply supported (*S*) elliptical plates

a/b	*h/b*	Mode sequences					
		1	2	3	4	5	6
1	0.001	4.936	13.89	13.89	25.61	25.61	29.71
	0.05	4.894	13.57	13.57	24.51	24.51	28.25
	0.10	4.779	12.72	12.72	22.01	22.01	25.03
	0.15	4.607	11.64	11.64	19.26	19.26	21.64
1.5	0.001	3.678	7.926	12.19	14.12	18.39	22.28
	0.05	3.658	7.817	11.94	13.78	17.81	21.45
	0.10	3.591	7.518	11.27	12.91	16.40	19.47
	0.15	3.489	7.101	10.39	11.80	14.71	17.23
2	0.001	3.300	5.910	9.580	11.53	14.37	15.69
	0.05	3.284	5.849	9.420	11.30	14.01	15.26
	0.10	3.229	5.677	8.991	10.69	13.10	14.19
	0.15	3.144	5.428	8.408	9.890	11.96	12.87
2.5	0.001	3.122	4.983	7.479	10.66	11.17	14.29
	0.05	3.105	4.939	7.382	10.47	10.96	13.94
	0.10	3.055	4.815	7.111	9.941	10.38	13.03
	0.15	2.978	4.631	6.730	9.239	9.615	11.89
3	0.001	3.009	4.456	6.321	8.642	10.94	11.44
	0.05	2.993	4.421	6.251	8.505	10.73	11.23
	0.10	2.946	4.321	6.054	8.153	10.18	10.61
	0.15	2.873	4.170	5.769	7.662	9.439	9.821

Table 5 Frequency parameters $(\omega b^2/4)\sqrt{\rho h/D}$, for clamped (*C*) elliptical plates

a/b	*h/b*	Mode sequences					
		1	2	3	4	5	6
1	0.001	10.21	21.25	21.26	34.87	34.87	39.76
	0.05	9.943	20.18	20.18	32.23	32.23	36.51
	0.10	9.249	17.78	17.78	27.05	27.05	30.28
	0.15	8.371	15.23	15.23	22.29	22.29	24.72
1.5	0.001	7.613	12.65	18.43	19.72	25.34	28.81
	0.05	7.449	12.24	17.59	18.83	23.86	27.05
	0.10	7.019	11.25	15.66	16.80	20.70	23.37
	0.15	6.453	10.06	13.53	14.59	17.52	19.73
2	0.001	6.844	9.874	13.99	17.46	19.24	22.01
	0.05	6.702	9.605	13.50	16.68	18.40	20.84
	0.10	6.328	8.929	12.33	14.89	16.48	18.28
	0.15	5.832	8.087	10.97	12.91	14.37	15.62
2.5	0.001	6.503	8.637	11.42	14.89	16.97	19.08
	0.05	6.370	8.418	11.07	14.34	16.23	18.25
	0.10	6.021	7.862	10.21	13.05	14.51	16.35
	0.15	5.554	7.158	9.178	11.57	12.59	14.29
3	0.001	6.310	7.954	10.02	12.54	15.55	16.67
	0.05	6.184	7.762	9.741	12.13	14.96	15.95
	0.10	5.848	7.269	9.033	11.13	13.57	14.27
	0.15	5.399	6.638	8.163	9.957	12.00	12.40

Table 6 Frequency parameters $(\omega b^2/4)\sqrt{\rho h/D}$, for *F-C* semi-elliptical plates

a/b	h/b	Mode sequences					
		1	2	3	4	5	6
1	0.001	8.603	19.67	28.86	36.33	45.09	57.71
	0.05	8.344	18.58	26.80	33.27	40.48	50.97
	0.10	7.818	16.43	22.81	27.77	32.82	40.32
	0.15	7.171	14.17	18.96	22.81	26.41	31.97
1.5	0.001	5.710	10.86	17.83	26.49	26.71	36.16
	0.05	5.593	10.48	16.97	24.75	25.01	33.06
	0.10	5.358	9.714	15.24	21.25	21.72	27.53
	0.15	5.051	8.806	13.36	17.77	18.46	22.54
2	0.001	4.822	7.905	12.02	17.24	23.63	25.06
	0.05	4.741	7.680	11.58	16.45	22.28	23.49
	0.10	4.570	7.240	10.69	14.85	19.64	20.26
	0.15	4.342	6.697	9.654	13.10	16.89	17.04
2.5	0.001	4.440	6.581	9.351	12.81	17.04	22.98
	0.05	4.373	6.417	9.054	12.32	16.25	21.48
	0.10	4.228	6.095	8.468	11.34	14.69	18.86
	0.15	4.028	5.690	7.765	10.22	13.00	16.28
3	0.001	4.234	5.864	7.908	10.42	13.46	17.95
	0.05	4.175	5.732	7.682	10.06	12.91	16.90
	0.10	4.042	5.468	7.235	9.358	11.84	15.09
	0.15	3.858	5.131	6.691	8.532	10.64	13.26

significant in elliptical plates but are significant in, for instance, rectangular plates (Xiang, et al. 1993).

5.3. Semi-elliptical plates

Four different boundary conditions *F-C*, *C-F*, *C-C* and *C-S* have been considered for the semi-elliptical Mindlin plate. The first letter denotes the straight edge condition while the latter denotes the curved edge condition. The first six natural frequencies have been computed and are presented in Tables 6-9 for χ=1.0, 1.5, 2.0, 2.5, 3.0 and h/b=0.001, 0.05, 0.10, 0.15.

The results for the semi-elliptical plates follow the same trend as the elliptical plates. The frequencies decrease with increasing aspect ratio, plate thickness and less restrained boundary conditions.

6. Conclusions

The problem of vibration of elliptical and semi-elliptical plates can be readily solved using the computerized Rayleigh-Ritz method. Natural frequencies for these plate shapes have been presented for various aspect ratios, plate thicknesses and boundary conditions. These results should be useful to engineering designers.

The penalty function method used to impose the *S*-type constraint in the Rayleigh-Ritz method

Table 7 Frequency parameters $(\omega b^2/4)\sqrt{\rho h/D}$, for *C-F* semi-elliptical plates

a/b	h/b	Mode sequences 1	2	3	4	5	6
	0.001	4.538	9.356	17.23	27.02	27.58	40.26
1	0.05	4.475	9.066	16.41	25.22	25.73	36.71
	0.10	4.325	8.455	14.78	21.58	22.31	30.66
	0.15	4.119	7.708	13.01	18.01	18.96	25.30
	0.001	4.389	7.462	11.99	18.03	25.14	25.67
1.5	0.05	4.330	7.271	11.56	17.17	23.56	24.06
	0.10	4.187	6.866	10.68	15.47	20.28	21.08
	0.15	3.990	6.360	9.650	13.62	16.99	18.10
	0.001	4.252	6.479	9.517	13.43	18.37	24.27
2	0.05	4.195	6.328	9.223	12.91	17.41	22.84
	0.10	4.060	6.010	8.617	11.86	15.69	19.79
	0.15	3.872	5.607	7.893	10.66	13.83	16.60
	0.001	4.141	5.873	8.121	10.93	14.50	18.68
2.5	0.05	4.086	5.745	7.892	10.55	13.78	17.75
	0.10	3.957	5.477	7.423	9.795	12.95	15.92
	0.15	3.778	5.134	6.853	8.910	11.28	14.01
	0.001	4.054	5.464	7.231	9.386	12.20	15.39
3	0.05	4.001	5.352	7.040	9.091	11.59	14.70
	0.10	3.878	5.116	6.653	8.493	10.66	13.33
	0.15	3.706	4.812	6.174	7.785	9.643	11.86

Table 8 Frequency parameters $(\omega b^2/4)\sqrt{\rho h/D}$, for *C-C* semi-elliptical plates

a/b	h/b	Mode sequences 1	2	3	4	5	6
	0.001	28.11	41.72	58.68	71.91	78.43	95.05
1	0.05	25.94	37.48	51.23	61.16	66.37	78.23
	0.10	21.68	30.15	39.74	45.69	49.72	56.88
	0.15	17.78	24.13	31.12	34.91	38.22	43.05
	0.001	25.43	32.43	41.21	51.86	65.79	67.07
1.5	0.05	23.55	29.60	37.02	45.74	55.74	57.38
	0.10	19.80	24.35	29.84	36.09	42.98	43.07
	0.15	16.30	19.78	23.96	28.64	32.92	33.69
	0.001	24.47	29.13	34.81	41.60	51.57	62.16
2	0.05	22.70	26.74	31.55	37.20	43.85	52.82
	0.10	19.13	22.18	25.78	29.92	34.58	40.39
	0.15	15.76	18.10	20.87	24.02	27.50	31.62
	0.001	23.98	27.48	31.73	36.78	45.22	53.40
2.5	0.05	22.26	25.30	28.84	33.07	38.22	45.78
	0.10	18.78	21.07	23.73	26.80	30.32	35.22
	0.15	15.49	17.24	19.29	21.62	24.23	27.65
	0.001	23.67	26.49	30.01	34.16	41.97	48.87
3	0.05	22.00	24.43	27.28	30.78	35.24	42.10
	0.10	18.58	20.41	22.52	25.04	27.96	32.44
	0.15	15.32	16.72	18.34	20.22	22.36	25.44

Table 9 Frequency parameters $(\omega b^2/4)\sqrt{\rho h/D}$, for *C-S* semi-elliptical plates

a/b	h/b	Mode sequences					
		1	2	3	4	5	6
1	0.001	19.50	31.43	46.60	58.51	64.62	79.75
	0.05	18.54	29.26	42.38	51.99	57.14	68.72
	0.10	16.40	24.97	34.83	41.27	45.23	52.91
	0.15	14.11	20.93	28.41	32.87	36.00	41.45
1.5	0.001	17.86	24.29	32.37	42.37	54.40	54.77
	0.05	17.04	22.89	30.08	38.77	48.67	48.93
	0.10	15.16	19.91	25.58	32.06	38.98	39.37
	0.15	13.11	16.92	21.37	26.32	31.07	31.74
2	0.001	17.19	21.56	26.83	33.44	41.31	53.41
	0.05	16.42	20.41	25.12	30.61	37.29	46.64
	0.10	14.65	17.92	21.72	26.04	31.21	36.97
	0.15	12.69	15.33	18.34	21.70	25.59	29.74
2.5	0.001	16.81	20.13	24.02	29.13	35.02	48.54
	0.05	16.07	19.12	22.64	27.07	32.27	40.47
	0.10	14.36	16.85	19.68	22.94	27.00	31.78
	0.15	12.46	14.47	16.74	19.26	22.37	25.66
3	0.001	16.57	19.25	22.36	26.73	31.58	44.04
	0.05	15.85	18.31	21.10	24.86	29.14	37.18
	0.10	14.18	16.19	18.48	21.16	24.71	28.93
	0.15	12.31	13.94	15.76	17.81	20.50	23.34

may also be used to impose any elastically restrained edge or point support. The penalty multiplier is equivalent to the elastic spring constants used in modelling these elastic restraints.

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