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Modal-based model reduction and vibration control for uncertain piezoelectric flexible structures

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Abstract. In piezoelectric flexible structures, the contribution of vibration modes to the dynamic response of system may change with the location of piezoelectric actuator patches, which means that the ability of actuators to control vibration modes should be taken into account in the development of modal reduction model. The spatial H_2 norm of modes, which serves as a measure of the intensity of modes to system dynamical response, is used to pick up the modes included in the reduction model. Based on the reduction model, the paper develops the state-space representation for uncertain flexible structures with piezoelectric material as non-collocated actuators/sensors in the modal space, taking into account uncertainties due to modal parameters variation and unmodeled residual modes. In order to suppress the vibration of the structure, a dynamic output feedback control law is designed by simultaneously considering the conflicting performance specifications, such as robust stability, transient response requirement, disturbance rejection, actuator saturation constraints. Based on linear matrix inequality, the vibration control design is converted into a linear convex optimization problem. The simulation results show how the influence of vibration modes on the dynamical response of structure varies with the location of piezoelectric actuators, why the uncertainties should be considered in the reductiom model to avoid exciting high-frequency modes in the non-collcated vibration control, and the possiblity that the conflicting performance specifications are dealt with simultaneously.

Keywords: spatial norm of modes; uncertain flexible structures; vibration control.

1. Introduction

The vibration control of flexible structures using piezoelectric materials as sensors and actuators has found many applications in aerospace, civil, and automotive field, such as large space structures, flexible manipulators, and tall buildings and so on, whose trend is to employ lightweight structures that are energy efficient and responsive. This makes the structure less stiff and therefore more susceptible to the vibration, which may last for a long time. The light-weight structures are typically characterized by poorly damped and clustered vibration modes with low resonant frequencies (Gawronski 1996). Such structures even become unstable since their open-loop poles are very close to the imaginary axis in the complex plane. Therefore, it is desirable to use active vibration control methods to attenuate the vibration of the structures. Recently, piezoelectric materials (Cao and Harley 1999) become very popular in the structural control problems due to their lightweight, large

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linear range and broad bandwidth. In literatures, active vibration control of structures with piezoelectric effect has been studied (Ha *et al.* 1992, Gao and Chen 2003, Narayanan and Balamurugan 2003, Kusculuoglu and Fallahi 2004, Yu and Wang 2007, Jiang *et al.* 2006, Fuller *et al*. 2002).

Flexible structures often include a large number of vibration modes. Using the model including all modes usually leads to a high-order controller, for which it is difficult to implement. There are several approaches that can be used for model reduction, such as balanced reduction and direct modes truncation and so on. The characteristics of vibration modes is not clear in the balanced-reduction model, so the direct modes truncation is the common practice to obtain a relatively low order model (Moheimani 2000). Although, generally speaking, low-frequency modes tend to contribute more significantly to structural vibration than high-frequency modes, the influence of vibration modes on the dynamical response of structure varies with the location of piezoelectric actuators, and the ability of actuators of to control vibration modes has to be taken into account in the reduction model by using modes truncation, which means that the reduction model should include the modes which significantly contribute to the dynamical response of structure, rather than only keep the first several low-frequency modes.

Model errors coming from the modes truncation also have to be considered in order to avoid control spillover and observation spillover. The interaction of two spillovers may lead to system instability, especially in the non-collocated control (Fuller *et al.* 1996, Clark *et al.* 1998). To deal with the problem, a variety of control design frameworks have been explored. Balas (1978) suggests that collocated actuators and sensors should be used, or a comb pre-filter could be built for the sensor signal. Meirovitch (1983) develops the independent modal space control where each mode is controlled independently. Besides, the variation in modal parameters (resonant frequencies and damping ratio) may degrade the performance of the controller (Bala 1995). Zhang (2001) designs a robust H_∞ control method with concerning the un-modeled dynamics. Taking into account uncertainties due to the variation in resonant frequencies and un-modeled high-frequency modes, Sridhar and Vittal (2000) achieve the disturbance rejection by using robust H_∞ control method with input limit.

Most of the time the controller for flexible structures is required to simultaneously satisfy different conflicting specifications, and therefore linear matrix inequalities (LMI) have recently emerged as a convenient and powerful tool to solve this problem and convert multi-objective control problem into convex optimization problem for which efficient algorithms and software exist (Carten *et al.* 1989, Chien *et al.* 2003). There are some applications of LMI for the vibration control of structures in the literatures such as (Sridhar and Vittal 2000) achieving the disturbance rejection by using robust H_∞ control method with input limit, (Hu and Ma 2004) designing a H_∞ output feedback control for the plate structure, (Xu and Chen 2004) giving the multi-objective vibration control for a certain beam structure, (Chen and Guo 2005) presenting the application of H_∞ method with control input constraints for active suspensions, and (Samuel and Vicente 2006), designing a H_∞ state feedback control for the plate structure.

In this paper, how to pick up the modes which significantly contribute to the dynamical response of structure in the mode truncation is investigated by using spatial H_2 norm of modes, and the ability of actuators to control vibration modes is considered in modeling a relatively low-order dynamic modal equation. Based on the dynamic modal state-space equation of the uncertain flexible structures, a dynamic output feedback controller is designed with satisfying the following: (1) Guaranteeing that the closed-loop system is internally stable and robust to model uncertainties; (2)

As for disturbance rejection, setting un upper bound on the H_∞ norm from external disturbance to the performance output; (3) In order to achieve the desired transient response performance, the damping ratio and decay rate requirement will be satisfied via regional constraints on the uncertain closed-loop pole locations; (4) To avoid actuator saturation, every control input satisfies $|V_{ai}(t)| \le V_{\max}$ (Meirovitch and Baruh 1983). Finally the validity of the modal-based vibration control is illustrated by a numerical example.

2. Model reduction using spatial H_2 norm of modes

2.1 Modelling in the modal space

Here the flexural vibration of an Euler-Bernoulli beam with K piezoelectric actuator patches is considered, neglecting the shear deformation and rotation of the beam. The beam has the dimensions of $l_\mathrm{b} \times w_\mathrm{b} \times h_\mathrm{b}$, and the jth actuating patch has the dimensions of $l_{\mathrm{a}j} \times w_{\mathrm{a}j} \times h_{\mathrm{a}j}$. Ignoring the influence of the small mass and stiffness of the piezoelectric thin patches on the structural dynamics, we have the partial differential equation for the flexural vibration of structure

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + \rho A\frac{\partial^2 y(x,t)}{\partial t^2} = \sum_{j=1}^{k}\frac{\partial^2 (M_{\mathrm{a}j}(x,t))}{\partial x^2} \tag{1}$$

Where, $y(x,t)$ is the spatially distributed transverse deflection of structure, namely the displacement response; EI is the flexural rigidity of structure; $M_{\mathrm{a}j} = K_j(H(x-x_{1j}) - H(x-x_{2j}))v_{aj}$ is the bending moment coming from the jth actuating patch; x_{2j}, x_{1j} are the location of the ends of the jth actuating patch respectively; $H(\bullet)$ Heaviside function; K_j depends on the performance and dimensions of beam and piezoelectric patches.

The boundary conditions are represented as

$$\begin{aligned} x = 0, \quad & B_1(y(x,t)) = 0 \\ x = l_\mathrm{b}, \quad & B_2(y(x,t)) = 0 \end{aligned} \tag{2}$$

Using the modal analysis and the given boundary conditions, the displacement response can be written as

$$y(x,t) = \sum_{i=1}^{\infty} \phi_i(x) q_i(t) \tag{3}$$

Where, $\phi_i(x)$, $i = 1, \ldots, \infty$ is the modal functions, and $q_i(t)$, $i = 1, \ldots, \infty$ is the modal coordinates.

Using the orthogonality of modal functions $\phi_i(t)$ and including the proportional modal damping ratio ζ_i, the Eq. (1) can be decoupled and written as

$$\ddot{q}_i(t) + 2\xi_i\omega_i\dot{q}_i(t) + \omega_i^2 q_i(t) = \sum_{j=1}^{\infty} b_{ij}v_{aj} \quad (i = 1, 2, \ldots, \infty) \tag{4}$$

Where, $q_i(t)$ is the ith modal coordinate; ξ_i, ω_i are the frequency and damping ratio of the ith mode respectively; $b_{ij} = K_j(\phi'(x_{2j}) - \phi'(x_{1j}))$, $\phi'(\bullet)$ is the first derivative with respect to x.

With the applied voltage to the actuators $v_\mathrm{a}(t) = [v_{\mathrm{a}1}(t), \ldots, v_{\mathrm{a}j}(t), \ldots, v_{\mathrm{a}k}(t)]$ as the inputs, and the displacement response $y(x,t)$ as the output, the transfer function of the structure can be written as

$$G(s,x) = \sum_{i=1}^{\infty} \frac{F_i \phi_i(x)}{s^2 + 2\xi_i \omega_i s + \omega_i^2} \tag{5}$$

Where, $F_i = [F_{i1}, F_{i2}, \ldots, F_{ij}, \ldots, F_{ik}]$, $F_{ij} = K_j(\phi_i'(x_{2j}) - \phi_i'(x_{1j}))$, $(j = 1, 2, \ldots, k;\ i = 1, 2, \ldots, \infty)$

2.2 H_2 norm of system

The H_2 norm of a system $G(s)$ can be written as

$$\|G(s)\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} tr((G(j\omega))^* G(j\omega)) d\omega \tag{6}$$

Using the state-space reprensentation of system $G(s)$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{7}$$

We have

$$\|G(s)\|_2^2 = tr(C^T C W_c) \quad \text{or} \quad \|G(s)\|_2^2 = tr(BB^T W_o) \tag{8}$$

Where, $(G(j\omega))^*$ is the conjugate transpose of the matrix $G(j\omega)$, is the trace of the matrix, W_c, W_o are the controllable and observable Gramian matrix, which can be obtained by solving the following Lyapunov equation

$$AW_C + W_C^T A + BB^T = 0 \quad \text{or} \quad AW_O + W_O^T A + C^T C = 0 \tag{9}$$

2.3 Spatial H_2 norm of modes and model reduction

The spatial H_2 norm of a system $G(s,x)$ with multi-input and spatially distributed output is defined as

$$\|G(s,x)\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_R tr((G(j\omega,x))^* G(j\omega,x)) \mathrm{d}\omega \mathrm{d}x \quad x \in \mathrm{R} \tag{10}$$

Using the orthogonality of the modal functions, that is $\int_R \rho A \phi_i(x) \phi_j(x) \mathrm{d}x = \delta_{ij}$, $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$, the spatial H_2 norm of a system $G(s,x)$ can be represented as

$$\begin{aligned} \|G(s,x)\|_2^2 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int tr((G(j\omega,x))^* G(j\omega,x)) \mathrm{d}\omega \mathrm{d}x \\ &= \sum_{i=1}^{\infty} \|G(s,x)\|_2^2 \end{aligned} \tag{11}$$

Where, $G_i(s,x) = \dfrac{[F_{i1}, \ldots, F_{ij}, \ldots, F_{ik}] \phi_i(x)}{s^2 + 2\xi_i \omega_i s + \omega_i^2}$ is the transfer function component corresponding to the ith mode of system.

It is obvious, from Eq. (11), that the spatial H_2 norm of a system can be represented as the superposition of spatial H_2 norm components $\|G(s,x)\|_2^2$, $i = 1,\ldots,\infty$ coming from independent modes of the system. The spatial H_2 norm component coming from an independent mode indicates the contribution of the mode to the dynamic response of structure. Comparing the spatial H_2 norms of independent modes, we include the modes which significantly contribute to the dynamical response of structure in the reduction model, rather than only the first several low-frequency modes.

Using $\int_R \phi_i(x)\rho A\phi_i(x)\mathrm{d}x = 1$, we can have the spatial H_2 norm component corresponding to the ith mode

$$\begin{aligned}\|G_i(s,x)\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int tr((G_i(j\omega,x))^*G(j\omega,x))\mathrm{d}\omega\mathrm{d}x \\ &= \sum_{j=1}^{k}\|G_{ij}(s,x)\|_2^2 \\ &= \rho A\sum_{j=1}^{k}\|g_{ij}(s)\|_2^2\end{aligned} \tag{12}$$

Where, $G_{ij}(s,x) = \dfrac{F_{ij}\phi_i(x)}{s^2+2\xi_i\omega_i s+\omega_i^2}$ is the component of $G_i(s,x)$ system corresponding to the jth actuator. The spatial H_2 norm corresponding to the ith mode can be represented as the superposition of k components related to k actuators. $\|g_{ij}(s)\|_2^2$ can be regarded as the H_2 norm of a SISO (single input and single output) system $g_{ij}(s) = \dfrac{F_{ij}}{s^2+2\xi_i\omega_i s+\omega_i^2}$, which can be obtain by using Eqs. (6)-(9). Comparing the spatial H_2 norm of modes $\|G_i(s,x)\|_2^2$, $i = 1,\ldots,\infty$, we can decide which modes will be included in the reduction model.

3. Uncertain modal state space representation

3.1 Nominal modal state space representation

Including external disturbances $w_j(t)$, $j = 1,\ldots,I$, the dynamic equation of structure with m modes an be written as

$$\begin{aligned}&\ddot{q}_i(t)+2\xi_i\omega_i\dot{q}_i(t)+\omega_i^2 q_i(t) = f_{\mathrm{c}i}(t)+f_{\mathrm{d}i}(t) \qquad i = 1,2,\ldots,m \\ &f_{\mathrm{c}i}(t) = \sum_{j=1}^{K} b_{ij}V_{\mathrm{a}j}(t) \\ &f_{\mathrm{d}i}(t) = \sum_{j=1}^{I}\tilde{b}_{ij}w_j(t)\end{aligned} \tag{13}$$

Where, $f_{\mathrm{c}i}(t), f_{\mathrm{d}i}(t)$ are the ith generalized modal control force and the ith generalized modal disturbance force; Similarly, $V_{\mathrm{a}j}(t)$ is the applied voltage to the jth actuating patch; b_{ij} is the influence coefficient of the jth actuating patch on the ith mode; $w_j(t)$ is the jth external disturbance; $\tilde{b}_{ij}$ is the influence coefficient of the jth disturbance on the ith mode.

In terms of the structural modes, the performance displacement outputs and the measured displacement outputs can be defined as

$$
\begin{aligned}
y_i(t) &= \sum_{j=1}^{m} c_{ij} q_j(t) \qquad i = 1,2,\ldots,p \\
z_i(t) &= \sum \tilde{c}_{ij} q_j(t) \qquad i = 1,2,\ldots,L
\end{aligned}
\tag{14}
$$

where $y_i(t)$ is the ith performance output; c_{ij} is the influence coefficient of the jth mode on the ith performance output; $z_i(t)$ is the ith measured output; $\tilde{c}_{ij}$ is the influence coefficient of the jth mode on the ith measured output.

Defining $x(t) = [q_1(t), \dot{q}_1(t), \ldots, q_i(t), \dot{q}_i(t), \ldots, q_m(t), \dot{q}_m(t)]^T$ as state variables, the state space model of the piezoelectric flexible structure can be written as

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + BV_a(t) + \tilde{B}w(t) \\
y(t) &= Cx(t) \\
z(t) &= \tilde{C}x(t)
\end{aligned}
\tag{15}
$$

where $x(t) \in R^{2m \times 1}$ is state vector; $V_a(t) = [V_{a1}(t), \ldots, V_{aK}(t)]^T \in R^{K \times 1}$ is the control input vector; $w(t) = [w_1(t), \ldots, w_I(t)]^T \in R^{I \times 1}$ is external disturbance vector; $y(t) \in R^{p \times 1}$ is performance output; $z(t) \in R^{L \times 1}$ is the measured output; $A = diag\{A_1, \ldots, A_i, \ldots, A_m\}$, $A_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\xi_i\omega_i \end{bmatrix}$.

3.2 Considering uncertainties

Firstly considering the variation in resonant frequencies and damping ratios. The variation in resonant frequencies and damping ratios is expressed as norm-bound uncertainties

$$
\begin{aligned}
\hat{\omega}_i &= \omega_i + \omega_{\Delta i}\delta_{1i}, \quad \|\delta_{1i}\| \le 1 \\
\hat{\xi}_i &= \xi_i + \xi_{\Delta i}\delta_{2i}, \quad \|\delta_{2i}\| \le 1
\end{aligned}
\tag{16}
$$

$$
\hat{A}_i = \begin{bmatrix} 0 & 1 \\ -\hat{\omega}_i^2 & -2\hat{\xi}_i\hat{\omega}_i \end{bmatrix}
\tag{17}
$$

Substituting Eq. (16) into Eq. (17) and neglecting the high-order small terms, we have

$$
\hat{A}_i = A_i + E_i\delta_i F_i
\tag{18}
$$

where $\delta_i = diag\{\delta_{1i}, \delta_{2i}\}$; $E_i = \begin{bmatrix} 0 & 1 \\ -\omega_{\Delta i} & -\xi_{\Delta i} \end{bmatrix}$ and $F_i = \begin{bmatrix} 2\omega_i & 2\xi_i \\ 0 & 2\omega_i \end{bmatrix}$ show the influence of variation in the ith resonant frequency and damping ratio on $\hat{A}_i$.

With Eq. (18), we have

$$
\hat{A} = diag\{\hat{A}_i, \ldots, \hat{A}_i, \ldots, \hat{A}_m\} = A + E\Delta_1 F
\tag{19}
$$

where $\Delta_1 = diag\{\delta_1, \ldots, \delta_i, \ldots, \delta_m\}$, $\|\Delta_1\| \le 1$; $E = diag\{E_1, \ldots, E_m\}$, $F = diag\{F_1, \ldots, F_m\}$.

Secondly considering the uncertainty due to un-modeled high-frequency modes, and the model error is represented by norm-bound addition uncertainty $\Delta G(s)$, we have

$$\hat{G}(s) = G(s) + \Delta G(s) \tag{20}$$

Weighting function $\Delta G(s)$ can be normalized by a weighting function $W(s)$.

$$\Delta G(s) = W(s) \cdot \Delta_2 \tag{21}$$

where Δ_2 is the unknown uncertainty satisfying $\|\Delta_2\|_\infty \le 1$; $W(s)$ has to form the upper bound of the un-modeled high-frequency modes.

The state-space representation for $W(s)$ can be written as

$$\begin{aligned} \dot{x}_r(t) &= A_r x_r(t) + B_r V_a(t) \\ h_2(t) &= C_r x_r(t) + D_r V_a(t) \end{aligned} \tag{22}$$

Using Eqs. (22), (19) and (15), the state-space representation for the system $\hat{G}(s)$ with modal parameters variation and un-modeled high-frequency modes can be written as

$$\begin{aligned} \dot{x}_o(t) &= A_o x_o(t) + B_o V_a(t) + \tilde{B}_o w(t) + E_1 g(t) \\ y(t) &= C_o x_o(t) + E_2 g(t) \\ z(t) &= \tilde{C}_o x_o(t) + E_3 g(t) \\ h(t) &= F_1 x_o(t) + F_2 V_a(t) \\ g(t) &= \Delta h(t) \\ \|\Delta\|_\infty &\le 1 \end{aligned} \tag{23}$$

where,

$$\dot{x}_o(t) = \begin{Bmatrix} \dot{x}(t) \\ \dot{x}_r(t) \end{Bmatrix},\ A_o = \begin{bmatrix} A & \\ & A_r \end{bmatrix},\ B_o = \begin{bmatrix} B \\ B_r \end{bmatrix},\ g(t) = \begin{Bmatrix} g_1(t) \\ g_2(t) \end{Bmatrix},\ E_1 = \begin{bmatrix} E \\ 0 \end{bmatrix}^T,\ \tilde{B}_o = \begin{bmatrix} \tilde{B} \\ 0 \end{bmatrix},\ C_o = \begin{bmatrix} C \\ 0 \end{bmatrix}^T,$$

$$E_2 = \begin{bmatrix} 0 \\ E_y \end{bmatrix}^T,\ \tilde{C}_o = \begin{bmatrix} \tilde{C} \\ 0 \end{bmatrix},\ E_3 = \begin{bmatrix} 0 \\ E_z \end{bmatrix}^T,\ F_1 = \begin{bmatrix} F & 0 \\ 0 & C_r \end{bmatrix},\ F_2 = \begin{bmatrix} 0 \\ D_r \end{bmatrix},\ \Delta = \begin{bmatrix} \Delta_1 & \\ & \Delta_2 \end{bmatrix}.$$

4. Control law

A dynamic output feedback law will be designed as the following

$$\begin{aligned} \dot{\zeta}(t) &= A_k \zeta(t) + B_k z(t) \\ V_a(t) &= C_k \xi(t) + D_k z(t) \end{aligned} \tag{24}$$

where $\zeta(t)$ is the state variable vector of the controller; A_k, B_k, C_k, D_k are unknown parameters of the controller. $D_k = 0$ means that the control law is strictly real.

With controller (24) and structure (23), the state space realization of the closed-loop system is written as

$$\begin{aligned} \dot{\hat{x}}(t) &= A_c\hat{x}(t) + B_{cw}w(t) + E_{cx}g(t) \\ y(t) &= C_c\hat{x}(t) + E_{cy}g(t) \\ h(t) &= F_{cx}\hat{x}(t) + F_{cg}g(t) \\ V_a(t) &= C_v\hat{x}(t) + D_vg(t) \\ g(t) &= \Delta h(t) \\ \|\Delta\|_\infty &\le 1 \end{aligned} \tag{25}$$

where, $\hat{x}(t) = \{x_o(t), \zeta(t)\}^T$, $A_c = \begin{bmatrix} A_o + B_oD_k\tilde{C}_o & B_oC_k \\ B_k\tilde{C}_o & A_k \end{bmatrix}$, $B_{cw} = \begin{bmatrix} \tilde{B}_o \\ 0 \end{bmatrix}$, $E_{cx} = \begin{bmatrix} E_1 + B_oD_KE_3 \\ B_kE_3 \end{bmatrix}$

$C_c = [C_o \ \ 0]$, $E_{cy} = E_2$, $F_{cx} = [F_1 + F_2D_k\tilde{C}_o, F_2C_k]$, $F_{cg} = F_2D_kE_3$, $C_v = [D_k\tilde{C}_o, C_k]$, $D_v = D_kE_3$

5. Design specifications

5.1 Stability and disturbance rejection

Considering the following norm-bound uncertainties due to modal parameters variation and unmodeled high-frequency modes

$$g^T(t)g(t) \le h^T(t)h(t) \tag{26}$$

Robust stability and disturbance rejection (an upper bound γ on the H_∞ norm from $w(t)$ to $y(t)$) of system (25) can be guaranteed if there exists $P = P^T > 0$ and *Lyapunov* function $V(\hat{x}(t)) = \hat{x}^T(t)P\hat{x}(t) > 0$ is taken, such that

$$dV(\hat{x}(t))/dt + y^T(t)y(t) - \gamma^2w^T(t)w(t) < 0 \tag{27}$$

Using Eq. (25), Eq. (26) and Eq. (27) can be, respectively, written as

$$\begin{bmatrix} \hat{x}(t) \\ g(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} -F_{cx}^TF_{cx} & -F_{cx}^TF_{cg} & 0 \\ -F_{cg}^TF_{cx} & I - F_{cg}^TF_{cg} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ g(t) \\ w(t) \end{bmatrix} \le 0 \tag{28}$$

$$\begin{bmatrix} \hat{x}(t) \\ g(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} A_c^TP + PA_c + C_c^TC_c & PE_{cx} + C_c^TE_{cy} & PB_{cw} \\ E_{cx}^TP + E_{cy}^TC_c & E_{cy}^TE_{cy} & 0 \\ B_{cw}^TP & 0 & -\gamma^2I \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ g(t) \\ w(t) \end{bmatrix} < 0 \tag{29}$$

Using S-procedure, Eq. (29) will hold in the case of Eq. (28) if and if only there exits $\beta \geq 0$, $P = P^T > 0$, such that

$$\begin{bmatrix} A_c^T P + PA_c + C_c^T C_c + \beta F_{cx}^T F_{cx} & PE_{cx} + C_c^T E_{cy} + \beta F_{cx}^T F_{cg} & PB_{cw} \\ E_{cx}^T P + E_{cy}^T C_c + \beta F_{cg}^T F_{cx} & E_{cy}^T E_{cy} + \beta F_{cg}^T F_{cg} - \beta I & 0 \\ B_{cw}^T P & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{30}$$

Recalling $\Delta = diag\{\delta_1 I_2, \ldots, \delta_m I_2, \Delta_2\}$, the conservative of design can be reduced by defining a matrix set $\tilde{S} = \{diag\{\tilde{S}_1, \ldots, \tilde{S}_m, \tilde{s}\}; \tilde{S}_i = R^{2\times 2}, \tilde{s} = R, \tilde{S}_i > 0, \tilde{s} > 0\}$. With Eq. (30), we have the sufficient condition for robust stability and disturbance rejection as follows: Considering norm-bound uncertainties due to modal parameters and unmodeled modes, robust stability and disturbance rejection (an upper bound γ on the H_∞ norm from $w(t)$ to $y(t)$) of closed-loop system (25) can be guaranteed remain if there exits $P = P^T > 0$, $S \in \tilde{S}$ and a scalar $\gamma > 0$, such that

$$\begin{bmatrix} A_c^T P + PA_c + C_c^T C_c + F_{cx}^T S F_{cx} & PE_{cx} + C_c^T E_{cy} + F_{cx}^T S F_{cg} & PB_{cw} \\ E_{cx}^T P + E_{cy}^T C_c + F_{cg}^T S F_{cx} & E_{cy}^T E_{cy} + F_{cg}^T S F_{cg} - S & 0 \\ B_{cw}^T P & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{31}$$

Eq. (31) can be written as

$$\begin{bmatrix} PA_c + A_c^T P & PE_{cx} & PB_{cw} \\ E_{cx}^T P & -S & 0 \\ B_{cw}^T & 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} F_{cx} & F_{cg} & 0 \\ C_c & E_{cy} & 0 \end{bmatrix}^T \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} F_{cx} & F_{cg} & 0 \\ C_c & E_{cy} & 0 \end{bmatrix} < 0 \tag{32}$$

Using Schur complement

$$\begin{bmatrix} PA_c + A_c^T P & PE_{cx} & PB_{cw} & F_{cx}^T & C_c^T \\ & -S & 0 & F_{cg}^T & E_{cy}^T \\ & & -\gamma^2 I & 0 & 0 \\ & sym & & -S^{-1} & 0 \\ & & & & -I \end{bmatrix} < 0 \tag{33}$$

Here the matrix inequality Eq. (33) is nonlinear in the unknown parameters P, A_k, C_k, B_k, D_k, and need to be reformulated in the new variables. Partitioning P and P^{-1} as, $P = \begin{bmatrix} p_1 & p_2 \\ p_2^T & p_3 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} \hat{p}_1 & \hat{p}_2 \\ \hat{p}_2^T & \hat{p}_3 \end{bmatrix}$, $p_1, \hat{p}_1 \in R^{2m\times 2m}$, then defining $\hat{F}_1 = \begin{bmatrix} \hat{p}_1 & I \\ \hat{p}_2^T & 0 \end{bmatrix}$, $P\hat{F}_1 = \hat{F}_2 = \begin{bmatrix} I & p_1 \\ 0 & p_2^T \end{bmatrix}$, and performing a congruence transformation with $diag(\hat{F}_1, I, I, I, I)$ on both inequalities Eq. (33), we obtain

$$\begin{bmatrix} A_o\hat{p}_1+\hat{p}_1A_o^T+B_oC_M+C_M^TB_o^T & A_o+B_oD_k\tilde{C}_o+A_M^T & E_1+B_oD_kE_3 & \tilde{B}_o & \hat{p}_1F_1^T+C_M^TF_2^T & \hat{p}_1C_o^T \\ & A_o^Tp_1+p_1A_o+\tilde{C}_o^TB_M^T+B_M\tilde{C}_o & p_1E_1+B_ME_3 & p_1\tilde{B}_o & F_1^T+\tilde{C}_oD_k^TF_2^T & C_o^T \\ & & -S & 0 & E_3^TD_k^TF_2^T & E_2^T \\ & & & -\gamma^2 I & 0 & 0 \\ sym & & & & -S^{-1} & 0 \\ & & & & & -I \end{bmatrix} < 0 \tag{34}$$

where,

$$\Phi_p = \begin{bmatrix} \hat{p}_1 & I \\ I & p_1 \end{bmatrix} > 0,\ p_2\hat{p}_2 = I - p_1\hat{p}_1,\ \begin{array}{l} A_M = p_1(A_o+B_oD_k\tilde{C}_o)\hat{p}_1 + p_2B_k\tilde{C}_o\hat{p}_1 + p_1B_oC_k\hat{p}_2^T + p_2A_k\hat{p}_2^T \\ B_M = p_1B_oD_k + p_2B_k \\ C_M = D_k\tilde{C}_o\hat{p}_1 + C_k\hat{p}_2^T \end{array}$$

5.2 Transient response performance

In order to achieve the desired transient performance, it is often necessary to place closed-loop poles in the specified region, which can be described by using LMI region. LMI region is defined as a subset D of the complex plane, and $D = \{s \in C{:}L + sM + \bar{s}M^T < 0\}$ can be used to denote a LMI region, where $L = L^T \in R^{m\times m}, M \in R^{m\times m}$. The characteristic function for LMI region can be expressed as $f_D(s) = L + sM + \bar{s}M^T$.

In order to achieve a specified decay rate and damping ratio in all controlled modes, the closed-loop poles need to be placed in the LMI region which is the intersection of the following two regions: 1) left half plane $\mathrm{Re}(s) < -\alpha$: $f_{D_1}(s) = s + \bar{s} + 2\alpha$ for which $L_1 = 2\alpha, M_1 = 1$, and 2) conic-sector with apex at the origin and inner angle 2θ: $f_{D_2}(s) = \begin{bmatrix} \sin\theta(s+\bar{s}) & \cos\theta(s-\bar{s}) \\ -\cos\theta(s-\bar{s}) & \sin\theta(s+\bar{s}) \end{bmatrix}$ for which $L_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $M_2 = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$ As a result, all controlled modes will decay in $e^{-\alpha t}$, and damping ratio will not less than $\xi = \cos\theta$.

LMI region has the property that the intersection of LMI regions is also an LMI region, so we have the desired LMI region as

$$f_{D_3}(s) = diag\{f_{D1}(s), f_{D2}(s)\},\ L_3 = \begin{bmatrix} 2\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & -\cos\theta & \sin\theta \end{bmatrix} \tag{35}$$

Poles of uncertain system (25) will belong to LMI region D_3 if there exists $P = P^T > 0$, such that

$$\begin{bmatrix} M_{D_3}(A_c, P) & M_{31}^T \otimes (PE_{cx}) & M_{32}^T \otimes F_{cx}^T \\ M_{31} \otimes (E_{cx}^T P) & -I & I \otimes F_{cg}^T \\ M_{32} \otimes F_{cx} & I \otimes F_{cg} & -I \end{bmatrix} < 0 \tag{36}$$

where, $M_{D_3}(A_c, P) = L_3 \otimes P + M_3 \otimes (PA_c) + M_3^T \otimes (PA_c^T)$, $M_{31}^T M_{32} = M_3$, $\otimes$ is Kronecker product. Similar to Eq. (33), the matrix inequality (36) is nonlinear in the unknown parameters P, A_k, C_k, B_k, D_k. Performing a congruence transformation with $diag(\hat{F}_1, I, I)$ on both inequalities (25), we obtain

$$\begin{bmatrix} L_3 \otimes \Phi_p + M_3 \otimes \Phi_A + M_3^T \otimes \Phi_A^T & M_{31}^T \otimes \Phi_B & M_{32}^T \otimes \Phi_C^T \\ & -I & I \otimes \Phi_D^T \\ & sym & -I \end{bmatrix} < 0$$

$$\Phi_p = \begin{bmatrix} \hat{p}_1 & I \\ I & p_1 \end{bmatrix}, \ \Phi_A = \begin{bmatrix} A_o \hat{p}_1 + B_o C_M & A_o + B_o D_k \tilde{C}_o \\ A_M & p_1 A_o + B_M \tilde{C}_o \end{bmatrix}, \ \Phi_B = \begin{bmatrix} E_1 + B_o D_K E_3 \\ p_1 E_1 + B_M E_3 \end{bmatrix}$$

$$\Phi_C = [F_1 \hat{p}_1 + F_2 C_M, F_2 D_k \tilde{C}_o], \ \Phi_D = F_2 D_k E_3, \ \hat{p}_2 p_2^T = 1 - \hat{p}_1 p_1 \tag{37}$$

5.3 Actuator saturation

In order to avoid actuator saturation, we need to set limit on every control input which will not be violated for a given set of disturbance and initial conditions. For defining the control input constraint, an ellipsoid χ of initial conditions is considered for the closed loop initial conditions as given below

$$\hat{x}_0^T P \hat{x}_0 < \lambda \tag{38}$$

where λ is a real number that defines the size of the ellipsoid, and $\hat{x}_0$ is the initial conditions for the closed loop system.

Let $S(\hat{x}_0, w)$ be the reachable set of state trajectories of the system in response to a given set of finite energy disturbance $W = \{w(t) | \int_0^\infty w^T(t) w(t) dt \le w_{\max} < \infty\}$ and a given set of initial conditions with ellipsoid χ.

The robust stability condition can be written as

$$\frac{dV(\hat{x}(t))}{dt} + y^T(t) y(t) - w^T(t) w(t) < 0 \tag{39}$$

Integrating over time 0 to t, we have

$$V(\hat{x}(t)) - V(\hat{x}(0)) + \int_0^t y^T y dt - \int_0^t w^T w dt < 0, \text{ for all } t \ge 0 \tag{40}$$

Because $V(\hat{x}(0)) = \hat{x}_0^T P \hat{x}_0 < \lambda$ and $\int_0^t y^T y dt \ge 0$, we can readily infer that for a given nonzero trajectory in the ellipsoid χ, each of the closed-loop trajectory $\hat{x}(t)$ in $S(\hat{x}_0, w)$ satisfies

$$\hat{x}^T(t) P \hat{x}(t) < \lambda + w_{\max}, \text{ for any } w(t) \in W, \text{ all } t \ge 0 \tag{41}$$

For zeros initial conditions

$$\hat{x}^T(t) P \hat{x}(t) < w_{\max}, \text{ for } w(t) \in W, \ t \ge 0 \tag{42}$$

For guaranteeing the control input constraint, each control input must satisfy

$$\|V_{aj}(t)\|_2^2 \le V_{\max}^2, \text{ for } t \ge 0,\ j = 1,\ldots,K \tag{43}$$

Using Eq. (15), Eq. (43) becomes

$$\hat{x}^T(t)C_{v,j}^T C_{v,j}\hat{x}(t) \le V_{\max}^2 \tag{44}$$

where $C_{v,j}$ is the jth row of the C_v.

Combining Eq. (42) and Eq. (44), the control input limit becomes

$$\frac{C_{v,j}^T C_{v,j}}{V_{\max}^2} < \frac{P}{w_{\max}},\quad j = 1,\ldots,K \tag{45}$$

By Eq. (45), control input limit can be written as matrix inequality constraints

$$\begin{bmatrix} P & C_{v,j}^T \\ C_{v,j} & \dfrac{V_{\max}^2}{w_{\max}} \end{bmatrix} > 0,\quad j = 1,\ldots,K \tag{46}$$

Performing a congruence transformation with $diag(\hat{F}_1, I)$ on both inequalities Eq. (46), we obtain

$$\begin{bmatrix} \hat{p}_1 & I & C_{M,j}^T \\ I & p_1 & 0 \\ C_{M,j} & 0 & \dfrac{V_{\max}^2}{w_{\max}} \end{bmatrix} > 0 \tag{47}$$

6. Multiobjective synthesis

In the previous section, specifications, such as robust stability and disturbance rejection, dynamic response performance and control input limit, have been as LMI constraints on the unknown controller state-space matrices and Lyapunov Matrix. So the controller design problem with conflicting specification requirements can be converted into a linear convex optimization problem as follows

$$\begin{aligned} &\text{Min} \quad \gamma \\ &\text{s.t.} \quad \text{Eq. (34),(37),(47)} \end{aligned} \tag{48}$$

7. Numerical example

First, to demenstrate that the influence of vibration modes on the dynamical response of structure varies with the location of piezoelectric actuators, the flexural vibration of a simply supported beam is considered. The piezoelectric actuator patch is located at $0.23l_b \to 0.49l_b$ and $0.1l_b \to 0.14l_b$

along the beam (l_b = 0.5 m is the length of the beam) respectively, and seven modes contributing most to the system dynamical response is hoped to be kept. Fig. 1 and Fig. 2 give the system frequency respones for system with displacement of different points (x = 0.04 m and x = 0.18 m) respectively along the beam as outputs, while the piezoelectric actuator patch is located at $0.23l_b \rightarrow 0.49l_b$. In figures, "A" denotes the real response of system, "B" the low-order response by using the direct truncation, "C" the low-order response by using the method given in this paper (1st~6th and 9th modes are kept) and "D" the low-order reponse by using the method given in the literature (Yu and Wang 2007). The comparision between Fig. 1 and Fig. 2 shows that the kept modes do not varies with the choosen output points when the piezoelectric actuator patch location is fixed. Fig. 3 gives the system frequency respones of system with displacement of the point (x = 0.04 m) as output, while the piezoelectric actuator patch is located at $0.1l_b \rightarrow 0.14l_b$, in which 1st~3rd and 5th~8th modes are kept different from Fig. 1 and Fig. 2. From the three figures, it can be seen that the influence of vibration modes on the dynamical response of structure varies with the

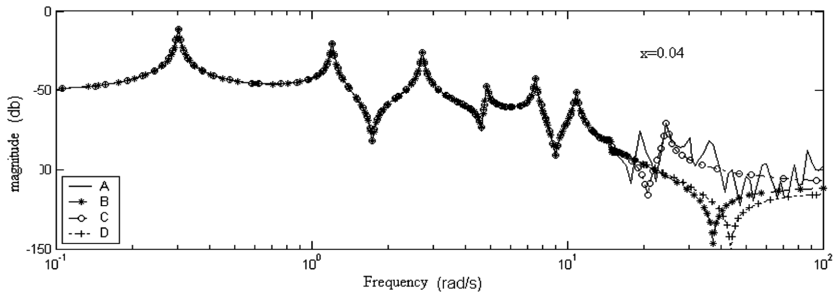

Fig. 1 Frequency responses of system (x = 0.04 m) with actuator patch located at $0.23l_b \rightarrow 0.49l_b$

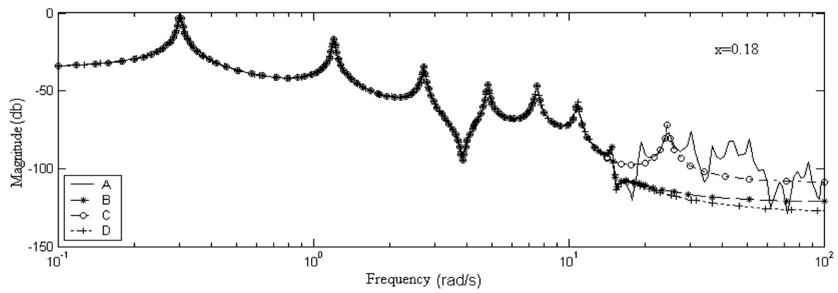

Fig. 2 Frequency responses of system (x = 0.18 m) with actuator patch located at $0.23l_b \rightarrow 0.49l_b$

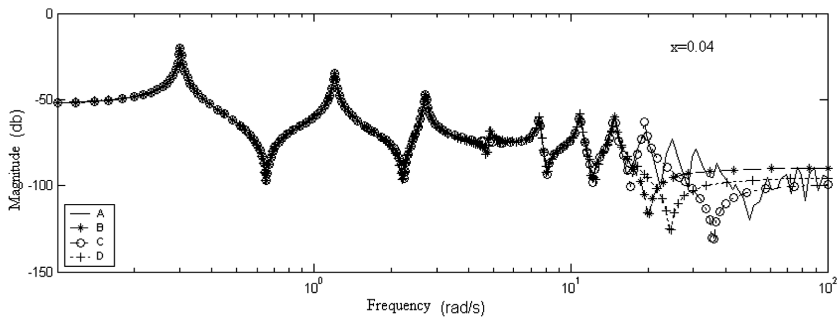

Fig. 3 Frequency responses of system (x = 0.04 m) with actuator patch located at $0.1l_b \rightarrow 0.14l_b$

location of piezoelectric actuators, which means the ability of actuators to control vibration modes has to be taken into account in modeling a relatively low-order dynamic modal equation.

Next, to demenstrate that the uncertainties should be considered to avoid ecxiting high-frequency modes in th non-collcated vibration control, the simulation is conducted to suppress the flexural vibration of a simply supported beam with non-collocated piezoelectric actuator/sensor patches bonded to it. The piezoelectric actuator patch is located at $0.1l_b \to 0.14l_b$ along the beam while the piezoelectric sensor is located at $0.58l_b \to 0.85l_b$ The performance displacement output is the displacement coming from some point along the beam. Based on the spatial H_2 norm of modes, the 1^{st}, 2^{nd}, 3^{rd}, 5^{th} modes are kept in the control design stage without considering the uncertainties. Fig. 4 and Fig. 5 give the impulse responses and frequency responses for the uncontrolled and controlled system without considering the uncertainties, from which it is evident that the high modes are exicted when the low frequenciec are controlled (Fig. 5), and the system vibration cannot be controlled (Fig. 4).

Finally, a dynamic output feedback uncertain H_∞ vibration control law is designed with transient response requirement and actuator saturation constraints by using linear matrix inequality method.

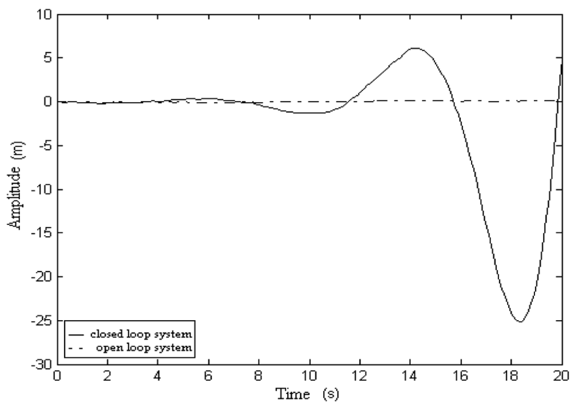

Fig. 4 Impulse responses without considering uncertainties

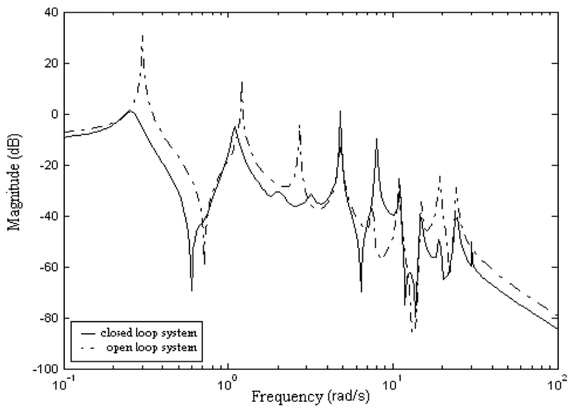

Fig. 5 Frequency responses without considering uncertainties

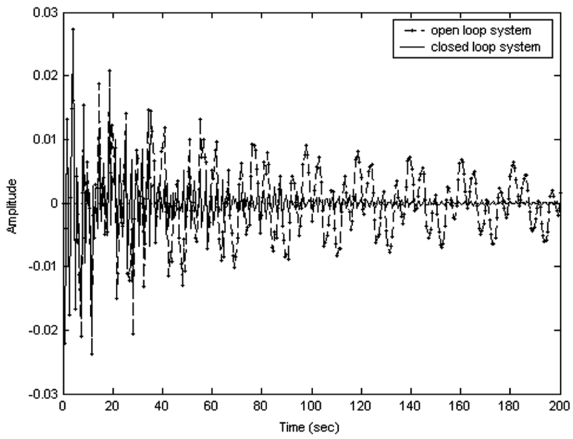

Fig. 6 Impulse responses with considering uncertainties

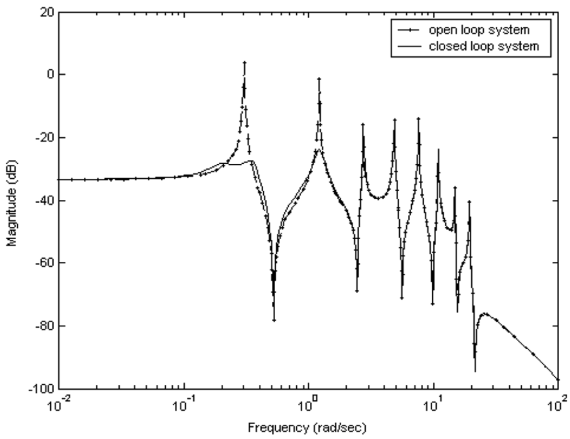

Fig. 7 Frequency responses with considering uncertainties

The residual modes is considered as norm-bound additive uncertainty, which is normalized by weighting function $W(s) = 0.01s^2/(1+s/8)^2$ (from actuator to performance output). A decay rate of 0.3, minimum damping ratio not less than 0.01 and the ±5 V control input limits are requried. Fig. 6 and Fig. 7 give the impulse responses and frequency responses for the uncontrolled and controlled system with considering the uncertainties, from which it can be seen that the low-frequency modes is suppressed without exciting the high-frequency modes, and therefore the spillover appearing in the non-collocated control without considering uncertainties is avoided.

8. Conlusions

(1) The influence of vibration modes on the dynamical response of structure varies with the location of piezoelectric actuators, and the ability of actuators to control vibration modes has to be taken into account in modeling a relatively low-order dynamic modal equation.

(2) The modal state-space representation is developed for uncertain flexible structures with non-collocated piezoelectric actuators and sensors bonded to it, taking into account uncertainties due to modal parameters variation and un-modeled residual high frequencies.

(3) A dynamical output feedback control law is designed by simultaneously considering the conflicting performance specifications, such as robust stability, transient response requirement, disturbance rejection, actuator saturation constraints.

(4) The results show that modal-based multiobjective vibration control law can suppress the low-frequency modes without exciting the high-frequency modes, and therefore the spillover appearing in the non-collocated control can be avoided.

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