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Axisymmetric vibrations of layered cylindrical shells of variable thickness using spline function approximation

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Abstract. Free axisymmetric vibrations of layered cylindrical shells of variable thickness are studied using spline function approximation techniques. Three different types of thickness variations are considered namely linear, exponential and sinusoidal. The equations of axisymmetric motion of layered cylindrical shells, on the longitudinal and transverse displacement components are obtained using Love's first approximation theory. A system of coupled differential equations on displacement functions are obtained by assuming the displacements in a separable form. Then the displacements are approximated using Bickley-spline approximation. The vibrations of two-layered cylindrical shells, made up of several types of layered materials and different boundary conditions are considered. Parametric studies have been made on the variation of frequency parameter with respect to the relative layer thickness, length ratio and type of thickness variation parameter.

Keywords: free vibration; cylindrical shell; spline method; variable thickness; eigenvalues.

1. Introduction

Layered circular cylindrical shells are widely used in the fields of shipbuilding, aerospace and other industries. Composite structures having high specific stiffness, better damping and shock absorbing characteristics, give good performance in industry. Liessa (1973) reported on vibration behavior of composite cylindrical shell structure of variable thickness with homogeneous wall. Dong (1968) and Greenberg (1980) studied the vibration of orthotropic composite cylindrical shells using Donnel's (1933) shallow shell theory. Reddy (1981) presented the finite element model on layered anisotropic plates and shells. Sakiyama *et al.* (2002) and Tsuiji and Sueoka (1989) analysed the vibration of cylindrical panel using Rayleigh-Ritz method. Later Toorani and Lakis (2006) studied the non-uniform composite cylindrical shells using hybrid finite element analysis, in which shear deformation theory and rotatory inertia are included and the thickness variation is considered

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in the circumferential direction. Hinton *et al.* (1995) used Mindlin-Reissner shell theory to analyze the variable thickness of folded plates and curved shells by applying the finite strip method.

Mizusawa and Kito (1995) used the spline strip method to study the vibration of cross-ply laminated cylindrical panels. Sivadas and Ganesan (1991, 1993) analyzed the free vibration of cylindrical shells of variable thickness in terms of linear and quadratic variations and discussed axisymmetric vibration of thick cylindrical shell for linear variation using FEM. Suzuki *et al.* (1993) used the power series solution to analyze the vibration of rotating circular cylindrical shells of variable thickness. Viswanathan and Navaneethakrishnan (2005) presented a paper using spline function approach, in which the conical shells of variable thickness are analyzed. The present work is to analyze the cylindrical shell of variable thickness using the spline method.

In the present study the free vibration of laminated circular cylindrical shells of variable thickness are analyzed using spline function techniques. The equations of motion are derived using Love's first approximation theory for homogeneous shells. The layers are considered to be thin, elastic, specially orthotropic or isotropic and assumed to be perfectly bonded together and to move without interface slip. The governing differential equations are obtained in terms of the reference surface displacements which are coupled in the longitudinal, circumferential and transverse displacement components. Assuming the displacement functions in a separable form, they reduce to a system of ordinary differential equations on a set of displacement functions which are functions of meridional co-ordinate only. Two types of layered materials are considered and two sets of boundary conditions are imposed in all. The equations have no closed form solution in general, so that the numerical solution techniques have to be resorted to.

In preference to a number of numerical methods available for such problems, like those of FEM (Konuralp Girgin 2006, Wang *et al.* 2006), Fourier series approach (2001) or Generalized differential quadrature (1996) a spline function technique is used. Bickley (1968) successfully tested the spline collocation method over a two point boundary value problem with cubic spline. Viswanathan and Navaneethakrishnan (2002, 2003, 2005) have also demonstrated this, along with its attractive features of elegance in handling and convergence. Recently Viswanathan and Lee (2007) studied the vibration of cross-ply plates including shear deformation theory using spline method. The advantage of this method is that a chain of lower order approximations, used here than the global higher order approximation.

In this work only the axisymmetric vibration is analyzed. Hence, the differential equations are reduced to the longitudinal and transverse displacement functions. Then the spline functions are approximated for the displacement functions with suitable order which are cubic and quintic. Collocation with these splines yield a set of field equations which, along with the equations of boundary conditions, reduce to a system of homogeneous simultaneous algebric equations on the assumed spline coefficients which results a generalized eigenvalue problem. The eigenvalue problem is solved for a frequency parameter using eigensolution technique to obtain as many frequencies as required, starting from the least. From the eigenvectors, the spline coefficients are calculated from which the mode shapes are constructed.

2. Formulation of the problem

The system of differential equations are derived for a thin shell of revolution which characterize the vibration comprising of isotropic and specially orthotropic layers based on Love's first

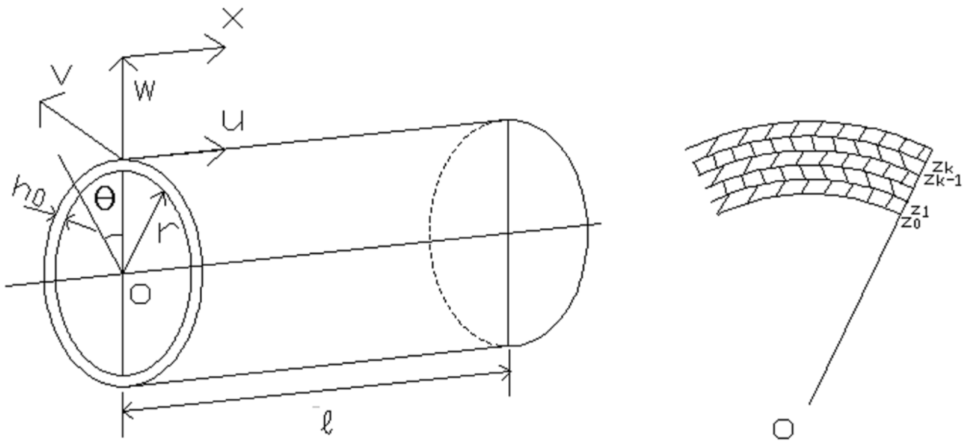

Fig. 1 Layered circular cylindrical shell of constant thickness: Geometry

approximation theory. The rotatory inertia and shear deformation are neglected. Also the general line of procedure of Ambartsumyan (1964) for the classical theory of thin shell is adopted. The assumption made in this case is that, each individual layer behaves macroscopically as a homogeneous orthotropic and linearly elastic material. Its material axes of symmetry parallel to the principal coordinates of the surface of the shell. Within the frame work of thin shell assumption, the layers may be of arbitrary thickness. They may be of arbitrary material properties and arranged with or without symmetry about the middle surface. The consecutive layer is assumed to be perfectly bonded together at their interface resulting in motion without slip. Since the layer can now be of variable thickness, the following assumption is made:

All the layers of the cylindrical shell vary in thickness either linearly, exponentially, sinusoidally, or in a combined way according to the same law of variation, the variation being gradual.

According to this, the material lines of orthotropy can still assumed to be parallel to the principal coordinate lines of the reference surface of the shell. Smaller the number of layers assumed, more realizable is this assumption. Only two layers are considered here for detailed study, for the reason explained earlier, it turns out that this assumption is well realizable. The coordinate system and the geometric parameters of the laminated cylindrical shells of constant thickness and the arrangement of its layers are shown in Fig. 1.

The thickness of the k-th layer is assumed in the form

$$h_k = h_{0k} g(x) \tag{1}$$

Where h_{0k} is a constant and $g(x)$ is assumed for suitable function of x with respect to the different thickness variation. The thickness of the layers is not completely independent. Their dependence is given by

$$\sum_k (z_k^2 - z_{k-1}^2)\rho_k = 0 \tag{2}$$

where ρ_k is the density of the k-th layer and z_k is the distance of the outer boundary of the k-th layer from the reference surface. The elastic coefficients corresponding to layers of uniform thickness with superscript 'c', one easily finds

$$A_{ij} = A_{ij}^c g(x), \quad B_{ij} = B_{ij}^c g(x), \quad D_{ij} = D_{ij}^c g(x) \tag{3}$$

where A_{ij}, B_{ij} and D_{ij} are extensional rigidity, coupling between bending and stretching and flexural rigidity respectively.

In this study, the thickness variation of each layer is assumed in the form

$$h(x) = h_0 g(x) \tag{4}$$

Where

$$g(x) = 1 + C_\ell \frac{x}{\ell} + C_e \exp\left(\frac{x}{\ell}\right) + C_s \sin\left(\frac{\pi x}{\ell}\right) \tag{5}$$

Here ℓ is the length of the cylinder.

The stress resultants and moment resultants are expressed in terms of the longitudinal, and transverse displacements u and w of the reference surface. The displacement v is neglected since, only the axisymmetric vibrations are studied in this work. The displacements are assumed in the separable form given by

$$\begin{aligned} u(x,t) &= U(x)e^{i\omega t} \\ w(x,t) &= W(x)e^{i\omega t} \end{aligned} \tag{6}$$

where x is the longitudinal coordinate, t is the time and ω is the angular frequency of vibration. Using Eq. (6) in the constitutive equations and the resulting expressions for the stress resultants and the moment resultants in the equilibrium equations, the governing differential equations of motion are obtained in the form

$$\begin{bmatrix} L_{11} & L_{13} \\ L_{31} & L_{33} \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix} = [0] \tag{7}$$

The operators appearing in Eq. (7) are

$$L_{11} = \frac{d^2}{dx^2} + \frac{g'}{g}\frac{d}{dx} + \lambda'^2$$

$$L_{13} = -s_4 \frac{d^3}{dx^3} - s_4 \frac{g'}{g}\frac{d^2}{dx^2} + s_2 \frac{1}{r}\frac{d}{dx} + \frac{g'}{gr}\left[s_2 + s_5 \frac{1}{r}\right]$$

$$L_{31} = s_4 \frac{d^3}{dx^3} + 2s_2 \frac{g'}{g}\frac{d^2}{dx^2} + \left[s_4 \frac{g''}{g} - \frac{s_2}{r}\right]\frac{d}{dx}$$

$$L_{33} = -s_7 \frac{d^4}{dx^4} - 2s_7 \frac{g'}{g}\frac{d^3}{dx^3} + \left[\frac{2s_5}{r} - s_7 \frac{g''}{g}\right]\frac{d^2}{dx^2} + \frac{g'}{g}\frac{2s_5}{r}\frac{d}{dx} - \frac{s_3}{r^2} + \frac{s_5}{r}\frac{g''}{g} + \lambda'^2 \tag{8}$$

where

$$s_2 = \frac{A^c_{12}}{A^c_{11}}, \quad s_3 = \frac{A^c_{22}}{A^c_{11}}, \quad s_4 = \frac{B^c_{11}}{A^c_{11}}, \quad s_5 = \frac{B^c_{12}}{A^c_{11}}, \quad s_6 = \frac{B^c_{22}}{A^c_{11}}, \quad s_7 = \frac{D^c_{11}}{A^c_{11}}$$
$$s_8 = \frac{D^c_{12}}{A^c_{11}}, \quad s_9 = \frac{D^c_{22}}{A^c_{11}}, \quad s_{10} = \frac{A^c_{66}}{A^c_{11}}, \quad s_{11} = \frac{B^c_{66}}{A^c_{11}}, \quad s_{12} = \frac{D^c_{66}}{A^c_{11}} \tag{9}$$

$$\lambda'^2 = \frac{R_0\omega^2}{A^c_{11}} \quad \text{is a frequency parameter} \tag{10}$$

$$R_0 = \sum_k \rho_{(k)}[z_k(0) - z_{k-1}(0)] = \sum_k \rho_k h_k(0) \quad \text{is the inertial coefficient.} \tag{11}$$

3. Method of solution

3.1 Modification of displacement equations

The differential equations on the displacement functions of Eq. (7) contain derivatives of third order in U and fourth order in W. Therefore the present form is not suitable to the solution procedure we propose to adopt. Hence, the equations are combined within themselves and a modified set of equations are derived. The modified equations are now become as 2nd order in U and 4th order in W and is given by

$$\begin{bmatrix} L_{11} & L_{13} \\ L^*_{31} & L^*_{33} \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix} = [0] \tag{12}$$

The new operators L^*_{31} and L^*_{33} are

$$L^*_{31} = s_4 \frac{g'}{g}\frac{d^2}{dx^2} + \left[s_4 \frac{g'^2}{g^2} - \frac{s_2}{r} - \lambda'^2 s_4 \right] \frac{d}{dx}$$

$$L^*_{33} = (s_4^2 - s_7)\frac{d^4}{dx^4} + (s_4^2 - 2s_7)\frac{g'}{g}\frac{d^3}{dx^3} + \left[\frac{2s_5}{r} - s_7\frac{g''}{g} - s_4\left\{ \frac{s_2}{r} - s_4\left(\frac{g''}{g} - \frac{g'^2}{g^2} \right) \right\} \right] \frac{d^2}{dx^2}$$
$$+ \frac{g'}{g}\left[\frac{2s_5}{r} - s_4 s_2 \frac{1}{r} \right] \frac{d}{dx} - \left[\frac{s_3}{r^2} - \frac{s_5}{r}\frac{g''}{g} + s_4 s_2 \frac{1}{r}\left(\frac{g''}{g} - \frac{g'^2}{g^2} \right) \right] + \lambda'^2 \tag{13}$$

3.2 Transformation

The parameters are non dimensionalized as

$\lambda = \ell\lambda'$, a frequency parameter

$\delta_k = \dfrac{h_k}{h}$, a relative thickness ratio of the kth layer

$L = \dfrac{\ell}{r}$, a length parameter (14)

$H = \frac{h_0}{r}$, ratio of thickness to radius

$R = \frac{r}{\ell}$, a radius parameter

$X = \frac{x}{\ell}$, $0 \le x \le \ell$, a distance coordinate

Here ℓ is the length of the cylinder, r is the radius, h_k is the thickness of the kth layer, h is the total thickness of the shell and h_0 is the constant thickness. Also define $\delta = \delta_1$ and $\delta_2 = 1 - \delta_1$, since we consider only two layers.

3.3 Thickness variation

The thickness $h_k(X)$ of the kth layer at the distance X from the origin o can be expressed as

$$h_k(X) = h_{0k}g(X) \tag{15}$$

Where
$$g(X) = 1 + C_\ell X + C_e \exp(X) + C_s \sin \pi X \tag{16}$$

The range of X lies between 0 and 1. i.e., $X \in [0, 1]$.

If $C_e = C_s = 0$, the thickness variation becomes linear, In this case it can be easily shown that $C_l = 1/\eta - 1$, where η is the taper ratio $h_k(0)/h_k(1)$. If $\eta = 1$, then $C_l = 0$ and the thickness becomes constant. If $C_\ell = C_s = 0$, the excess thickness over uniform thickness varies exponentially and if $C_\ell = C_e = 0$, the excess thickness varies sinusoidally. The thickness of the layer at $X = 0$ is h_{0k} for the first and third cases, but the thickness is $h_{0k}(1 + C_e)$ for the second case.

3.4 Spline collocation procedure

The displacement functions $U(X)$ and $W(X)$ are approximated using spline collocation procedure to solve the problem assuming in the same way as in Viswanathan and Navaneethakrishnan (2003).

$$U^*(X) = \sum_{i=0}^{2} a_i X^i + \sum_{j=0}^{N-1} b_j (X - X_j)^3 H(X - X_j)$$

$$W^*(X) = \sum_{i=0}^{4} e_i X^i + \sum_{j=0}^{N-1} f_j (X - X_j)^5 H(X - X_j) \tag{17}$$

Here, $H(X - X_j)$ is the Heaviside step function. The function U^* is approximated using cubic spline and V^* is approximated using quintic spline, since the system of differential Eq. (9) are of second degree in U and of fourth degree in W.

The range of X is divided into N subintervals, at the points $X = X_s$, $s = 1, 2, \ldots, N-1$. The width of each interval is $1/N$ and $X_s = s/N$, $s = 0, 1, \ldots, N$, since the knots X_s are chosen equally spaced. The assumed splines must agree with functions they are approximated at the nodes, which follows that these splines satisfy the differential equations given by Eq. (9), at all $X = X_s$. This gives the result s in the homogeneous system of $(2N + 2)$ equations in the $(2N + 8)$ spline coefficients.

The boundary conditions are used as follows: (i) both the edges clamped (C-C), (ii) both the edges hinged (H-H). Each of the boundary conditions gives six more equations thus giving a total of $(2N+8)$ equations, in the same number of unknowns. The resulting field and boundary conditions gives raise to the generalized eigenvalue problem of the form

$$[M]\{q\} = \lambda^2[P]\{q\} \tag{18}$$

where $[M]$ and $[P]$ are matrices of order $(2N+5)\times(2N+5)$, $\{q\}$ is a matrix of order $(2N+5)\times 1$. This is treated as a generalized eigenvalue problem in the eigenparameter λ and the eigenvector whose elements are the spline coefficients.

4. Results and discussion

4.1 Convergence and comparative study

A convergence study has been studied for the frequency parameter value to choose the number of subintervals N of the range of X. The material properties are taken from Elishakoff and Stavsky (1976). The program was run for several cases of parametric values, material combinations, thickness variations, for value of $N = 4$ onwards. There was some improvement in λ with increasing the value of N, but the improvement came down steadily. It can be seen that the choice of $N = 14$ is adequate since for the next value of N the percent change in values of λ, are very low, the maximum being 0.35%. The results are not furnished here since for want of space.

Table 1 shows the comparison of the frequency parameter λ for various length parameter obtained by present method with those results obtained by Sivadas and Ganesan (1993) for thin shells. The fundamental frequency λ converted in to the suitable parameter $\bar{\lambda}^2 = \rho a^2\omega^2/E_2$. The following parametric values are used: radius $a = 0.1$ m, Young's modulus $E = 2\times10^{11}$ N/m^2, Poisson ratio $\nu =$ 0.3, mass density $\rho = 7800$ kg/m^3 from Sivadas and Ganesan (1993). The maximum percentage changes between present value and available result is 6.66%. So, the agreement of the current results is quite good, providing the credibility to the method of analysis and results.

Table 1 Comparative study of axisymmetric vibration of cylindrical shell of linear variation in thickness under C-C boundary condition with Sivadas and Ganesan (1993)
$h^* = a/h_{av}$, $h_0 = h_{av}/(1 + k/2)$, $\mu = l/a$ and $h = h_0[1 + k(2X - 1)]$
$h^* = 50$

Fundamental frequency parameter	$\mu = l/a$				
	10	5	3	2	1.2
λ	1.988	1.704	1.794	1.740	1.254
$\tilde{\lambda}$	0.208	0.357	0.627	0.912	1.095
$\bar{\lambda}$	0.195	0.375	0.650	0.925	1.110
Difference (%)	6.66	4.8	3.5	1.43	1.35

λ: present value
$\tilde{\lambda}$: present value converted to Sivadas and Ganesan parameter
$\bar{\lambda}$: Sivadas and Ganesan parameter

4.2 Discussion

Figs. 2-5 depicts the nature of the values of the frequency parameter λ with respect to the increase of the relative thickness ratio δ for the layered cylindrical shells, whose thickness varies linearly $(C_\ell \neq 0, C_e = C_s = 0)$, exponentially $(C_e \neq 0, C_\ell = C_s = 0)$ and sinusoidally $(C_s \neq 0, C_\ell = C_e = 0)$ for different values of thickness variation parameter, with two fixed values L = 1.5 (length parameter) and H = 0.02 (thickness parameter). The first three meridional modes $(m = 1, 2, 3)$ are considered here and in all the studies that follow.

Fig. 2 shows the nature of the frequency under the clamped-clamped (C-C) boundary conditions for two kinds of two layered materials made up of HSG-SGE and HSG-PRD materials. Thus, when

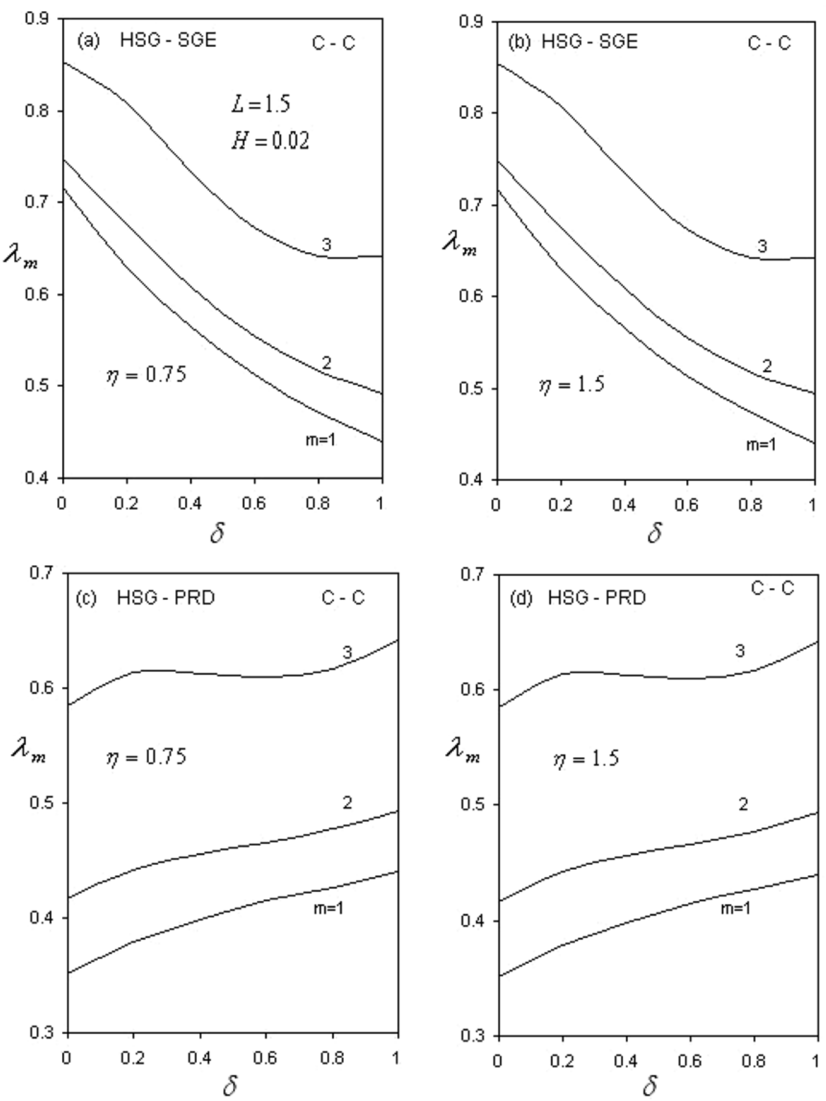

Fig. 2 Variation of frequency parameter with relative layer thickness: Cylindrical shells of linear variation in thickness of layers under clamped-clamped boundary conditions

δ = 0 the inner layer disappears then the shell is in homogeneous. When δ = 1 the outer layer disappears, again the shell is in homogeneous. Figs. 2(a) and (b) corresponds to a shell whose inner and outer layers are made up of HSG-SGE materials. In this case when δ = 0, the shell becomes homogeneous made up of SGE material and when δ = 1, the shell is again homogeneous made up of HGE material. The values of taper ratio is fixed as η = 0.75 and η = 1.5 in Figs. 2(a) and (b) respectively. From this figure it can be seen that the value of $\lambda_m(m = 1,2,3)$ decreases as δ increases for all the three modes. Figs. 2(c) and (d) corresponds to a shell whose inner and outer layers are made up of HSG-PRD materials. Here the value of $\lambda_m(m = 1,2,3)$ increases as δ increases. The shell is homogeneous for the extreme values of δ, equal to 0 or 1. The value of $\lambda_m(m = 1,2,3)$ is highest for the homogeneous SGE shells (at δ = 0 in Figs. 1(a) or (b)) and the

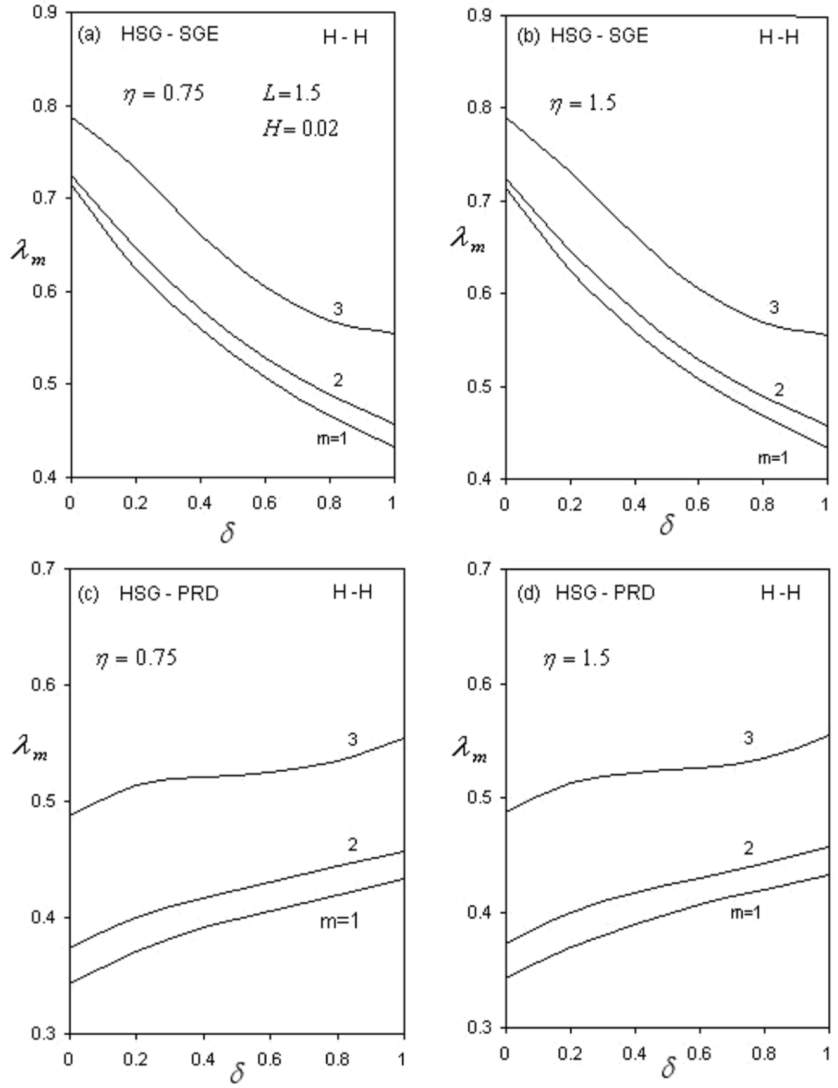

Fig. 3 Variation of frequency parameter with relative layer thickness: Cylindrical shells of linear variation in thickness of layers under hinged - hinged boundary conditions

least for homogeneous PRD shells (at $\delta = 0$ in Figs. 2(c) or (d)) and assumes a value in between them for homogeneous HSG shells (at $\delta = 1$ in Figs. 2(a), (b), (c) or (d)). It is clearly shows that it is possible to attain a desired frequency, between these two extreme values by suitably choosing the value of δ.

Figs. 3(a) and (b) shows the effect of the thickness ratio δ on the frequency parameters $\lambda_m (m = 1, 2, 3)$ with fixed values of η for the shells made of HSG-SGE materials. Both the ends are hinged (H-H). Figs. 3(c) and (d) is drawn on frequency parameters $\lambda_m (m = 1, 2, 3)$ for the layered materials HSG-PRD. The behavior of the frequency parameters is similar as in Fig. 2. But the frequencies are higher for C-C conditions comparing with corresponding study of H-H

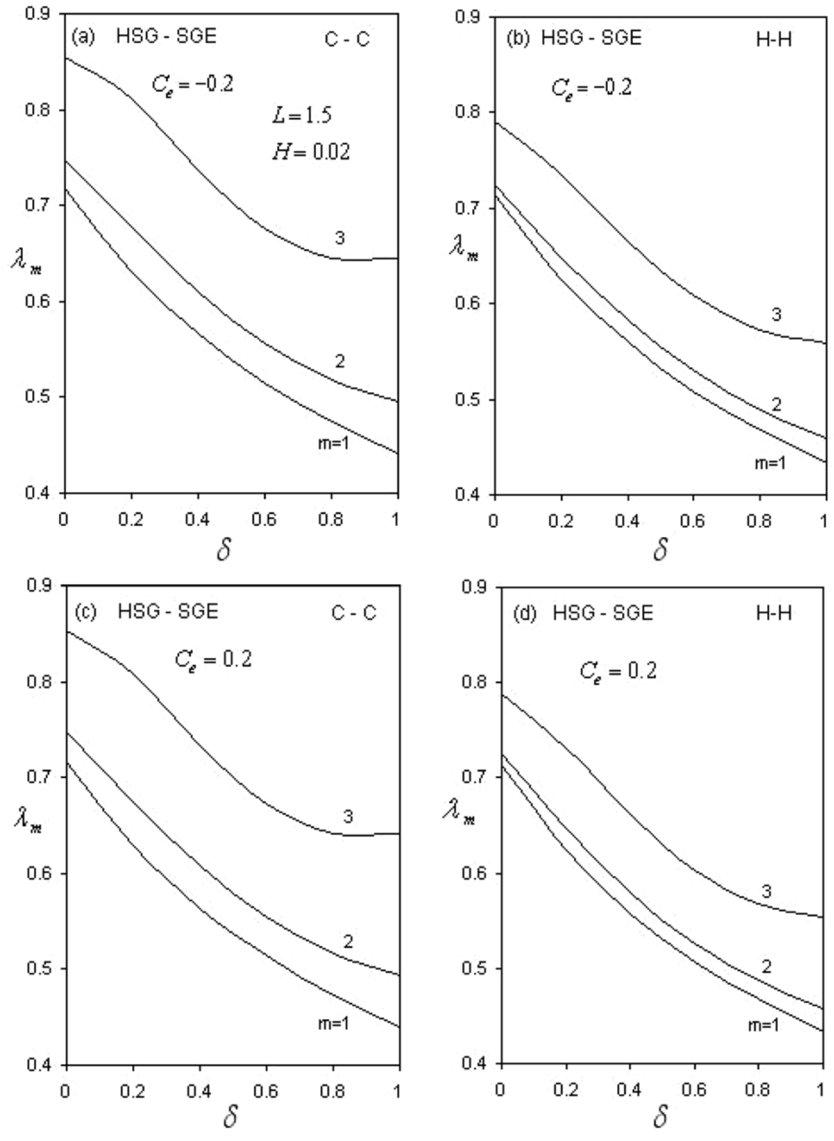

Fig. 4 Variation of frequency parameter with relative layer thickness: Cylindrical shells of exponential variation in thickness of layers. Layer materials: HSG-SGE

conditions. This reveals that the designer can choose the required structure by choosing the value of δ and combination of layered materials along with necessary boundary conditions.

Fig. 4 corresponds to exponential variation in thickness of layers $(C_e \neq 0, C_\ell = C_s = 0)$. The cases C_e = 0.2 and −0.2 are studied. Accordingly the thickness of the layers at any point is higher or lower than the thickness at x = 0. The frequencies as seen in the corresponding figures are correspondingly higher and lower. Similar remarks apply to Fig. 5, which pertain to sinusoidal variation in thickness. The meridional section of the layers is convex or concave according as C_s = ± 0.25. The range of thickness parameter chosen carefully so that the thickness does not vanish or become negative anywhere and the thin shell assumption are valid.

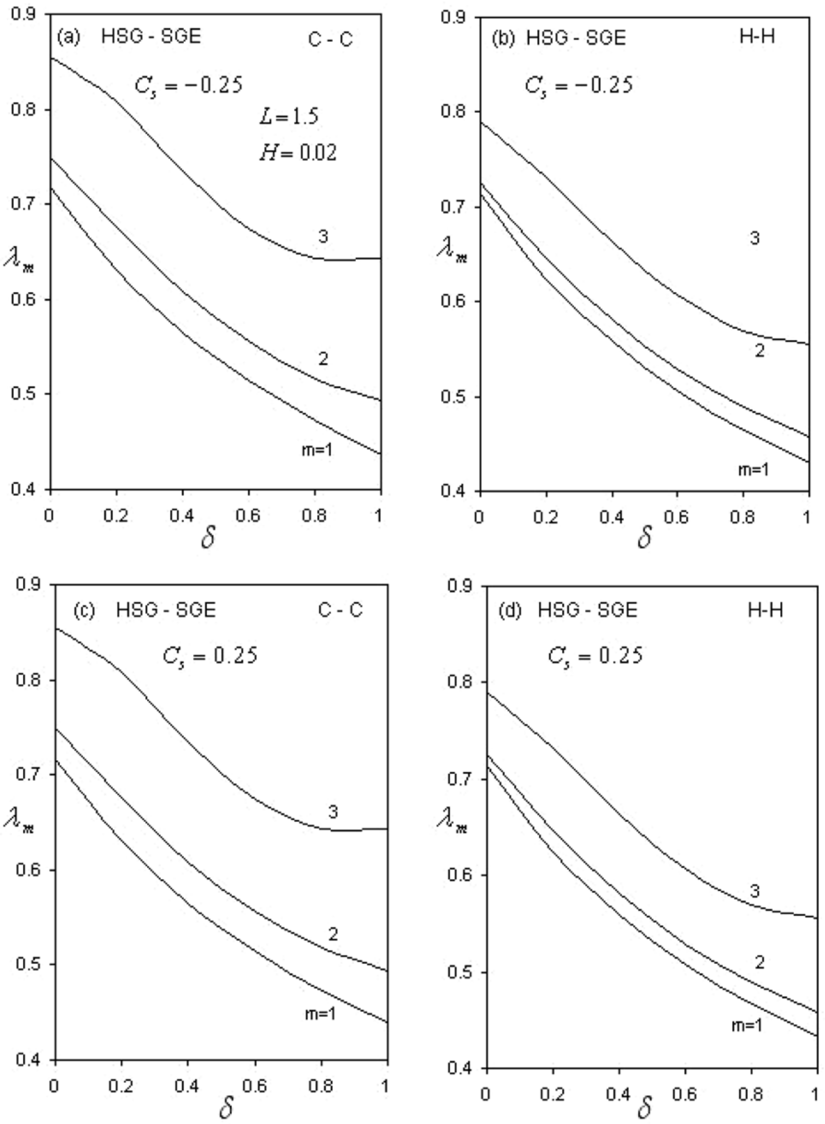

Fig. 5 Variation of frequency parameter with relative layer thickness: Cylindrical shells of sinusoidal variation in thickness of layers . Layer materials: HSG-SGE

In Figs. 6 and 7 the influence of the nature of the variation of thickness of the layers of the shell on its vibrational behavior is studied. An HSG-SGE shell held under two types of boundary

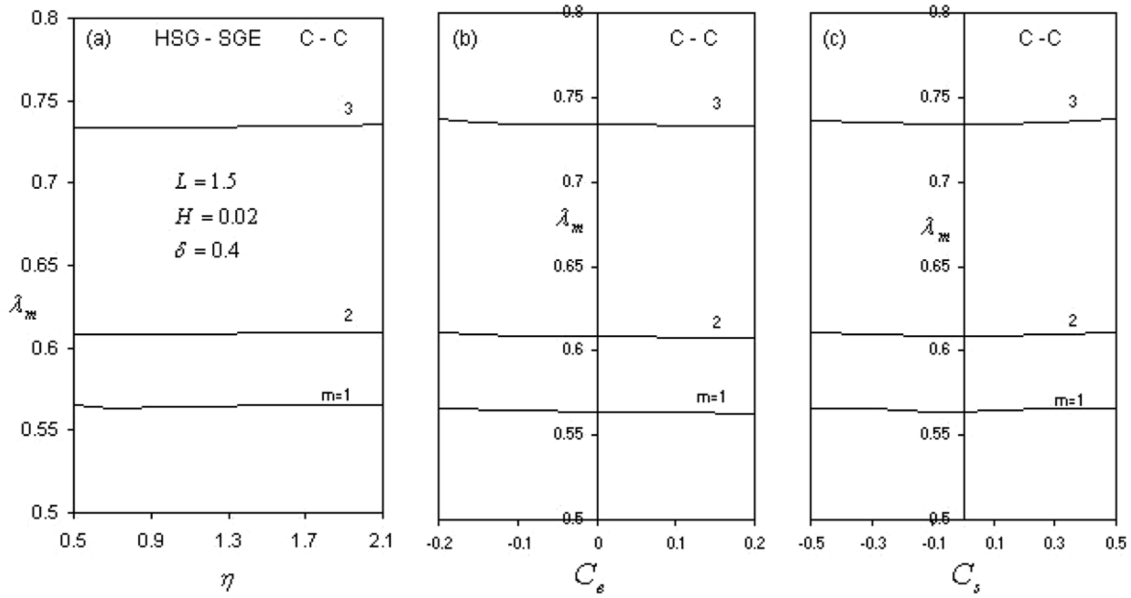

Fig. 6 (a) Effect of taper parameter on frequency parameter, (b) Effect of coefficient of exponential variation on frequency parameter, (c) Effect of coefficient of sinusoidal variation on frequency parameter. C-C boundary conditions.

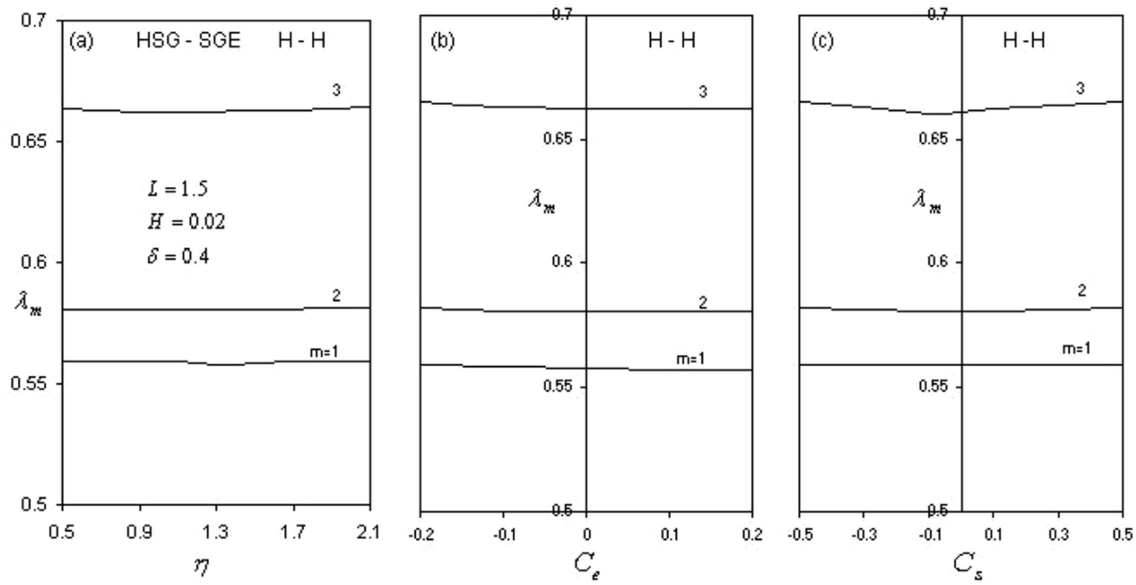

Fig. 7 (a) Effect of taper parameter on frequency parameter, (b) Effect of coefficient of exponential variation on frequency parameter, (c) Effect of coefficient of sinusoidal variation on frequency parameter. H-H boundary conditions

conditions with three types of variation in thickness of layers is considered, with L = 1.5, H = 0.02 and δ = 0.4. Fig. 6(a) relates to linear variation in thickness of layers under C-C boundary conditions. The thickness is constant when the taper ratio η = 1. Variation of $\lambda_m(m = 1, 2, 3)$ with respect to η for $0.5 \leq \eta \leq 2.1$ is studied. It is seen that λ_m is almost constant for all the values of η. The percent changes induced in λ_1, λ_2, λ_3 over the range of values of η considered under C-C boundary condition is 0.0376%, 0.201%, 0.247%. The effect of exponential variation in thickness of layers is analyzed in Fig. 6(b). When C_e = 0, thickness is uniform. The thickness at the end $x = \ell$ of the cylinder is higher or lower than the thickness at the other end x = 0 according as $C_e \lessgtr 0$. The percent changes induced in λ_1, λ_2, λ_3 over the range of values of C_e under C-C boundary condition is 0.487%, 0.419%, 0.466%. The effect of sinusoidal variation in thickness of layers on frequency parameters is studied in Fig. 6(c). These effects are almost similar to those due to the exponential variation just discussed. Here the coefficient of thickness variation is considered over the range [−0.5, 0.5].

In Fig. 7 the influence of the taper ratio η, the coefficient of exponential variation of thickness C_e and the coefficient of sinusoidal variation C_s on λ_m are depicted, along with the effect of the H-H boundary conditions. The effect of λ_m is almost same for all the cases of linear and exponential variation as described in Fig. 6. But the variation of $\lambda_m(m = 3)$ decrease slowly between $-0.5 \leq C_s \leq -0.2$ and steady between $-0.1 \leq C_s \leq 0.1$ and then increase afterwards. In this variation the C-C boundary conditions contribute slightly higher values to the influence of the coefficients of thickness variation on frequencies than the H-H conditions contributing values to the influence of the coefficients of thickness variation on frequencies.

The frequency parameter λ is explicitly a function of the length ℓ of the cylinder. Hence, when studying the influence of the length of the cylinder on its vibrational behavior, the actual frequency ω and not λ is considered. Fig. 8 describes how the length parameter L affects ω (in 10^3 Hz) for HSG-SGE layered cylindrical shells under C-C conditions, with H = 0.02 and δ = 0.4. Linear,

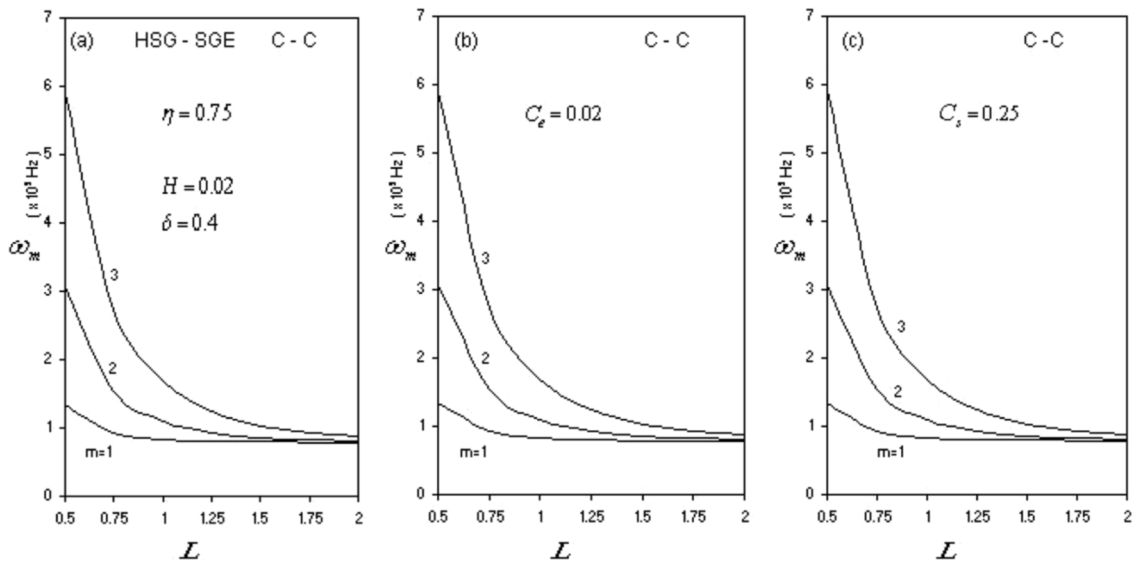

Fig. 8 Variation of frequency parameter with length parameter under C-C boundary conditions . Layer materials: HSG-SGE. (a) Linear variation, (b) Exponential variation, (c) Sinusoidal variation in thickness of layers

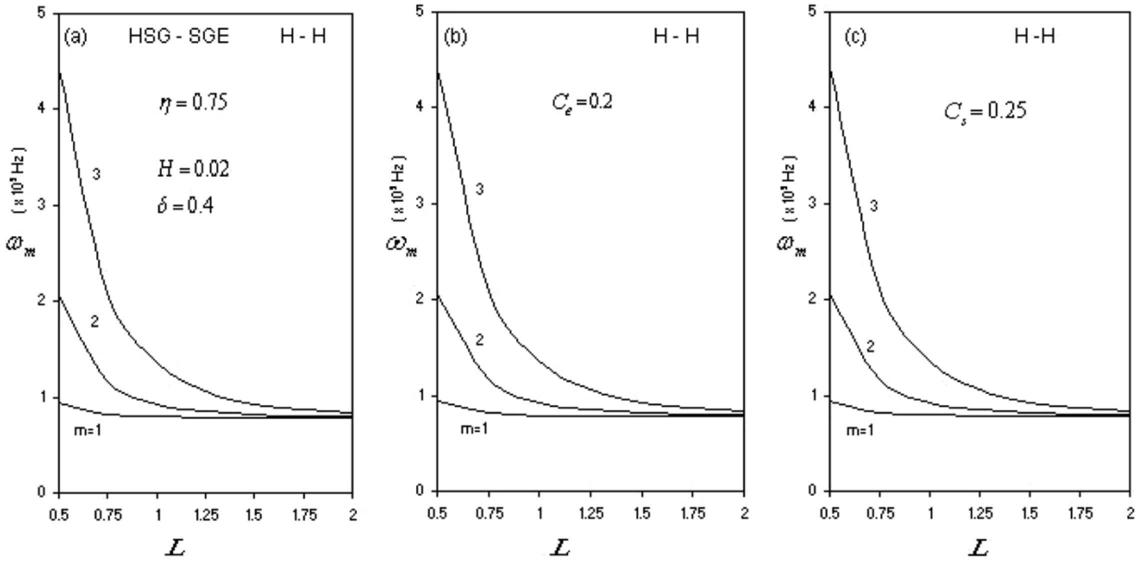

Fig. 9 Variation of frequency parameter with length parameter under H-H boundary conditions. Layer materials: HSG-SGE. (a) Linear variation, (b) Exponential variation, (c) Sinusoidal variation in thickness of layers

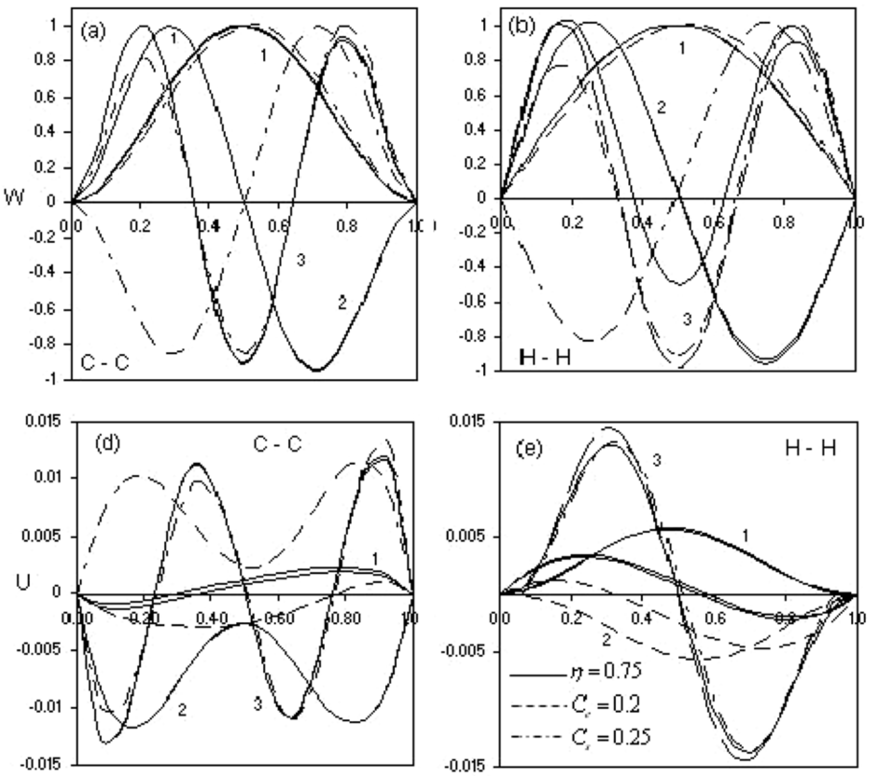

Fig. 10 Mode shapes of axisymmetric vibration of cylindrical shells of different types of variation in thickness of layers. (a), (d) C-C boundary conditions, (b), (e) H-H boundary conditions

exponential and sinusoidal thickness variations are analyzed. It can be seen that ω decreases as L increases. The decrease is fast for very short shells, the rate of decrease increasing with higher modes. There is no rapid change in the rate of decrease of ω in the interval $0.65 < L < 0.95$ after which the decrease is very low. The fundamental frequencies are almost constant for $L > 0.95$. The similar situations arise in the Fig. 9 which is analyzed for H-H boundary conditions considering the materials made up of HSG-SGE.

Certain mode shapes of HSG-SGE shell are presented in Fig. 10. All the two types of boundary conditions and all the three types of variation in thickness of layers are considered as indicated. The transverse displacements predominate in all of them. Both the transverse and radial displacements are normalized with respect to the maximum transverse displacements. Up to the third modes of vibration are presented. The displacement curves of any particular mode for all the three types of thickness variation are seen close to each other. The shells considered are of $L = 1.5$, $H = 0.02$ and $\delta = 0.4$ with thickness coefficients $\eta = 0.75$, $C_e = 0.2$ and $C_s = 0.25$.

5. Conclusions

The vibrational behavior of layered cylindrical shell of variable thickness is studied. Linear, exponential and sinusoidal variations are discussed assuming two types of layered materials for two layered shells. Frequencies may vary with the relative thickness of layers, the nature of variation of their thickness, the length ratio of the cylindrical shell and the boundary conditions. The frequency parameter values tend to decrease, in general, with increase of length of the shell. This provides scope for one to choose suitable thickness proportions among the given materials to achieve the desired vibrational behaviour.

The study also shows the elegance and usefulness of the spline function collocation method for boundary value problems.

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Notation

A_{ij} : Elastic coefficients representing the extensional rigidity
B_{ij} : Elastic coefficients representing the coupling between bending and stretching
C_e : Coefficient of exponential variation
C_ℓ : Coefficient of linear variation
C_s : Coefficient of sinusoidal variation
C_{ij} : Elastic coefficients representing the flexural rigidity
L_{ij} : Differential operator occurring in the equations of motion
L_{ij}^* : Differential operator occurring in the equations of motion
N : Number of intervals of spline interpolation
R_0 : Inertial coefficient of a leyered shell

U : Extensional displacement function
W : Normal displacement function
X : Non-dimensional meridional distance co-ordinate
$h_k(x)$: Thickness of the k-th layer of the shell at any x
h_0 : Constant thickness
ℓ : Length of the cylindrical shell
m : Meridional mode number
r : Radius of the cylinder
t : Time coordinate
u : Meridional displacement of the deformed reference surface
v : Circumferential displacement of the deformed reference surface
w : Normal displacement of the deformed reference surface
x : Meridional coordinate of any point on the shell
z : Normal coordinate of any point on the shell
z_k : Distance of the top of the k-th layer from the reference surface
δ, δ_1 : Relative layer thickness h_1/h
λ : Nondimensional frequency parameter
ρ : Mass density of the material of the shell
θ : Circumferential coordinate
ω : Circular frequency of motion
η : Taper ratio in the case of linear variation of thickness

Abbreviations

C-C : Both the ends fully clamped
H-H : Both the ends hinged
HSG : High strength graphite epoxy
PRD : PRD-49-111 Epoxy
SGE : S-glass epoxy