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Stochastic analysis of external and parametric dynamical systems under sub-Gaussian Lévy white-noise

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Abstract. In this study stochastic analysis of non-linear dynamical systems under α-stable, multiplicative white noise has been conducted. The analysis has dealt with a special class of α-stable stochastic processes namely sub-Gaussian white noises. In this setting the governing equation either of the probability density function or of the characteristic function of the dynamical response may be obtained considering the dynamical system forced by a Gaussian white noise with an uncertain factor with $\alpha/2$-stable distribution. This consideration yields the probability density function or the characteristic function of the response by means of a simple integral involving the probability density function of the system under Gaussian white noise and the probability density function of the $\alpha/2$-stable random parameter. Some numerical applications have been reported assessing the reliability of the proposed formulation. Moreover a proper way to perform digital simulation of the sub-Gaussian α-stable random process preventing dynamical systems from numerical overflows has been reported and discussed in detail.

Keywords: Lévy white noise; stochastic differential calculus; Fokker-Planck equation; sub-Gaussian white noise.

1. Introduction

Normal white noises are very popular stochastic processes and they have been used to model several type of physical phenomena. The main feature of such processes is that they may be defined as formal time derivative of Wiener processes. In this setting the powerful machinery established with Itô stochastic differential calculus (Itô 1956) may be used to yield probabilistic characterization of dynamical response of systems driven by normal white noises in terms of stochastic moments. On the other hand an alternative probabilistic characterization may be provided solving the Fokker-Planck-Kolmogorov (FPK) differential equation yielding the conditional probability density function (PDF) of the response or the corresponding Fourier transform, namely the characteristic function of the response (CF) (Stratonovich 1967, Lin 1976). However, several real phenomena observed in physics, seismology, electrical engineering, economics and in some other research fields show evident non-Gaussianity observed in heavy tail distributions or in the impulsive nature of the recorded samples. The need for non-Gaussian models to describe the fluctuations exhibited by such a phenomena has risen the interest in the so-called α-stable Lévy processes (Samorodnitsky *et al.* 1994, Griguriu 1995a,b). This kind of stochastic processes are characterized by the knowledge of

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the four parameters α, σ, β, μ, which are, respectively, the stability index, the scale factor, the skewness and the shift. Variation of the stability index $\alpha \in (0, 2]$ yields a wide class of stochastic processes including the Gaussian white noise obtained for $\alpha = 2$. Some interesting applications of α-stable Lévy processes may be also found in fractal models of materials at micro-scale (see e.g. Carpinteri *et al.* 2003).

On one hand linear and non-linear systems driven by external α-stable Lévy processes (formal time derivative of the Lévy motion processes) have been treated in the past either in terms of PDF via the fractional Einstein-Smoluchowsky (ES) equation obtained with the aid of fractional differential calculus (see e.g. Hilfer 2000) or in terms of CF (Chechkin *et al.* 2002, Grigoriu 2004). On the other hand closed-form expressions for the stationary CF have been obtained only for some specific values of the stability index (α = 1, 1/2) and some numerical procedures have been proposed in terms of equivalent linearization (Grigoriu 2000) or by means of numerical wavelet approach (Di Paola *et al.* 2005).

The main challenge in the analysis of dynamical systems in presence of α-stable Lévy white noises is related to the divergence of statistical moments of the α-stable random variables L_α, namely $E[L_\alpha^p] = \infty$ if $p \geq \alpha$. Unbounded statistical moments are strictly related with the heavy-tails distributions of the random variables L_α (Samorodnitsk *et al.* 2003) and some sharp jumps at the onset of possible realization of α-stable Lévy noises must be expected. This feature makes Monte-Carlo methods of non-linear dynamical systems highly unstable since numerical overflows in numerical simulations. In order to overcome this major drawback some truncations of the PDF of the α-stable random variables L_α have been recently proposed (Sokolov *et al.* 2004) in which a power law with exponent $5 - \alpha$ has been used to truncate the PDF of an α-stable distribution. In this context a modified fractional FPK equation has been obtained and the probability density function of the response converge towards a Gaussian density in the central part.

Some remarkable contribution to the theory of stochastic differential equation (SDE) driven by stable noises has recently been reported in scientific literature (Bass *et al.* 2006, Imkeller *et al.* 2006, Schertzer *et al.* 2001, Yanovsky *et al.* 2000).

In this paper analysis of dynamical systems under parametric α-stable excitation has been reported in presence of a special class of symmetric α-stable excitation namely the sub-Gaussian α-stable processes. Such processes are defined by the product of a zero mean normal process $W_2(t)$ with prescribed second-order correlation and the square root of an $\alpha/2$-stable random variable $(\alpha \leq 2)$ $A_{\alpha/2}$, totally skewed to the right $(\beta = 1)$ and independent of $W_2(t)$. The Gaussian factor $W_2(t)$ is dubbed *underlying Gaussian process* and the sub-Gaussian white noise may be defined as the formal derivative of an α-stable sub-Gaussian Wiener process $dB_\alpha(t) = A_{\alpha/2}^{1/2} dB_2(t)$. Analysis of dynamical systems under parametric-type excitation is performed by a 1-dimensional integral involving the PDF of the $\alpha/2$-stable variable and the PDF of the response of the dynamical system subjected to Gaussian white noise $W_2(t)$. This latter consideration stems out assuming that the dynamical system is forced by a Gaussian white noise $W_2(t)$ with uncertain parameter $A_{\alpha/2}^{1/2}$.

Some numerical applications involving non-linear parametric oscillator have been reported contrasting the PDF obtained by the solution of the governing equation with the estimate obtained via Monte-Carlo simulation suitably modified to deal with α-stable excitation. In more detail we introduce a proper truncation of heavy tails of the forcing α-stable random process neglecting the second-order probability of occurrence of outcomes of order $n_\alpha/\Delta t^{1/2}$ with Δt the integration step of numerical integration and n_α a real number smaller than $(1/\Delta t)^{1/2}$. The proposed Monte-Carlo analysis yields accurate estimation of PDF and CF once opportune choices of n_α have been selected preventing numerical overflows of the dynamical system.

2. The case of α-stable white noise external excitation

Let us consider a scalar, non-linear dynamical system under parametric white noise with the equation of motion written as

$$\dot{Z} = f(Z,t) + g(Z,t)\tilde{W}_\alpha(t) \tag{1}$$

with $f(Z,t)$ and $g(Z,t)$ deterministic non-linear functions of the random process Z and of time t. The random process $\tilde{W}_\alpha(t)$ in Eq. (1) is defined by the formal derivative of $B_\alpha(t)$ ($\tilde{W}_\alpha = dB_\alpha(t)/dt$) which is a random process with the following properties

i) It has independent, stationary increments following the α-stable distribution, that is

$$B_\alpha(t) - B_\alpha(s) \sim S_\alpha((t-s)^{1/2}/\sqrt{2}, 0, 0) \tag{2a}$$

ii) The CF of an increment of $dB_\alpha(t) = A^{1/2} dB_2(t)$, takes the form

$$\phi_{dB_\alpha}(\theta) = \exp[-(dt/2)^{\alpha/2}|\theta|^\alpha] \tag{2b}$$

where $dB_2(t)$ represents the increment of the Wiener process and A is an $\alpha/2$ stable random variable totally skewed on the right $A \sim S_{\alpha/2}((\cos[\pi\alpha/4]^{2/\alpha}), 1, 0)$ independent of $B_2(t)$ then $B_\alpha(t)$ is a sub-Gaussian random process.

iii) For $\alpha = 2$, $B_\alpha(t) \to B_2(t)$ that is $A = 1$ with probability 1 and then the non-normal α-stable process reverts to normal white noise process, or in other words the Wiener process is a particular case of the random process $B_\alpha(t)$.

It has to be emphasized that the Lévy motion process $L_\alpha(t)$ whose formal derivative is the well-known Lévy white noise $W_\alpha(t)$ possesses the properties i) and iii) but the process $dL_\alpha(t)$ exhibits a quite different CF, that is $\phi_{dL_\alpha}(\theta) = \exp[-dt|\theta|^\alpha]$, that is the scales of $dL_\alpha(t)$ and $dB_\alpha(t)$ are different. Only selecting $\alpha = 2$ the two process have exactly the same scale (for $\alpha \equiv 2$, $dL_\alpha \equiv dB_\alpha \equiv dB_2$).

Let us assume that the dynamical system reported in Eq. (1) written in Itô form is forced by an increment $dL_\alpha(t)$ instead of $dB_\alpha(t)$. Let us also suppose that the excitation is external namely the non-linear function $g(Z,t) = g(t)$. In this context the equation ruling the evolution of the PDF is the so-called Einstein-Smoluchowsky (ES) equation involving Riesz-Weil fractional derivative in the diffusion term, that is

$$\frac{\partial p_Z(z,t)}{\partial t} = -\frac{\partial}{\partial z}(p_Z(z,t)f(z,t)) + g^r(t)\frac{\partial^\alpha}{\partial |z|^\alpha} p_Z(z,t) \tag{3}$$

in which $\partial/\partial|z|^\alpha$ is the symmetric fractional space derivative which is defined for a "sufficiently well-behaved" function through its Fourier transform $\Im[\bullet]$ (Samko *et al.* 1987)

$$\Im\left[\frac{\partial^{\alpha} p_Z(z,t)}{\partial |z|^{\alpha}}\right] = -|\theta|^{\alpha}\phi_Z(\theta,t) \tag{4}$$

or in terms of the Riemann-Liouville derivatives as

$$\frac{\partial^{\alpha} p_Z(z,t)}{\partial |z|^{\alpha}} = -\frac{1}{2\cos(\pi\alpha/2)}[D_{+}^{\alpha} p_Z(z,t) + D_{-}^{\alpha} p_Z(z,t)] \tag{5}$$

where $\alpha > 0$ and if $0 < \alpha < 1$ then Riemann-Liouville derivatives reads

$$D_{+}^{\alpha} p_Z(z,t) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial z}\int_{-\infty}^{z}\frac{p_Z(\xi,t)}{(z-\xi)^{\alpha}}d\xi;\quad D_{-}^{\alpha} p_Z(z,t) = -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial z}\int_{z}^{\infty}\frac{p_Z(\xi,t)}{(\xi-z)^{\alpha}}d\xi \tag{6a,b}$$

else for $\alpha > 1$

$$D_{\pm}^{\alpha} p_Z(z,t) = \frac{(\pm 1)^n}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial z^n}\int_{-\infty}^{z}\xi^{n-\alpha-1} p_Z(z \mp \xi, t)d\xi \tag{7}$$

with $n-1 < \alpha \le n \in \mathbb{N}$. For $\alpha = 1$ $D_{\pm}^{\alpha} p_Z(z,t)$ is the Hilbert transform of the probability density function $p_Z(z,t)$. The derivatives in Eqs. (6), (7) are characterized in the Fourier transform space as

$$\Im[D_{\pm}^{\alpha} p_Z(z,t)] = (\mp i\theta)^{\alpha}\phi_Z(\theta,t) \tag{8}$$

where

$$(-i\theta)^{\alpha} = |\theta|^{\alpha}\exp\left[\mp\frac{i\alpha\pi}{2}\mathrm{sgn}(\theta)\right] \tag{9}$$

and with some algebraic manipulations of Eq. (3) by means of Eq. (4) the spectral counterpart of the ES equation is provided in the form

$$\frac{\partial\phi_Z(\theta,t)}{\partial t} = i\theta E[f(Z,t)\exp(i\theta Z)] - |\theta|^{\alpha}\phi_Z(\theta,t) \tag{10}$$

Such an equation in the following termed as *spectral* ES has been derived in the scientific literature (Dittlevsen 2005, Grigoriu *et al.* 2005). The ES equation holds for Lèvy white noise input that is in presence of increments of Lèvy white noise $dL_{\alpha}(t)$ with scale *dt*. The question is how to solve the case in which the input is the sub-Gaussian white noise with increment $dB_{\alpha}(t)$? Answer to this question will be provided in the next section.

3. The sub-Gaussian α-stable input process

In this section the study of dynamical systems forced by sub-Gaussian Lèvy process will be reported. In sec.3.1 the analysis will be reported for the case of external excitation exploiting the proposed method. Analysis of parametric-type excitation will be reported in sec.3.2.

3.1 The sub-Gaussian external excitation

Let us recast the equation of motion in Eq. (1) for external excitation, $g(Z,t) = g(t) = 1$, without loss of generality in Itô form as

$$dZ = f(Z,t)dt + dB_\alpha(t) \tag{11}$$

with $dB_\alpha(t) = W_\alpha dt$ representing a sub-Gaussian α-stable increment of Lèvy motion. In the following this kind of α-stable increment denoted dB_α will be obtained by the product $dB_\alpha(t) = A_{\alpha/2}^{1/2} dB_2(t)$, namely by the product of the square root of an $\alpha/2$-stable random variable and a time increment of Brownian motion $dB_2(t) = W_2(t)dt$. Random process $W_2(t)$ is the normal white noise characterized by the correlation $R_{W_2}(t_1,t_2) = q_2\delta(t_2-t_1)$ and $\delta(\bullet)$ is a Dirac delta function, q_2 is the strength of the white noise.

In this case the evolution in terms of CF or the equation ruling the PDF may be obtained in two steps (Di Paola *et al.* 2008). In the first step the CF or PDF function of the dynamical system excited by a normal white noise with deterministic amplitude

$$d\tilde{Z} = f(\tilde{Z},t)dt + a^{1/2}dB_2(t) \tag{12}$$

will be obtained. In this case the CF equation and the corresponding Fokker-Planck equations are written as

$$\frac{\partial\phi_{\tilde{Z}}(\theta;a,t)}{\partial t} = i\theta E[e^{i\theta\tilde{Z}} f(\tilde{Z}(a,t))] - \frac{aq_2}{2}\theta^2\phi_{\tilde{Z}}(\theta;a,t) \tag{13}$$

more explicitely if $f(Z,t)$ takes a polynomial form of the type $f(Z,t) = \sum_{j=1}^{N} c_j(t)Z^j$ then Eq. (13) is written as

$$\frac{\partial\phi_{\tilde{Z}}(\theta;a,t)}{\partial t} = i\theta\sum_{j=1}^{N}(-1)^j c_j(t)\frac{\partial^j\phi_{\tilde{Z}}(\theta;a,t)}{\partial\theta^j} - \frac{aq_2\theta^2}{2}\phi_{\tilde{Z}}(\theta;a,t) \tag{14}$$

or in terms of the Fokker-Planck-Kolmogorov (FPK) equation

$$\frac{\partial p_{\tilde{Z}}(z;a,t)}{\partial t} = -\frac{\partial}{\partial z}(f(z(a,t)),t)p_{\tilde{Z}}(z;a,t) + \frac{q_2}{2}a\frac{\partial^2 p_{\tilde{Z}}(z;a,t)}{\partial z^2} \tag{15}$$

As soon as the CF and PDF have been obtained Eqs. (13)-(15) may be considered as a stochastic differential equation with random coefficient $A^{1/2}$. In this perspective the PDF and CF of the dynamical response will be provided solving the 1-dimensional integrals

$$p_Z(z;t) = \int_0^\infty p_A(a)p_{\tilde{Z}}(z;a,t)da \tag{16}$$

$$\phi_Z(\theta;t) = \int_0^\infty p_A(a)\phi_{\tilde{Z}}(\theta;a,t)da \tag{17}$$

If linear system is considered superposition principle holds and letting $a = 1$ in Eq. (11) the response of the differential equation $\dot{\tilde{Z}} = C_1\tilde{Z} + W_2(t)$ is Gaussian then the response process $Z(t) = A_{\alpha/2}^{1/2}\tilde{Z}(t)$ is sub-Gaussian α-stable.

Multi-degree of freedom linear dynamical systems under sub-Gaussian excitations may be studied with similar arguments. In more detail introducing a n-dimensional vector $\mathbf{Z}(t)$ collecting the

Lagrangian parameters of the system $\mathbf{Z}(t) = [Z_1(t)\ Z_2(t)\ \dots\ Z_n(t)]$, the CF of the response is provided as

$$E[\exp(i\boldsymbol{\theta}^T\mathbf{Z})] = \phi_{\mathbf{Z}}(\boldsymbol{\theta}) = \exp[-|\boldsymbol{\theta}^T\mathbf{R}_{\tilde{\mathbf{Z}}}\boldsymbol{\theta}|^{\alpha/2}] \tag{18}$$

with $\boldsymbol{\theta} = [\theta_1\ \theta_2\ \dots\ \theta_n]$ and $\mathbf{R}_{\tilde{\mathbf{Z}}}$ is the correlation matrix of the Gaussian vector process $\tilde{\mathbf{Z}}(t)$ ($\mathbf{R}_{\tilde{\mathbf{Z}}} = E[\tilde{\mathbf{Z}}\tilde{\mathbf{Z}}^T]$) evaluated with well-known methods (Falsone 1994).

3.2 The sub-Gaussian parametric excitation

In this section analysis of a parametric dynamical system driven by α-stable sub-Gaussian white noise input will be considered.

Let us consider the equation of motion of a scalar dynamical system in the form

$$\dot{Z} = f(Z,t) + g(Z,t)A^{1/2}W_2(t) \tag{19}$$

As a first step the coefficient $A^{1/2}$ is considered a deterministic parameter and the differential equation is rewritten in Itô form as

$$d\tilde{Z} = m(\tilde{Z},t)dt + a^{1/2}g(\tilde{Z},t)dB_2(t) \tag{20}$$

In which the drift term $m(\tilde{Z},t)$ has been modified taking into account the Wong-Zakaj (WZ) or Stratonovich (S) correction term (Stratonovich 1967) as

$$m(\tilde{Z},t) = f(\tilde{Z},t) + \frac{q_2 a}{2}g'(\tilde{Z},t)g(\tilde{Z},t) \tag{21}$$

Itô differential rule of functions $\psi(\tilde{Z},t)$ of the dynamic response reads

$$d\psi(\tilde{Z},t) = \frac{d\psi}{\partial t}dt + \frac{d\psi}{\partial\tilde{Z}}d\tilde{Z} + \frac{1}{2}\frac{\partial^2\psi}{\partial\tilde{Z}^2}(d\tilde{Z})^2 \tag{22}$$

The third term in the right-hand side of Eq. (22) is essential since $dB_2(t)$ is of order of magnitude $(dt)^{1/2}$ and term $(d\tilde{Z})^2$ is of the same order of the first term. By letting $\psi(\tilde{Z},t) = \exp[i\theta\tilde{Z}]$ in Eq. (22), taking mathematical expectation and accounting for the non-anticipative property of Itô calculus $E[\psi(\tilde{Z},t)dB_2^k] = E[\psi(\tilde{Z},t)]E[dB_2^k]$ the differential equation ruling the evolution of the CF is readily written as

$$\frac{\partial}{\partial t}\phi_{\tilde{Z}}(\theta;a,t) = i\theta E[\exp(i\theta\tilde{Z})m(\tilde{Z},t)] - \frac{\theta^2 q_2 a}{2}E[\exp(i\theta\tilde{Z})g(\tilde{Z},t)] \tag{23}$$

The FPK equation ruling the evolution of the PDF of the response $\tilde{Z}(t)$ may be obtained by inverse Fourier transform of Eq. (23) yielding

$$\frac{\partial}{\partial t}p_{\tilde{Z}}(\tilde{z};a,t) = -\frac{\partial}{\partial\tilde{z}}[p_{\tilde{Z}}(\tilde{z};a,t)m(\tilde{z},t)] + \frac{q_2}{2}a\frac{\partial^2[p_{\tilde{Z}}(\tilde{z};a,t)g^2(z,t)]}{\partial\tilde{z}^2} \tag{24}$$

Solutions of Eqs. (23), (24) for the CF or the PDF of $\tilde{Z}(t)$, respectively, will be used in Eq. (17) to yield the CF and the PDF of the dynamical system in Eq. (19).

The proposed method to deal with parametrically excited scalar dynamical systems may be extended straightforwardly to analysis of multi-degree of freedom systems. In this context let us suppose to consider a nonlinear, parametrically excited multi degree of freedom system ruled by the differential equation system as

$$\dot{\mathbf{Z}}(t) = \mathbf{f}(\mathbf{Z}(t), t) + \mathbf{g}(\mathbf{Z}(t), t) A_{\alpha/2}^{1/2} W_2(t) \tag{25}$$

with $\mathbf{Z}(t)$ an $(n \times 1)$ vector collecting the dynamic response $\mathbf{f}(\mathbf{Z}, t)$ is an n-vector collecting linear and nonlinear function of the response and $\mathbf{g}(\mathbf{Z}, t)$ is a $(n \times 1)$ vector of parametric functions of the response.

The PDF or the CF of the system in Eq. (25) is provided by method already described considering the dynamical system $\tilde{\mathbf{Z}}(t)$, excited by a parametric normal white noise with deterministic amplitude $a^{1/2}$. The corresponding Itô equation reads

$$d\tilde{\mathbf{Z}}(t) = \mathbf{f}(\tilde{\mathbf{Z}}, t)dt + a^{1/2}\mathbf{g}(\tilde{\mathbf{Z}}, t)dB_2(t) \tag{26}$$

The FPK equation associated to the dynamical system in Eq. (26) reads

$$\frac{\partial p_{\tilde{\mathbf{Z}}}(\tilde{\mathbf{z}}, t)}{\partial t} = -\nabla_{\tilde{\mathbf{z}}}^{T}(\mathbf{m}(\tilde{\mathbf{z}}, t)p_{\tilde{\mathbf{Z}}}(\tilde{\mathbf{z}}, t)) + \frac{aq_2}{2}\nabla_{\tilde{\mathbf{z}}}^{T^{[2]}}(\mathbf{g}^{[2]}(\tilde{\mathbf{z}}, t)p_{\tilde{\mathbf{Z}}}(\tilde{\mathbf{z}}, t)) \tag{27}$$

where $\nabla_{\tilde{\mathbf{Z}}}^{T} = [\partial/\partial\tilde{z}_1 \;\; \partial/\partial\tilde{z}_2 \; \ldots \; \partial/\partial\tilde{z}_1]$ is the gradient operator, $[\bullet] \otimes [\bullet]$ is the Kronecker product and $[\bullet]^{[2]} = [\bullet] \otimes [\bullet]$ is the Kronecker power. The drift term in Eq. (27) $\mathbf{m}(\tilde{\mathbf{Z}}, t)$ is represented by

$$\mathbf{m}(\tilde{\mathbf{z}}, t) = \mathbf{f}(\tilde{\mathbf{z}}, t) + \frac{aq_2}{2}(\nabla_{\tilde{\mathbf{z}}}^{T}\mathbf{g}(\tilde{\mathbf{z}}, t))\mathbf{g}(\tilde{\mathbf{z}}, t) \tag{28}$$

in which the WZ correction term has been incorporated.

Solution of the FPK equation associated to vector $\tilde{\mathbf{Z}}(t)$ yields the multidimensional PDF $p_{\tilde{\mathbf{Z}}}(\tilde{\mathbf{z}}, t; a)$ and the PDF of the vector $\mathbf{Z}(t)$ may be obtained solving the one-dimensional integral

$$p_{\mathbf{Z}}(\mathbf{z}, t) = \int_0^{\infty} p_A(a) p_{\tilde{\mathbf{Z}}}(\tilde{\mathbf{z}}, t; a) da \tag{29}$$

Probabilistic characterization in terms of CF is similar and it has not been reported for brevity's sake.

4. Order of the increments of sub-Gaussian Lévy white noise

The proposed method of analysis has been assessed via Monte-Carlo simulation. Monte-Carlo analysis of dynamical systems in presence of stable Lèvy white noises deserves some considerations about the order of increments of the excitation process. In more detail let us recast Eq. (11) in discrete form as

$$Z(t_{k+1}) = f(Z_k, t_k)\Delta t + \Delta B_{\alpha}(t_k) \tag{30a}$$

with the stochastic increment $\Delta B_{\alpha}(t_k)$ reported in the second term of Eq. (30a) given as

$$\Delta B_\alpha(t_k) = L_\alpha(t_k)\Delta t^{1/2} = A_{\alpha/2}^{1/2}\Delta B_2(t_k) = A_{\alpha/2}^{1/2}G(t_k)\Delta t^{1/2} \tag{30b}$$

where $L_\alpha(t_k)$ is the realization of an α-stable sub-Gaussian random variable obtained by product of a Gaussian random variable $G(t_k)$ and an $\alpha/2$-stable random variable $A_{\alpha/2}^{1/2}$. At first glance the increments $\Delta B_\alpha(t_k)$ in Eq. (30a) seems of order $\Delta t^{1/2}$ as for normal white noise and then all the rules of the Itô calculus may be used for. However the order of $\Delta B_\alpha(t_k)$ is quite different from $\Delta t^{1/2}$ as it may be noticed by the CF of $\Delta B_\alpha(t_k)$ which reads

$$\phi_{\Delta B_\alpha}(\theta) = \exp[-(\Delta t)^{\alpha/2}|\theta|^\alpha\bar{\sigma}_G^\alpha] \tag{31}$$

where $\bar{\sigma}_G = \sigma_G/\sqrt{2}$ and σ_G is the standard deviation of the Gaussian random variable $G(t_k)$.

From Eq. (31) we may observe that statistical moments of $\Delta B_\alpha(t)$ do not exists since $\phi_{\Delta B_\alpha}(\theta)$ exhibits a discontinuity in zero which is related to the presence of heavy tails of the PDF of $p_{L_\alpha}(l_\alpha)$.

On the other hand probability that $L_\alpha(t_k)$ is larger than $(n_\alpha/\Delta t^{1/2})$ is provided by the expression

$$P(|L_\alpha(t_k)| > (n_\alpha/\Delta t^{1/2\alpha})) = C_\alpha[n_\alpha/\Delta t^{1/2}]^{-\alpha} = C_\alpha\left(\frac{n_\alpha}{\Delta t^{1/2}}\right)^{-\alpha}; \quad (\Delta t \to 0) \tag{32}$$

where $C_\alpha = \left(\int_0^\infty x^{-\alpha}\sin(x)dx\right)^{-1}$ is a real number dependent of the stability index α ranging from zero ($\alpha = 2$) and one ($\alpha = 0$) and the real coefficient n_α in Eq. (32) is selected with requirement $n_\alpha \le 1/\Delta t^{1/2}$. This latter condition is necessary to maintain the order of probability in Eq. (32) always of order $\Delta t^{1/2}$, neglecting probabilities of occurrence of $L_\alpha(t_k)$ with higher order (Di Paola *et al.* 2007).

This latter assumption lead us to argue that very small probability of occurrence of $L_\alpha(t_k)$ are excluded from the analysis introducing a truncation of the tails of the PDF $p_{L_\alpha}(l_\alpha)$ for $L_\alpha(t_k) > (n_\alpha/\Delta t^{1/2})$. In this context the statistical moments of increments $dB_\alpha(t)$ may be evaluated with the expression

$$E[dB_\alpha^{2j}] = \lim_{\Delta t\to 0} E[\Delta B_\alpha^{2j}] = \lim_{\Delta t\to 0} 2\Delta t^j\int_0^{(n_\alpha/\Delta t^{1/2})} l_\alpha^{2j} p_{L_\alpha}(l_\alpha)dl_\alpha = K_{2j}(n_\alpha)dt^{\alpha/2} \tag{33a}$$

$$E[dB_\alpha^{2j-1}] = 0 \tag{33b}$$

As for example the order of increments of the sub-Gaussian Cauchy random process ($\alpha = 1$) reads

$$E[dB_1^{2j}] = \lim_{\Delta t\to 0} 2\Delta t^j\int_0^{(n_\alpha/\Delta t^{1/2})} l_\alpha^{2j}\frac{1}{\pi(l_\alpha^2+1)}dl_\alpha = \frac{2n_\alpha^{2j-1}}{(2j-1)\pi}dt^{1/2} \tag{34a}$$

$$E[dB_1^{2j-1}] = 0 \tag{34b}$$

From a different perspective we may observe that requirement in Eq. (32) yields the geometrical condition that the area under the impulse dB_α between the time instants t and $t + dt$ is of order

$n_\alpha dt^{1/2}$ as in the case of the Wiener process in which area is of order $B_2 dt^{1/2}$ with B_2 a Gaussian random variable.

The proposed model to evaluate the stochastic moments of the increments of the sub-Gaussian may be used to represent the characteristic function of the increments as

$$\phi_{dB_\alpha}(\theta) = \exp[-(dt/2)^{\alpha/2}|\theta|^\alpha] \simeq 1 - \left(\frac{dt}{2}\right)^{\alpha/2}|\theta|^\alpha = \exp\left[-\sum_{j=0}^{\infty}\frac{(i\theta)^{2j}}{2j!}E[dB_\alpha^{2j}]\right] \quad (35)$$

which yields, introducing Eqs. (33a,b) into the latter Eq. (35) and performing manipulation the expression of $|\theta|^\alpha$ as

$$|\theta|^\alpha = \lim_{n_\alpha\to\infty}\sum_{j=1}^{\infty}\frac{(i\theta)^{2j}K_{2j}(n_\alpha)}{2j!} \quad (36)$$

specifying values of $\alpha = 1$ Eq. (36) yields

$$|\theta| = \lim_{n_\alpha\to\infty} 2\sum_{j=1}^{\infty}\frac{(i\theta)^{2j}}{2j!}\frac{n_\alpha^{2j-1}}{\pi(2j-1)} = \lim_{n_\alpha\to\infty}\left[\frac{e^{-n_\alpha^2\theta^2}}{n_\alpha\sqrt{\pi}} - \frac{1}{n_\alpha\sqrt{\pi}} + \theta\,\mathrm{erf}(n_\alpha\theta)\right] \quad (37)$$

with the function $\mathrm{erf}(x) = 2\int_0^x e^{-\xi^2}d\xi$. Representation of the term $|\theta|$ for different values of the coefficient n_α has been reproduced in Fig. 1 and it may be observed that as soon as n_α increases the convergence of Eq. (37) become more and more accurate.

The proposed representation of sub-Gaussian Levy process has been applied to perform Monte-Carlo simulation to represent the stationary characteristic function for the linear system $\dot{Z} = aZ + bW_\alpha(t)$ $(a = -2, b = 1)$ for different values of the coefficient n_α. It may be observed (Fig. 2) that the exact CF (Grigoriu 1995a); Di Paola *et al.* 2003) is obtained setting values of n_α = 25 in the Monte-Carlo simulation whereas with n_α = 5 the characteristic functions are not coinciding.

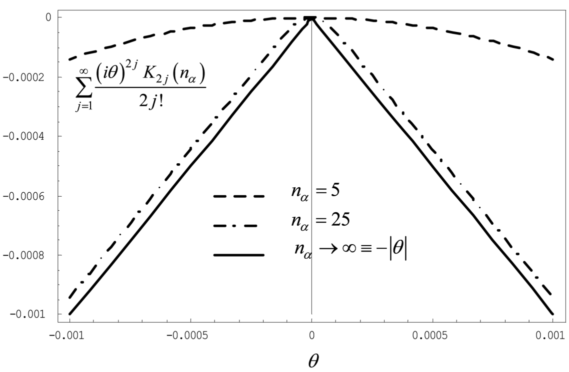


Fig. 1 Convergence of series in Eq. (33) to the argument $|\theta|$ in the expression of $\phi_{dB_\alpha}(\theta)$ (Eq. 31) for different values of n_α

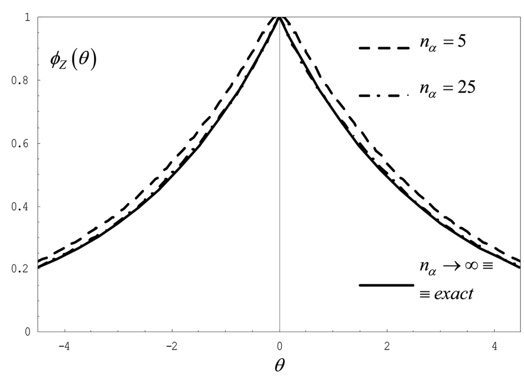


Fig. 2 Stationary characteristic function $\phi_Z(\theta)$ of a linear system under sub-Gausian white noise obtained via Monte-Carlo simulation for different values of n_α contrasted with exact expression

It is worth noticing that in the framework of Monte Carlo simulation, the heavy tails of the PDF may yield some very large deviates of $L_\alpha(t_k)$ leading to a numerical overflow. Monte-Carlo analysis will be conducted, preventing numerical instabilities, with the proposed clip of the tails of the PDF assuming convenient value of the coefficient n_α. In more detail the coefficient n_α will be selected such that the truncation of the series in Eq. (36) yields accurate results in terms of the $|\theta|^\alpha$ function and we may disregard all values of $L_\alpha \geq (n_\alpha/\Delta t^{1/2})$.

5. Numerical application

Non-linear dynamical systems under external and parametric sub-Gaussian white noises have been examined in this section. Monte-Carlo simulation has been performed with the proposed truncation of the tails of the PDF of the random variable $L_\alpha \geq (n_\alpha/\Delta t^{1/2})$.

Let us consider the nonlinear cubic oscillator ruled by the differential equation

$$\dot{Z} = -\rho Z^3 + W(t); \quad \rho > 0 \tag{38}$$

with selected parameters $\rho = -1$; $S_0 = 1$; $q_2 = 2\pi S_0$. Under the assumption that $W(t)$ is a normal white noise the FPK equation is written as

$$\frac{\partial p_Z(z,t)}{\partial t} = \frac{\partial}{\partial z}(-\rho z^3 p_Z(z,t)) + \frac{q_2}{2}\frac{\partial^2}{\partial z^2}p_z(z,t) \tag{39}$$

The steady-state solution of this equation is

$$p_Z(z,t) = C\exp[-\rho z^4/2q_2] \tag{40}$$

where constant C may be obtained from normalization condition $(C = \sqrt[4]{(\rho q_2)2^3}/\Gamma(1/4))$ with $\Gamma(\bullet)$ the well-known Gamma function. On the other hand Fourier transform of Eq. (39) the CF equation is provided as (see Eq. (18) with $a = 1, C_3 = -1$)

$$\frac{\partial \phi_Z(z,t)}{\partial t} = \theta\frac{\partial^3 \phi_Z(\theta,t)}{\partial \theta^3} - \frac{\theta^2 q_2}{2}\phi_Z(\theta,t) \tag{41}$$

the steady-state response is readily obtained using the appropriate boundary conditions which for this case reads

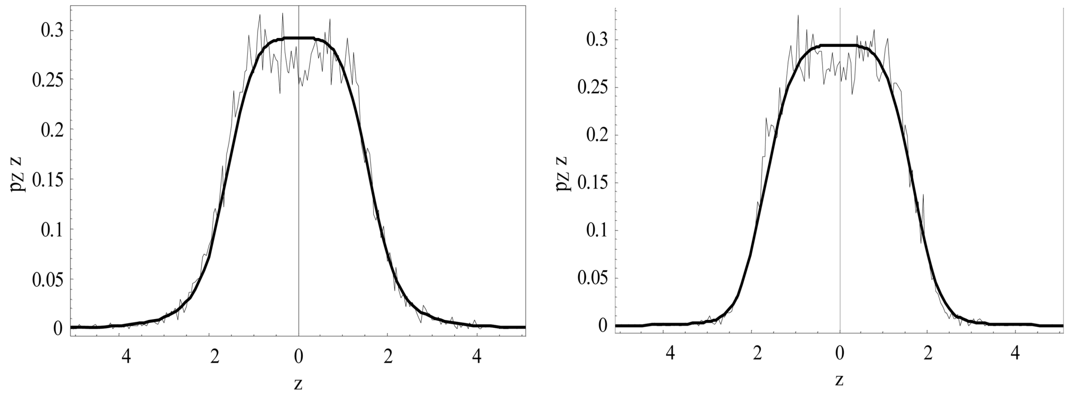
$$\phi(0) = 1 \tag{42a}$$

$$\left.\frac{d^2\phi_Z(\theta)}{d\theta^2}\right|_{\theta=0} = -E[Z^2] \tag{42b}$$

which yields a CF that coincides with the Fourier transform of the PDF reported in Eq. (40).

In the framework of $W_2(t)$ Lévy white noise the Einstein-Smoluchowsky equation is written as

$$\frac{\partial p_Z(z,t)}{\partial t} = -Q\left(\frac{Q}{\rho}\right)^{\frac{2-\alpha}{2+\alpha}}\frac{\partial}{\partial z}(\rho z^3 p_Z(z,t)) + Q\frac{\partial^\alpha}{\partial |z|^\alpha}p_Z(z,t) \tag{43}$$



(a) (b)

Fig. 3 (a) PDF via Eq. (46) contrasted with MC simulation $\alpha = 1.5$, (b) PDF via Eq. (46) contrasted with MC simulation $\alpha = 1.8$

where Q is the strength of the Lèvy noise in the following set $Q = 1$ for sake of simplicity. Fourier transform of Eq. (43) maps the PDF in the Fourier domain yielding the differential equation for the CF in the form

$$\frac{\partial \phi_Z(\theta, t)}{\partial t} = \theta\rho \frac{\partial^3 \phi_Z(z,t)}{\partial \theta^3} - |\theta|^\alpha \phi_Z(\theta, t) \tag{44}$$

with the steady-state solution in the form

$$\phi_Z(\theta) = \frac{2}{\sqrt{3}} \exp\left(-\frac{|\theta|}{2}\right) \cos\left(\frac{\sqrt{3}}{2}\theta - \frac{\pi}{6}\right) \tag{45}$$

whose inverse Fourier transform is the PDF of the oscillator in Eq. (38) as

$$p_Z(z) = \frac{1}{\pi((1/\rho)^4 - (z/\rho)^2 + z^4)} \tag{46}$$

In the case of sub-Gaussian oscillator $dB_\alpha = dt^{1/2} A^{1/2} W_2(t)$ where $W_2(t)$ is a realization of a Gaussian random process. The PDF corresponding of to the generic realization of the random variable A is (Di Paola *et al.* 2003)

$$p_{\tilde{Z}}(z;a) = C(a) \exp[-\rho z^4 / 2aq_2] \tag{47}$$

where the normalization condition is $C(a) = \sqrt[4]{2^3 (q_2 \rho / a)} / \Gamma(a/4)$ and then

$$p_Z(z) = \int_0^\infty p_A(a) C(a) \exp[-\rho z^4 / 2aq_2] da \tag{48}$$

Now let us suppose that a new state variable is introduced as a nonlinear transformation as $Y = Z^2$. In this case the equation of motion of the nonlinear dynamical system reported in Eq. (35) is transformed in the parametric-type differential equation

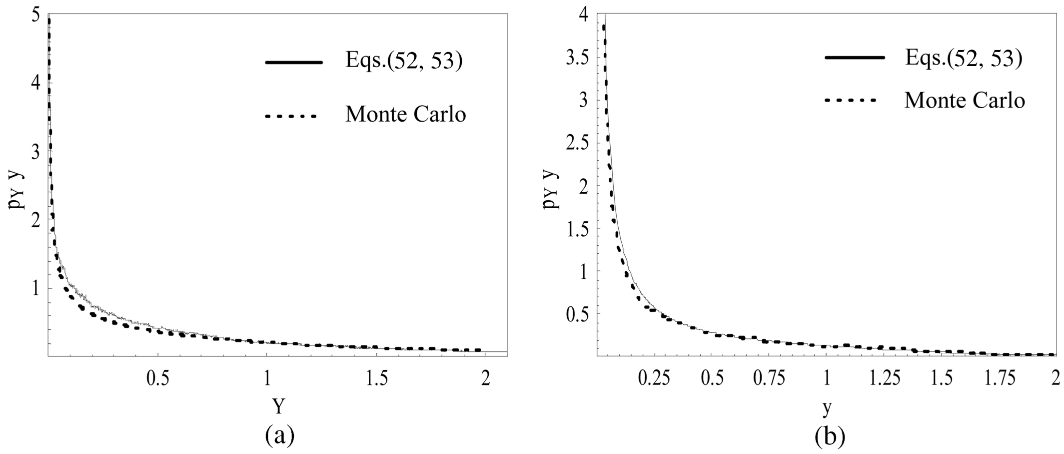


Fig. 4 (a) PDF via Eqs. (52), (53) contrasted with M.C. simulation α = 0.8, (b) PDF via Eqs. (52), (53) contrasted with M.C. simulation α = 1.8

$$\dot{Y} = -2\rho Y^2 + A^{1/2}\sqrt{Y}W_2(t) \tag{49}$$

which is formally similar to the parametric differential equation reported in Eq. (19). Probabilistic characterization of dynamical system reported in Eq. (49) is obtained, according to sect.4, solving the FPK equation ruling the evolution of the dynamical system driven by the underlying Gaussian process $W_2(t)$

$$\dot{\tilde{Y}} = -2\rho\tilde{Y}^2 + a^{1/2}\sqrt{\tilde{Y}}\,W_2(t) \tag{50}$$

The FPK equation for the PDF of Eq. (50) is of the form

$$\frac{\partial p_{\tilde{Y}}(\tilde{y}, t;a)}{\partial t} = \frac{\partial}{\partial \tilde{y}}[(2\rho\tilde{y}^2 + aq_2)p_{\tilde{Y}}(\tilde{y}, t;a)] + 2aq_2\frac{\partial^2}{\partial \tilde{y}^2}[\tilde{y}p_{\tilde{Y}}(\tilde{y}, t;a)] \tag{51}$$

with the stationary solution provided, under the assumption of zero probability flow (Cai *et al.* 1995), by the PDF in the form

$$p_{\tilde{Y}}(\tilde{y};a) = \frac{C(a)}{\sqrt{\tilde{y}}}\exp[-\rho\tilde{y}^2/2aq_2] \tag{52}$$

with constant $C(a)$ obtained via normalization condition. The probabilistic characterization of the dynamical response $Y(a)$ of dynamical system in Eq. (48) is obtained via Monte-Carlo estimation

$$p_Y(y) = E[p_{\tilde{Y}}(\tilde{y};a)] \cong \frac{1}{N}\sum_{j=1}^{N} p_{\tilde{Y}}(\tilde{y}, a^{(j)}) \tag{53}$$

with $p_{\tilde{Y}}(\tilde{y}, a^{(j)})$ represents the pdf obtained for the jth realization of the $\alpha/2$-stable variable A. Observation of Figs. (4a,b) shows the coincidence of the benchmark solution via simulation and the proposed approach in the exploited range of occurrence of the random process $Y(t)$.

6. Conclusions

In this paper the analysis of parametrically excited non-linear dynamical systems excited by α-stable processes has been carried out. The governing equation of the probability density function of parametrically excited dynamical systems has been never formulated for α-stable excitation and in this paper the α-stable process considered belongs to the special class of sub-Gaussian process. Sub-Gaussian stochastic processes are provided by the product of an $\alpha/2$-stable variable $A \sim S_{\alpha/2}$ $((\cos[\pi\alpha/4])^{2/\alpha}, 1, 0)$ totally skewed to the right and a white noise with Gaussian distribution. In this framework the differential equation governing the evolution of the probability density function, or its Fourier transform counterpart, namely the characteristic function, may be obtained considering that $\alpha/2$-stable variable A is an uncertain parameter multiplying a normal white noise. The governing equation of the probability density function is the well-known Fokker-Planck-Kolmogorov equation, for parametric-type excitation in which the uncertain parameter A is involved.

The probability density function of the dynamic response is then obtained in two steps:

i. Evaluation of the probability density function of the parametrically excited dynamical system under normal white noise treating $\alpha/2$-stable variable A as a parameter.

ii. Evaluation of a simple integral involving the probability density function of the $\alpha/2$-stable variable A and the probability density function of the system under the normal white noise.

The proposed methodology has been used to evaluate the dynamical response of a non-linear system under parametric-type excitation contrasting the response with an estimate via Monte-Carlo simulation suitable modified to prevent numerical instabilities due to large deviates of α-stable excitation.

References

Cai, G.Q. and Lin, Y.K. (1995), *Probabilistic Structural Dynamics: Advanced Theory and Applications*, McGraw-Hill, NY.

Carpinteri, A., Chiaia, B. and Cornetti, P. (2003), "On the mechanics of quasi brittle-materials with a fractal microstructure", *Eng. Frac. Mech.*, **70**, 2321-2349.

Chechkin, A., Gonchar, V., Klafter, J., Metzler, R. and Tanatarov, L. (2002), "Stationary states of nonlinear oscillators driven by Lévy noise", *Chem. Phys.*, **284**, 233-251.

Di Paola, M. and Failla, G. (2005), "Stochastic response of linear and nonlinear systems to α-stable Lèvy white noises", *Prob. Eng. Mech.*, **20**(2), 128-135.

Di Paola, M. and Sofi, A. (2008), "Linear and nonlinear systems under sub-Gaussian (α-stable) input", *Prob. Eng. Mech. (submitted)*.

Di Paola, M., Pirrotta, A. and Zingales, M. (2007), "Ito calculus extended to systems driven by alpha-stable Lévy white noises (A novel clip on the tails of Lévy motion)", *Int. J. Nln. Mech*., **42**, 1046-1054.

Falsone, G. (1994), "Cumulants and correlations for linear systems under non-stationary delta-correlated processes", *Prob. Eng. Mech.*, **9**, 157-165.

Grigoriu, M. (1995a), "Linear systems subject to non-Gaussian α-stable processes", *Prob. Eng. Mech.,* **10**, 23-34.

Grigoriu, M. (1995b), "Linear and nonlinear systems with non-Gaussian white noise input", *Prob. Eng. Mech.,* **10**, 171-179.

Grigoriu, M. (2000), "Equivalent linearization for systems driven by Lèvy white noise", *Prob. Eng. Mech.*, **15**, 285-190.

Grigoriu, M. (2004), "Characteristic function equations for the state of dynamic systems with Gaussian, poisson and Lévy white noise", *Prob. Eng. Mech.*, 449-461.

Hilfer, R. (ed.) (2000), *Fractional Calculus in Phyisics*, World Scientific, Singapore.

Lin, Y.K. (1976), *Probabilistic Theory of Structural Dynamics,* Kriegher, FL.

Samko, S.G., Kilbas, A.A. and Marichev, O.I. (1993), *Fractional Integrals and Derivatives: Theory and Applications*, Gordon & Breach Science Publishers, Amsterdam, NL.

Samorodnitsky, G. and Grigoriu, M. (2003), "Tails of solutions of certain nonlinear stochastic differential equations driven by heavy tailed Lèvy motions", *Stoch. Proc. Appl.*, **105**, 69-97.

Samorodnitsky, G. and Taqqu, M.S. (1994), *Stable non-Gaussian Random Processes*, *Stochastic Models with Infinite Variance*, Chapman and Hall, UK.

Sokolov, I.M., Chechkin, A.V. and Klafter, J. (2004), "Fractional diffusion equation for a power-law-truncated Lèvy process", *Phys. A*, **336**, 245-251.

Stratonovich, R.L. (1967), *Topics in the Theory of Random Noise*, Gordon & Breach, NY.