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Technical Note

Natural frequencies of thin plates by means of the finite element method using a conditioned conforming quadrilateral element

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1. Introduction

Rectangular finite elements are commonly used for the analysis of the transverse behaviour of elastic flat plates only in those cases where the contour is made up by orthogonal segments. In this aspect the work by Bogner, Fox and Schmit (1966) is very well known. They developed a conforming rectangular element that is extremely efficient for the analysis of both, static and dynamic situations in the case of thin plates. These authors included the second mixed derivative of the transverse displacement as a fourth degree of freedom in each node of the rectangular element. The reason for the rather small popularity of the Bogner element is based on the fact that it is not possible to generalize its formulation to quadrangular irregular shapes due to the loss of continuity (Irons and Ahmal 1980).

Recently a *p*-version of the finite element method has been used (Sidi 2006) to determine the bending natural frequencies of a cantilever flexible plate mounted on the periphery of a rotating hub using a rectangular element. The second mixed derivative of the transversal displacement is included in the mentioned reference as the fourth nodal degree of freedom.

Another new quadrilateral four nodes sixteen degree of freedom thin plate element was presented by Huang *et al.* (2002) based of the similarity theory between plane elasticity and plate bending.

In this note the author presents a quadrangular element useful for the static and dynamic analysis of simple, non-rectangular domains such as trapezoidal and rhomboidal plates. The proposed quadrangular finite element possesses four degrees of freedom per node and it is formulated using shape functions that are similar to the ones employed by Bogner.

On the other hand the quadrilateral finite elements developed by Fraejis de Veubeke (1968) are also highly efficient but require a rather complex formulation. It is important to point out that the Veubeke's element has been included in the well-known professional software ALGOR Professional Mech (2001) for linear analysis of thin plates.

The main goal of the present work consists in the development of an algorithm of simple

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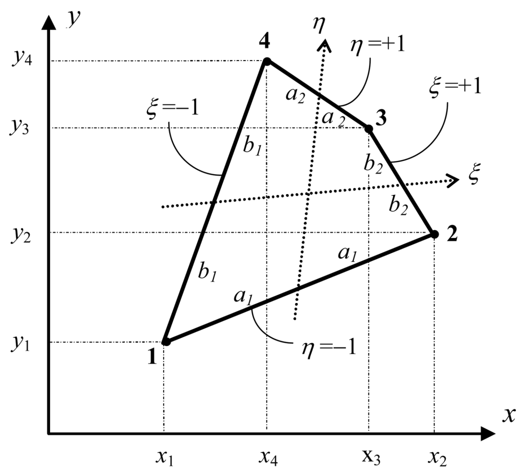

Fig. 1 Quadrangular element in the physical x-y plane

computational implementation but which yields good numerical results with a rather small mesh density of finite elements. This study presents results of natural frequencies which compare satisfactorily with professional ALGOR solutions.

2. Formulation of the algorithm

The four degrees of freedom at each node of the quadrangular element are the transverse displacement (w), its first two partial derivatives ($\partial w/\partial x$, $\partial w/\partial y$) and the second mixed derivative ($\partial^2 w/\partial x\partial y$). The sixteen degrees of freedom are grouped in the vector of the nodal displacements

$$\{\delta\}^t = [w_1 \;\; (\partial w/\partial x)_1 \;\; (\partial w/\partial y)_1 \;\; (\partial^2 w/\partial x\partial y)_1 \;\; \ldots \;\; (\partial^2 w/\partial x\partial y)_4] \tag{1}$$

Fig. 1 shows the quadrangular element in the physical plane x-y indicating the local pair of axes that coincide with the median axes in which adequate non-dimensional coordinates ξ and η have been introduced. The figure indicates also the local numbering of the nodes and the notation adopted for the sizes of the sides.

The coordinate transformation is defined by means of the following bilinear functions

$$x = \frac{1}{4}(1-\xi)(1-\eta)x_1 + \frac{1}{4}(1+\xi)(1-\eta)x_2 + \frac{1}{4}(1+\xi)(1+\eta)x_3 + \frac{1}{4}(1-\xi)(1+\eta)x_4 \tag{2a}$$

$$y = \frac{1}{4}(1-\xi)(1-\eta)y_1 + \frac{1}{4}(1+\xi)(1-\eta)y_2 + \frac{1}{4}(1+\xi)(1+\eta)y_3 + \frac{1}{4}(1-\xi)(1+\eta)y_4 \tag{2b}$$

With shape functions of the form $N(\xi, \eta)$ grouped in the row-matrix $[N]$, the transverse displacement is expressed in terms of the adimensional variables

$$w(\xi, \eta) = [N]\{\delta\} \tag{3}$$

Based upon the complete polynomial of third degree the sixteen shape functions for the Bogner rectangular element of sides $2a$ and $2b$ may be deduced (Rossi 1997). The empiric formulation proposed by the author consists in replacing a and b in the shape functions of the rectangle by lineal functions for the quadrilateral element

$$a = \frac{1}{2}(a_1 + a_2) - \frac{1}{2}(a_1 - a_2)\eta \tag{4a}$$

$$b = \frac{1}{2}(b_1 + b_2) - \frac{1}{2}(b_1 - b_2)\xi \tag{4b}$$

It is proved that the element is a conditioned conforming quadrilateral element (CCQE) because it fulfils present the conditions of continuity of the displacement and its first derivatives on adjacent sides of contiguous elements when the quadrilateral region is modeled with a mesh generated with two bundles of straight lines, with centres in the points of intersection of the opposed sides of the region. It is important to point out that a change of direction in a node shared by to two or more elements causes the loss of continuity.

Using the well-known classical dynamic stiffness finite element method (Bathe 1982) the author has used the quadrature method of Gauss-Legendre (Burden and Faires 2002) and the inverse iteration method (Chandrupatla and Belegundu 1991) when implementing the computational algorithm.

3. Numerical results

The numerical tests performed to the proposed element were carried out by means of the determination of the natural frequencies of rhomboidal and trapezoidal plates with available solutions calculated by the author with the professional Algor software using the Veubeke's element. In all cases it was taken $\upsilon = 0.3$ and values of the frequency coefficient $\Omega = \omega a^2(\rho h/D)^{1/2}$ were calculated, where a is one of the plate's size and D is the flexural rigidity of the plate.

The first example is the case of rhomboidal plates with a skew of 45º, simply supported in their contours. The regular meshes of finite elements were generated by means of parallel straights to the sides of the rhomboidal plates with n and m divisions along the sides a and b, respectively. In this study one case was solved, with relationship $b/a = 3$. Table 1 depicts the results obtained with the proposed element (CCQE) and the professional software ALGOR.

The next case is a trapezoidal isosceles plate with bases a and b, and height d. It was solved with $d/a = 0.5$ and $b/a = 0.4$. The design of the mesh consisted in dividing the bases of the trapezoid in n segments of equal size and the non parallel sides in m segments also of the same longitude. Table 2 depicts the corresponding results for simply supported boundary conditions obtained by the author with the proposed element, and with ALGOR by means of a model with similar mesh design (n = 140, m = 105) formed by 14700 elements.

Table 1 Values of Ω_1 and Ω_2 for simply supported rhomboidal plates (skew angle 45º)

b/a	Element	Mesh $n \times m$	Number of nodes	DOF	Ω_1	Ω_2
3	CCQE	12 × 36	481	1820	21.05	24.93
		16 × 48	833	3196	21.05	24.91
		20 × 60	1281	4956	21.04	24.89
	ALGOR	80 × 240	19521	57915	21.04	24.89

Table 2 Values of Ω_i (i = 1...6) for a simply supported isosceles trapezoidal plate

Mesh $n \times m$	Number of nodes	DOF	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
12 × 9	130	448	63.340	127.89	179.32	228.07	253.57	350.49
20 × 15	336	1228	63.252	127.83	179.19	227.99	253.34	350.27
28 × 21	638	2392	63.215	127.81	179.14	227.96	253.25	350.23
ALGOR 140 × 105	14946	44062	63.213	127.81	179.14	227.98	253.27	350.27

4. Conclusions

The intention of this study is to present the formulation of a conditioned conforming quadrangular finite element for the dynamic analysis of thin plates demonstrating its utility in the particular cases of quadrangular regions adequately meshed where the conditions of continuity are satisfied. In these cases the obtained numerical results are excellent with a rather small mesh density of finite elements. The author does not recommend its use with irregular meshes where the conditions of conformity are not satisfied in a complete fashion.

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