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Impact of uncertain natural vibration period on quantile of seismic demand

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Abstract. This study investigates effect of uncertainty in natural vibration period on the seismic demand. It is shown that since this uncertainty affects the acceleration and displacement responses differently, two ratios, one relating peak acceleration responses and the other relating the peak displacement responses, are not equal and both must be employed in evaluating and defining the critical seismic demand. The evaluation of the ratios is carried out using more than 200 strong ground motion records. The results suggest that the uncertainty in the natural vibration period impacts significantly the statistics of the ratios relating the peak responses. By using the statistics of the ratios, a procedure and sets of empirical equations are developed for estimating the probability consistent seismic demand for both linear and nonlinear systems.

Keywords: peak linear elastic responses; peak inelastic responses; probability; seismic demand; uncertainty.

1. Introduction

The demand of earthquake excitations on structures is commonly represented by the peak displacement, velocity and acceleration of a series of single-degree-of-freedom (SDOF) systems. These peak quantities are related through the natural vibration period. Given the natural vibration period and the damping ratio, if one of the peak responses is known, the other two peak responses can be calculated directly. The peak responses can be represented in a tri-partite plot (or four-way logarithmic plot) (Veletsos and Newmark 1960, Chopra 2000), and in the acceleration-displacement response spectra (ADRS) format (Freeman *et al.* 1975) which was developed to aid the performance-based design and assessment (or capacity spectrum method) (ATC 1996). The evaluation of the peak linear elastic responses for a given set of ground motion records can be carried out using one of the well-known time step integration methods (Chopra 2000). These peak responses together with seismic risk analysis and safety consideration (Cornell 1968, Newmark and Rosenblueth 1970, McGuire 1974) are used to select the design response spectrum for linear elastic

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SDOF systems. The peak responses obtained from the design response spectrum are divided by the yield reduction factor to provide the minimum required yield strength for designing inelastic systems. The yield reduction factor, which depends on the ductility factor, is assessed using peak linear elastic and peak inelastic responses obtained from strong ground motion records.

The design response spectrum, traditionally, is defined as the product of the standard design spectrum and return period values of a peak ground motion parameter such as the peak ground acceleration or peak ground velocity. This approach may lead to the design response spectrum to be probability-inconsistent for structures of different natural vibration periods. To overcome this inconsistency, the use of the so-called uniform hazard spectra (UHS), which ensure that the probabilities of exceedance of peak responses of linear elastic SDOF systems are uniform for all possible natural vibration periods, has been considered in the literature (McGuire 1974, Frankel *et al*. 1996, Adams *et al*. 1999, Wang and Hong 2005).

The assessment of the yield reduction factor has been presented by many including Veletsos and Newmark (1960), Krawinkler and Nassar (1992), Vidic *et al*. (1994), Miranda (2000) and Riddell *et al*. (2002). This yield reduction factor is selected based on the statistics of the ratios between the peak linear elastic responses, and the yield or the peak inelastic responses. These ratios vary from record to record since the seismic excitation is a stochastic process. Further, it should be noted that the predicted structural properties such as the natural vibration period T_n and the damping ratio ξ are uncertain as well (Haviland 1976). Since these uncertainties affect the peak or yield responses, they should be considered for assessing the ratios, hence, the seismic design responses or seismic demand. The consideration of these uncertainties for evaluating the displacement-based ratios was presented by Hong and Jiang (2004) for elastoplastic hysteretic systems. Based on the results obtained by using more than 200 strong ground motion records, they indicated that the uncertainty in T_n affects significantly the statistics of the ratios for the displacement responses, while the influence of the uncertainty in ξ on the estimated ratios is relatively small. They also provided simple empirical equations for evaluating the mean and the coefficient of variation of the ratios for peak displacement responses.

In the present study, it is shown that if there is uncertainty in T_n, one must use two sets of ratios, one developed based on the acceleration responses and the other developed based on the displacement responses, to assess the seismic demand. This is because that the uncertainty in T_n affects the displacement and the acceleration responses in different ways. Evaluation and discussion of the statistics of these ratios are presented by considering more than 200 strong ground motion records. Using the obtained statistics of the ratios, a procedure and sets of empirical equations for estimating quantiles of the seismic demand are given. Details for deriving these empirical equations are presented in the following section.

2. Ratios between the peak linear elastic and inelastic responses

2.1 Relation between peak elastic and inelastic responses

Let $D_E(T_n, \xi)$, $D_I(T_n, \xi, \mu)$ and $D_y(T_n, \xi, \mu)$ denote the peak linear elastic, the peak inelastic and the required yield displacement for given ductility factor μ and damping ratio ξ. Further, let $A_E(T_n, \xi)$, and $A_y(T_n, \xi, \mu)$ denote the peak (pseudo-) acceleration responses corresponding to $D_E(T_n, \xi)$, and $D_y(T_n, \xi, \mu)$, respectively. Given a strong ground motion record, by definition the following holds

$$D_E(T_n, \xi) = \left(\frac{T_n}{2\pi}\right)^2 A_E(T_n, \xi) \tag{1}$$

and

$$D_y(T_n, \xi, \mu) = \left(\frac{T_n}{2\pi}\right)^2 A_y(T_n, \xi, \mu) \tag{2}$$

Consider that T_n and ξ are uncertain with mean values denoted by m_T and m_ξ, respectively. Define the ratios $R_y(\mu)$ and $R_\mu(\mu)$ based on the peak displacement responses by the following equations

$$R_y(\mu) = D_E(m_T, m_\xi) / D_y(T_n, \xi, \mu) \tag{3}$$

and

$$R_\mu(\mu) = D_I(T_n, \xi, \mu) / D_E(m_T, m_\xi) \tag{4}$$

These ratios are commonly considered in the literature, especially the ratio $R_y(\mu)$ that is known as yield (displacement) reduction factor. Also, define the ratios $\psi_y(\mu)$ and $\psi_\mu(\mu)$ based on the peak acceleration responses by

$$\psi_y(\mu) = A_E(m_T, m_\xi) / A_y(T_n, \xi, \mu) \tag{5}$$

and

$$\psi_\mu(\mu) = \mu\, A_y(T_n, \xi, \mu) / A_E(m_T, m_\xi) \tag{6}$$

The ratio $\psi_y(\mu)$ can be thought as yield force reduction factor. Using the above equations, it can be shown that $\psi_y(\mu) = (T_n/m_T)^2 R_y(\mu)$ and that $\psi_\mu(\mu) = (m_T/T_n)^2 R_\mu(\mu)$. This indicates that if there is no uncertainty in T_n, the ratios defined based on the peak displacement responses are equal to those that are defined based on the peak acceleration responses. However, if there is uncertainty in T_n, $\psi_y(\mu)$ is not equal to $R_y(\mu)$ and, $\psi_\mu(\mu)$ is not equal to $R_\mu(\mu)$ and the means of $\psi_y(\mu)$ and $\psi_\mu(\mu)$ are not equal to the means of $R_y(\mu)$ and $R_\mu(\mu)$. Further, $E(\psi_y(\mu))$ is not equal to the product of $E((T_n/m_T)^2)$ and $E(R_y(\mu))$, and $E(\psi_\mu(\mu))$ is not equal to the product of $E((m_T/T_n)^2)$ and $E(R_\mu(\mu))$, where $E(\bullet)$ denotes the expectation. This is because that $R_y(\mu)$ and $R_\mu(\mu)$ are not independent of T_n. Therefore, one must use two sets of ratios, one for the peak displacement responses and the other for the peak acceleration responses, to characterize the inelastic peak responses if there is uncertainty in T_n.

Note that the linear elastic response spectrum derived based on the seismic risk analysis is usually available by considering T_n as a deterministic quantity. Using this spectrum and the ratios defined in Eq. (3) to Eq. (6), one can conveniently evaluate the peak linear elastic and nonlinear responses with or without uncertainty in T_n. Note also that since the displacement ductility factor m equals the ratio of the peak inelastic displacement to the yield displacement, based on the definition given in Eqs. (3) and (4) one has $R_y(\mu) = \mu / R_\mu(\mu)$. Therefore, one only needs to provide probabilistic characterization of $R_y(\mu)$ or $R_\mu(\mu)$ to evaluate $D_I(T_n, \xi, \mu)$ and $D_y(T_n, \xi, \mu)$. Similarly, one only needs to provide probabilistic characterization of $\psi_y(\mu)$ or $\psi_\mu(\mu)$ to evaluate $A_y(T_n, \xi, \mu)$. The use of $R_\mu(\mu)$ and $\psi_\mu(\mu)$ is considered in the present study, and the assessment of $\psi_\mu(\mu)$ is given in the following.

Since $\psi_\mu(\mu)$ equals $R_\mu(\mu)$ if there is uncertainty in ξ only and, the influence of the uncertainty in ξ on $R_\mu(\mu)$ is less significant than that due to the uncertainty in T_n (Hong and Jiang 2004), therefore, in the present study the uncertainty in ξ is ignored. Further, for numerical evaluation through out this study, it is considered that ξ equals 0.05, and that the structure system can be modeled as a SDOF elastoplastic hysteretic system.

2.2 Statistics of $\psi_\mu(\mu)$

As mentioned previously, the assessment of the yield displacement reduction factor $R_\mu(\mu)$ considering the uncertainty in T_n and ξ was given in Hong and Jiang (2004). Their recommended statistical characterization of $R_\mu(\mu)$ is adopted in the present study. In their assessment, a total of 230 (components of) records, that were used by Miranda (2000) and are found in the database prepared by Silva (2001), were considered. For consistency, the same set of records and procedure used in Hong and Jiang (2004) for assessing $R_\mu(\mu)$ are used to evaluate $\psi_\mu(\mu)$. These records are for the Kern County, Borrego Mountain, San Fernando, Imperial Valley, Morgan Hill, Whittier, Loma Prieta, Landers, Northridge earthquakes.

For the probabilistic analysis, the assumption, that the ratio of the measured to predicted T_n is a lognormal variate with coefficient of variation (cov) values ranging from about 0.2 to 0.4, is adopted. This assumption is based on the studies given by Haviland (1976), and Davenport and Hill-Carroll (1986). Further, only ξ equal to 0.05 is considered in the present study.

To show the impact of the uncertainty in T_n on $\psi_\mu(\mu)$, consider that the mean of the actual to the predicted vibration period equal to unit, and its cov, v_T, equal to 0.2. Let $m_\psi(\mu, m_T, v_T)$ denote the mean of $\psi_\mu(\mu)$ obtained by considering that T_n has a mean of m_T and a cov of v_T. The calculated values of $m_\psi(m_T, v_T, \mu)$ for v_T = 0.2 are shown in Fig. 1 and compared with those obtained without considering the uncertainty in T_n, (i.e., for v_T = 0.0). For a particular value of the ductility factor and a mean of T_n (denoted by $E(T_n)$ or m_T), the values of $m_\psi(m_T, v_T, \mu)$ presented in the figure were obtained based on 50 × 230 simulated sample points (i.e., 50 simulation cycles and 230 records).

The results shown in Fig. 1 suggest that in most cases the mean of $\psi_\mu(\mu)$ obtained by considering the uncertainty in T_n, $m_\psi(m_T, v_T, \mu)$, is higher than that without considering the

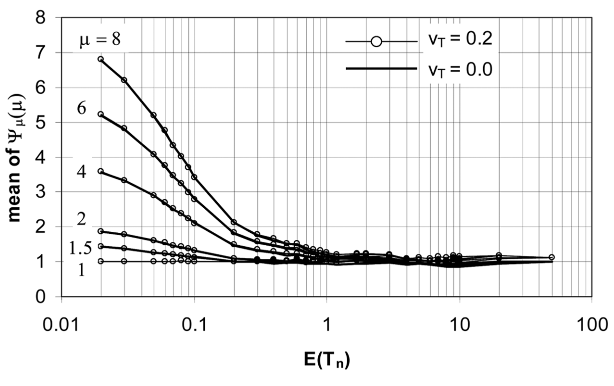


Fig. 1 Mean of the ratio $\psi_\mu(\mu)$ for v_T = 0.2

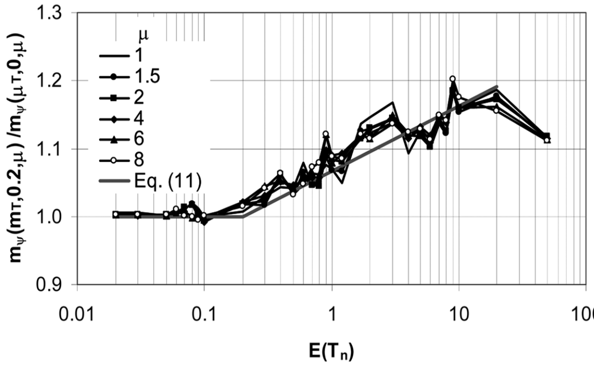
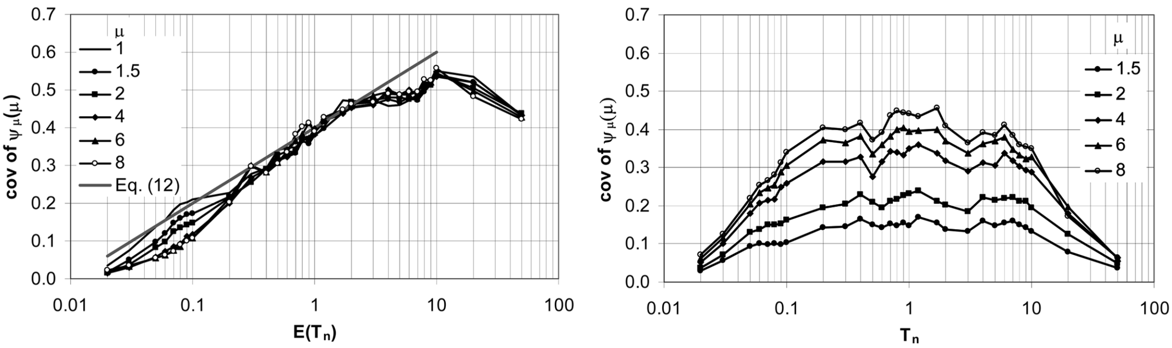


Fig. 2 Ratio of the mean of $\psi_\mu(\mu)$ with uncertainty in T_n to that with out uncertainty in T_n, $m_\psi(m_T, 0.2, \xi)/m_\psi(m_T, 0, \xi)$



(a) (b)

Fig. 3 Coefficient of variation of $\psi_\mu(\mu)$; (a) for $v_T = 0.2$, (b) for $v_T = 0.0$

uncertainty. To better appreciate this, the ratio between $m_\psi(m_T, 0.2, \mu)$ and $m_\psi(m_T, 0, \mu)$ is calculated and shown in Fig. 2. The results depicted in the figure suggest that $m_\psi(m_T, 0.2, \mu)/m_\psi(m_T, 0, \mu)$ remains almost equal to 1.0 for m_T less than 0.2 (sec), increases almost as a linear function of $\ln(m_T)$ for m_T ranging from about 0.2 to 10 and decreases for m_T greater than 10. This observed trend appears to be independent of the ductility factor.

It must be emphasized that through out this study a wide range of vibration period values is considered for plotting since such plots could be used to guide the selection of the empirical relations for the mean and cov of the ratios. However, since the records are band filtered at about 0.1 to 0.5 Hz and about 20 to 40 Hz, the conclusions drawn from the analysis results should be limited to about 0.05 to 10 s.

The coefficient of variation of $\psi_\mu(\mu)$ by considering the uncertainty in T_n is shown in Fig. 3(a) for $v_T = 0.2$. Comparison of the results shown in this figure and those shown in Fig. 3(b) for $v_T = 0.0$ suggests that the consideration of the uncertainty in T_n reduces the differences between the cov of $\psi_\mu(\mu)$ for different ductility levels. In such a case, the cov of $\psi_\mu(\mu)$ could be approximately considered to be independent of μ. The highest cov value of $\psi_\mu(\mu)$ that is approximately equal to 3 times the cov of T_n occurs at about m_T equal to 20 (sec). Note that cov of $\psi_\mu(\mu)$ for $v_T = 0.2$ is equal to that of for $\psi_\mu(\mu)$ for $v_T = 0.0$ if T_n is very small.

The above simulation analysis was repeated for v_T equal to 0.4. The obtained results are presented

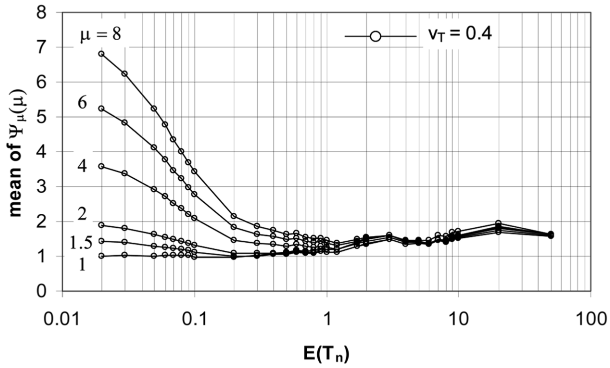


Fig. 4 Mean of the ratio $\psi_\mu(\mu)$ for v_T = 0.4

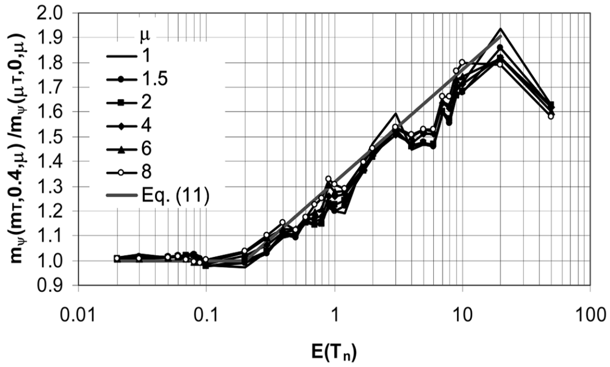


Fig. 5 Ratio of the mean of $\psi_\mu(\mu)$ with uncertainty to that with out uncertainty, $m_\psi(m_T, 0.4, \xi)/m_\psi(m_T, 0, \xi)$

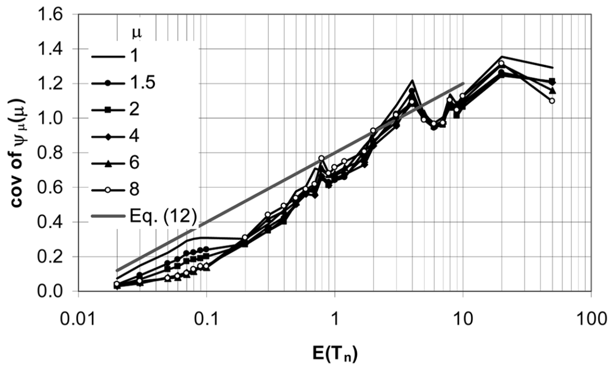


Fig. 6 Coefficient of variation of $\psi_\mu(\mu)$ for v_T = 0.4

in Figs. 4 to 6. In all cases, the observations drawn from Figs. 1 to 3 seem to be equally applicable to the results presented in these figures.

2.3 Empirical equations for $\psi_\mu(\mu)$

For the mean of $R_\mu(\mu)$, $m_R(m_T, v_T, \mu)$, the empirical equation given in Hong and Jiang (2004) was

$$m_R(m_T, v_T, \mu) = 1/(1+(1/(\mu+c_1 v_T)-1)\exp(-c\mu^{-\gamma} m_T^{\alpha})), \quad m_T \geq 0.02 \tag{7}$$

where c_1, c, γ and α are model parameters determined from regression analysis. The set of parameters (c_1, c, γ, α) equals (0, 15.16, 0.833, 1.152), (0.61, 14.04, 0.785, 1.180), and (1.14, 13.93, 0.631, 1.367) for v_T = 0, 0.2 and 0.4, respectively. For the cov of $R_\mu(\mu)$, $v_R(\mu, m_T, v_T)$, their suggested approximations were

$$v_R(T_n, 0, \mu) = \begin{cases} v \times \ln(T_n/0.01)/\ln(10) & \text{for} \quad 0.05 \leq T_n < 0.1 \\ v & \text{for} \quad 0.1 \leq T_n < 10 \end{cases} \tag{8}$$

for $v_T = 0$ where $v = (\mu-1)^{1/3}/5$, and

$$v_R(m_T, v_T, \mu) = \begin{cases} 2v_T + v_T \times \ln(m_T/0.01)/\ln(10) & \text{for} \quad 0.05 \leq m_n < 0.1 \\ 3v_T \times \ln(100/m_T)/\ln(1000) & \text{for} \quad 0.1 \leq m_n < 10 \end{cases} \tag{9}$$

for v_T greater than 0.2 but less than 0.4.

Note that $m_\psi(m_T, 0, \mu) = m_R(m_T, 0, \mu)$ (for $v_T = 0$) because in such a case $\psi_\mu(\mu)$ equals $R_\mu(\mu)$ that was discussed in Section 2.1. Recall that it was observed that $m_\psi(m_T, v_T, \mu)/m_\psi(m_T, 0, \mu)$ is almost independent of μ, and appears to be piecewise linear functions of $\ln(m_T)$. Based on these, one may use the following simple empirical expression to approximate $m_\psi(m_T, v_T, \mu)$

$$m_\psi(m_T, v_T, \mu) = h(m_T, v_T) m_R(m_T, 0, \mu) \tag{10}$$

where $h(m_T, v_T)$ is to be determined based on the results shown in Fig. 2. As a first attempt, a crude approximation given in the following for $v_T > 0$ can be used

$$h(m_T, v_T) = \begin{cases} 1 & \text{for} \quad 0.05 \leq m_T < 0.2 \\ 1 + a \times \ln(m_T/0.2)/\ln(100) & \text{for} \quad 0.2 \leq m_T < 10 \end{cases} \tag{11}$$

and $a = 0.04\exp(7.8 v_T)$ which was obtained by considering the results for v_T = 0.2 and 0.4 as well as those for v_T = 0.3. Comparison of the empirical equation given in Eq. (11) and the simulated results is presented in Figs. 2 and 5 for v_T = 0.2 and 0.4, respectively.

Since the cov of $\psi_\mu(\mu)$, $v_\psi(m_T, v_T, \mu)$, varies almost linearly as a function of $\ln(m_T)$ for m_T less than 20 (sec), we suggest the following very simple approximation

$$v_\psi(m_T, v_T, \mu) = 3v_T \ln(m_T/0.01)/\ln(1000), \quad \text{for} \quad 0.02 \leq m_T < 10 \tag{12}$$

This equation is compared with the simulation results and shown in Figs. 3 and 6, for v_T = 0.2 and 0.4, respectively.

3. Seismic demand with consistent probability levels

3.1 Seismic demand

The evaluation of the peak linear elastic responses based on the uniform hazard spectra (UHS)

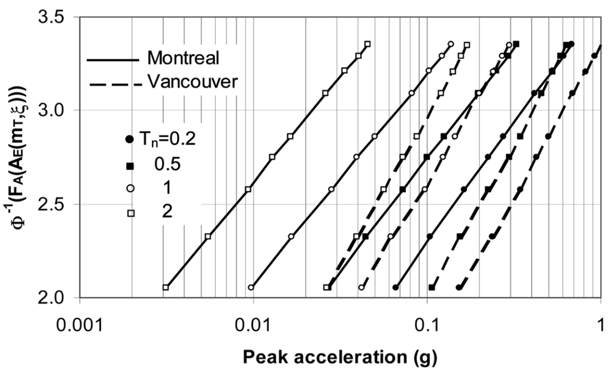


Fig. 7 Probability distribution of peak acceleration response for Montreal and Vancouver shown in lognormal probability paper

approach for the US and Canada can be found in many studies including Sobel (1995), WGC (1995), Frankel (1996) and Adams *et al.* (1999). These spectra did not incorporate the effect of uncertainty in T_n. The results of Sobel (1995) and WGS (1995) that are focused on the nuclear power plant sites in the US and Canada appear to suggest that the peak linear elastic responses at different frequencies can be modeled as lognormal variates. To further inspect the adequacy of this assumption, the results of the peak acceleration responses provided by Halchuk (2002) for Vancouver and Montreal are plotted in the lognormal probability paper and shown in Fig. 7. In the figure, $\Phi^{-1}(\bullet)$ denotes the inverse of the standard normal distribution function, and $F_A(A_E(m_T, \xi))$ represents the probability distribution function of the annual maximum peak linear elastic acceleration response $A_E(m_T, \xi)$. In here, ξ rather than m_ξ is used because ξ is treated as deterministic variable in this study. The results shown in Fig. 7 suggest that the assumption that $A_E(m_T, \xi)$ is a lognormal variate is adequate (at least in the tail region) since its probability distribution plotted in the lognormal probability paper follows almost a straight line. A simple distribution fitting analysis indicates that the cov of $A_E(m_T, \xi)$ for m_T varying from 0.1 to 2 (sec) is approximately equal to 2.0 to 2.5 for Vancouver and equal to 4 to 12 for Montreal. Note that the cov of $D_E(m_T, \xi)$, v_D, equals the cov of $A_E(m_T, \xi)$, v_A, since $D_E(m_T, \xi) = (m_T/(2\pi))^2 A_E(m_T, \xi)$.

Let q denote the annual probability of non-exceedance. Given the probability distribution function of $A_E(m_T, \xi)$, one can evaluate the q-quantile of $A_E(m_T, \xi)$, A_{Eq} (i.e., probability of $A_E(m_T, \xi)$ less than equal to A_{Eq} is equal to q), which can be used to define the UHS with probability of non-exceedance equal to q. One can also use the q-quantile of $D_E(m_T, \xi)$, D_{Eq}, to define the UHS as well because A_{Eq} can be uniquely determined from D_{Eq} for a structure with natural vibration period equal to m_T. Note that for simplicity of notation the dependence of A_{Eq} and D_{Eq} on m_T and ξ is dropped.

The peak responses D_{Eq} and A_{Eq} can be represented in the ADRS format as illustrated in Fig. 8. The radial line shown in the figure represents the natural vibration period m_T.

If one uses the acceleration-based ratio $\psi_\mu(\mu)$ (see Eq. (6)), the obtained yield acceleration is

$$A_y(T_n, \xi, \mu) = \psi_\mu(\mu) A_E(m_T, \xi)/\mu \tag{13}$$

whereas if one uses the displacement-based ratio $R_\mu(\mu)$ (see Eq. (4)), the peak displacement responses are

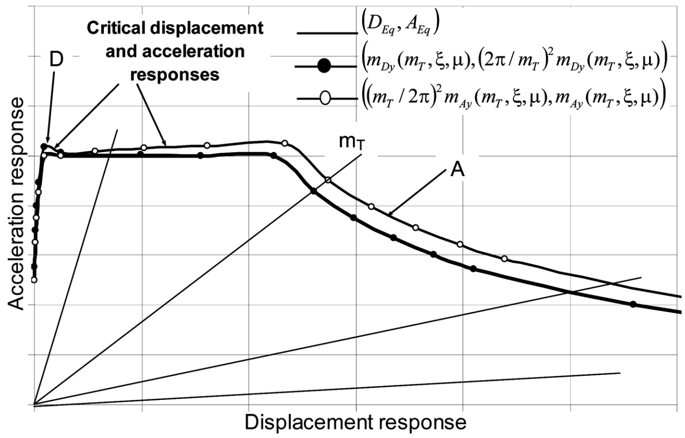


Fig. 8 Illustration of the seismic demand in the acceleration-displacement response format

$$D_I(T_n, \xi, \mu) = R_\mu(\mu) D_E(m_T, \xi) \tag{14a}$$

and

$$D_y(T_n, \xi, \mu) = R_\mu(\mu) D_E(m_T, \xi)/\mu \tag{14b}$$

It should be noted that when μ equals one, $D_y(T_n, \xi, \mu)$ and $A_y(T_n, \xi, \mu)$ represent the peak linear elastic displacement and acceleration responses, respectively. From Eqs. (13) and (14b), we have

$$A_y(T_n, \xi, \mu) = \frac{\psi_\mu(\mu)}{R_\mu(\mu)} \left(\frac{2\pi}{m_T}\right)^2 D_y(T_n, \xi, \mu) \tag{15}$$

This indicates that $A_y(T_n, \xi, \mu)$ is not equal to $(2\pi/m_T)^2 D_y(T_n, \xi, \mu)$ because $\psi_\mu(\mu)$ is not equal to $R_\mu(\mu)$ that was shown in Section 2.0. Similarly, one can show that the conditional expectation of $A_y(T_n, \xi, \mu)$ for given quantile of $A_E(m_T, \xi)$, $m_{Ay}(m_T, \xi, \mu)$

$$m_{Ay}(m_T, \xi, \mu) = (m_\psi(m_T, v_T, \mu)/\mu) A_{Eq} \tag{16a}$$

is usually not equal to $(2\pi/m_T)^2 m_{Dy}(m_T, \xi, \mu)$ where $m_{Dy}(m_T, \xi, \mu)$ denotes the conditional expectation of $D_y(T_n, \xi, \mu)$ given by

$$m_{Dy}(m_T, \xi, \mu) = (m_R(m_T, v_T, \mu)/\mu) D_{Eq} \tag{16b}$$

One can also show that the quantile of $A_y(T_n, \xi, \mu)$, A_{yq}, is not equal to, $(2\pi/m_T)^2 D_{yq}$, where D_{yq} represents the quantile of $D_y(T_n, \xi, \mu)$.

It must be emphasized that $m_{Ay}(m_T, \xi, \mu)$ and $m_{Dy}(m_T, \xi, \mu)$ refer to the expectation over T_n and conditioned on that $D_E(m_T, \xi)$ and $A_E(m_T, \xi)$ are equal to D_{Eq} and A_{Eq}, respectively. They do not represent the means of $D_y(T_n, \xi, \mu)$ and $A_y(T_n, \xi, \mu)$, rather they represent an approximation to the quantiles of $D_y(T_n, \xi, \mu)$ and $A_y(T_n, \xi, \mu)$, respectively. This will be shown in detail shortly after.

Now, consider that the seismic demand is formed by $m_{Ay}(m_T, \xi, \mu)$ and $m_{Dy}(m_T, \xi, \mu)$ for a structure that has an uncertain (or unknown) natural vibration period with predicted mean equal to m_T, and that one's task is to design a structure or to check the adequacy of a designed structure. If the acceleration response capacity of the structure is exactly equal to the demand $m_{Ay}(m_T, \xi, \mu)$, it is equivalent to say that the displacement capacity of the structure is equal to $(2\pi/(m_T))^2$ $m_{Ay}(m_T, \xi, \mu)$. In other words, the verification of the adequacy of the structure by using $m_{Ay}(m_T, \xi, \mu)$ for the acceleration response is equivalent to that by using $(m_T/(2\pi))^2\, m_{Ay}(m_T, \xi, \mu)$ for the displacement response. For μ = 1, this acceleration-response seismic demand criterion formed by $((m_T/(2\pi))^2\, m_{Ay}(m_T, \xi, \mu),\ m_{Ay}(m_T, \xi, \mu))$ is schematically shown in Fig. 8 and identified as curve A.

If the structure to be designed or checked has displacement response capacity equal to the displacement seismic demand $m_{Dy}(m_T, \xi, \mu)$, it is equivalent to say that the acceleration capacity of the structure is equal to $(2\pi/m_T)^2\, m_{Dy}(m_T, \xi, \mu)$. That is, the verification of the adequacy of the structure by using $m_{Dy}(m_T, \xi, \mu)$ for the displacement response is equivalent to that by using $(2\pi/m_T)^2\, m_{Dy}(m_T, \xi, \mu)$ for the acceleration response. For μ = 1, this displacement-response seismic demand criterion formed by $(m_{Dy}(m_T, \xi, \mu), (2\pi/m_T)^2 m_{Dy}(m_T, \xi, \mu))$ is again schematically shown in Fig. 8 but identified as curve D. Note this curve almost coincides with that defined by (D_{Eq}, A_{Eq}) except for very small displacements.

The acceleration-response seismic demand criterion and the displacement-response seismic demand criterion do not always coincide. If $m_{Ay}(m_T, \xi, \mu)$ is greater than $(2\pi/m_T)^2\, m_{Dy}(m_T, \xi, \mu)$, $(m_T/(2\pi))^2\, m_{Ay}(m_T, \xi, \mu)$ must be greater than $m_{Dy}(m_T, \xi, \mu)$, and vice versa. A small difference between $m_{Ay}(m_T, \xi, \mu)$ and $(2\pi/m_T)^2\, m_{Dy}(m_T, \xi, \mu)$ could result in a large discrepancy between $(m_T/(2\pi))^2\, m_{Ay}(m_T, \xi, \mu)$ and $m_{Dy}(m_T, \xi, \mu)$ for large values of m_T. For very small values of m_T, a slight difference between $m_{Dy}(m_T, \xi, 1)$ and $(m_T/(2\pi))^2\, m_{Ay}(m_T, \xi, \mu)$ can lead to a significant difference between $(2\pi/m_T)^2\, m_{Dy}(m_T, \xi, \mu)$ and $m_{Ay}(m_T, \xi, \mu)$.

The above indicates that a structure that just meets the acceleration-response seismic demand criterion (or displacement-response seismic demand criterion) may not satisfy the displacement-response seismic demand criterion (or acceleration-response seismic demand criterion). Therefore, an inadequate structure could be considered to be acceptable depending on the selected checking criterion. To overcome this problem, one must require the acceleration response capacity of the structure be greater than or equal to the maximum of $m_{Ay}(m_T, \xi, \mu)$ and $(2\pi/m_T)^2\, m_{Dy}(m_T, \xi, \mu)$, which is equivalent to say that the displacement response capacity of the structure is greater than or equal to the maximum of $(m_T/(2\pi))^2 m_{Ay}(m_T, \xi, \mu)$ and $m_{Dy}(m_T, \xi, \mu)$. This seismic demand criterion for designing and checking the structure, $(D_y, A_y)_{mc}$, can be expressed as

$$(D_y, A_y)_{mc} = (D_{ymc}, A_{ymc})$$
$$= \left(\max\left(\left(\frac{m_T}{2\pi}\right)^2 m_{Ay}(m_T, \xi, \mu) m_{Dy}(m_T, \xi, \mu)\right),\ \max\left(m_{Ay}(m_T, \xi, \mu)\left(\frac{2\pi}{m_T}\right)^2 m_{Dy}(m_T, \xi, \mu)\right)\right) \tag{17}$$

Substituting Eq. (16) into Eq. (17) gives

$$(D_y, A_y)_{mc} = \phi_{mc}(m_T, \xi, \mu) A_{Eq}\left(\left(\frac{m_T}{2\pi}\right)^2, 1\right) \tag{18a}$$

where the non-dimensional multiplication factor $\phi_{mc}(m_T, \xi, \mu)$ is given by

$$\phi_{mc}(m_T, v_T, \mu) = \max(m_\psi(m_T, v_T, \mu), m_R(m_T, v_T, \mu))/\mu \tag{18b}$$

and $m_\psi(m_T, v_T, \mu)$ and $m_R(m_T, v_T, \mu)$ are given in Eqs. (7) and (10), respectively. In deriving this equation, $A_{Eq} = (2\pi/m_T)^2 D_{Eq}$ is used. Eq. (18) indicates that $(A_y(T_n, \xi, \mu))_{mc} = (2\pi/m_T)^2 (D_y(T_n, \xi, \mu))_{mc}$ which is the consequence of the definition of seismic demand criterion given in Eq. (17).

It should be noted that whether the ADRS or any other plotting format is used the quantities given in Eq. (17) (or Eq. (18)) should be employed in defining the critical seismic demand and the use of $m_{Ay}(m_T, \xi, \mu)((m_T/(2\pi))^2, 1)$ or $m_{Dy}(m_T, \xi, \mu)(1, (2\pi/m_T)^2)$ as critical seismic demand should be avoided because of the problem discussed previously.

Similar conclusions to the above can be drawn if the uncertainty in $\psi_\mu(\mu)$, $R_\mu(\mu)$ and T_n is considered, and the quantiles of peak acceleration and displacement responses are employed in defining the seismic demand. In such a case, the critical seismic demand denoted by $(D_y, A_y)_{qc}$ are given by

$$(D_y, A_y)_{qc} = (D_{yqc}, A_{yqc})$$

$$= \left(\max\left(\left(\frac{m_T}{2\pi}\right)^2 (A_y(m_T, \xi, \mu))_q, D_y(m_T, \xi, \mu))_q\right),\ \max\left(A_y(m_T, \xi, \mu))_q, \left(\frac{2\pi}{m_T}\right)^2 (D_y(m_T, \xi, \mu))_q\right)\right) \quad (19)$$

3.2 Evaluation of quantiles

Since $A_E(m_T, \xi)$ is considered to be lognormally distributed, the q-quantile of $A_E(m_T, \xi)$, A_{Eq}, is given by

$$A_{Eq} = \frac{m_A}{\sqrt{1 + v_A^2}} \exp(\beta_q \sqrt{\ln(1 + v_A^2)}) \quad (20)$$

where m_A represents the mean of $A_E(m_T, \xi)$; and $\beta_q = \Phi^{-1}(q)$.

If $\psi_\mu(\mu)$ is considered to be lognormally distributed and statistically independent of $A_E(m_T, \xi)$, it can be shown (Benjamin and Cornell 1970) that the quantile of $A_y(T_n, \xi, \mu)$, A_{yq} is given by

$$A_{yq} = \frac{m_A m_\psi(m_T, v_T, \xi)}{\mu\sqrt{1 + v_{\psi A}^2}} \exp(\beta_q \sqrt{\ln(1 + v_{\psi A}^2)}) = \frac{h_\psi(m_T, v_T, \xi, q)}{\mu} A_{Eq} \quad (21)$$

where

$$h_\psi(m_T, v_T, \xi, q) = \frac{m_\psi(m_T, v_T, \xi)}{\sqrt{1 + (v_\psi(m_T, v_T, \xi))^2}} \exp(\beta_q(\sqrt{\ln(1 + v_{\psi A}^2)} - \sqrt{\ln(1 + v_A^2)})) \quad (22)$$

and $1 + v_{\psi A}^2 = (1 + (v_\psi(m_T, v_T, \xi))^2)(1 + v_A^2)$. Eqs. (20) and (21) that are used for evaluating the quantile of $A_y(T_n, \xi, \mu)$ are equally applicable for evaluating of the quantile of $D_y(T_n, \xi, \mu)$, except that whenever A and ψ appear in these equations, they are replaced by D and R, respectively.

Comparison of Eq. (16) and Eq. (20) indicates that A_{yq} and D_y are not equal to $m_{Ay}(m_T, \xi, \mu)$ and $m_{Dy}(m_T, \xi, \mu)$, respectively. Therefore, one should not expect that the inelastic yield response spectra, that is obtained by using a linear elastic design response spectrum with a return period T (= $1/(1 - q)$) times the mean of $\psi_\mu(\mu)$ (or mean of $R_\mu(\mu)$), has the return period equal to T.

Substituting Eqs. (21) and (22) into Eq. (19) results in

$$(D_{yqc}, A_{yqc}) = \phi_{qc}(m_T, v_T, \mu, q) A_{Eq}\left(\left(\frac{m_T}{2\pi}\right)^2, 1\right) \tag{23a}$$

where the non-dimensional multiplication factor $\phi_{qc}(m_T, v_T, \mu, q)$ is given by

$$\phi_{qc}(m_T, v_T, \mu, q) = \max(h_\psi(m_T, v_T, \mu, q), h_R(m_T, v_T, \mu, q))/\mu \tag{23b}$$

Note that Eq. (23a) gives $A_{yqc} = (2\pi/m_T)^2 D_{yqc}$ which is the consequence of the definition of seismic demand criterion given in Eq. (19). The above equations are applicable even v_T equals zero. The error by using Eq. (18) to approximate Eq. (23) depends not only on the statistics of the $\psi_\mu(\mu)$ and $R_\mu(\mu)$ but also on q.

3.3 Numerical results

For the numerical analysis, the values of the cov of peak linear elastic responses v_A varying from 2 to 12 will be considered, and the cov of v_T equal to 0, 0.2 and 0.4 are employed.

First, the non-dimensional multiplication factor $\phi_{mc}(m_T, v_T, \mu)$ given in Eq. (18b) is evaluated. The obtained values of $\phi_{mc}(m_T, v_T, \mu)$ are shown in Fig. 9. The results shown in Fig. 9(a) correspond to the case when there is no uncertain in T_n. In such a case, since $m_\psi(m_T, 0, \mu) =$

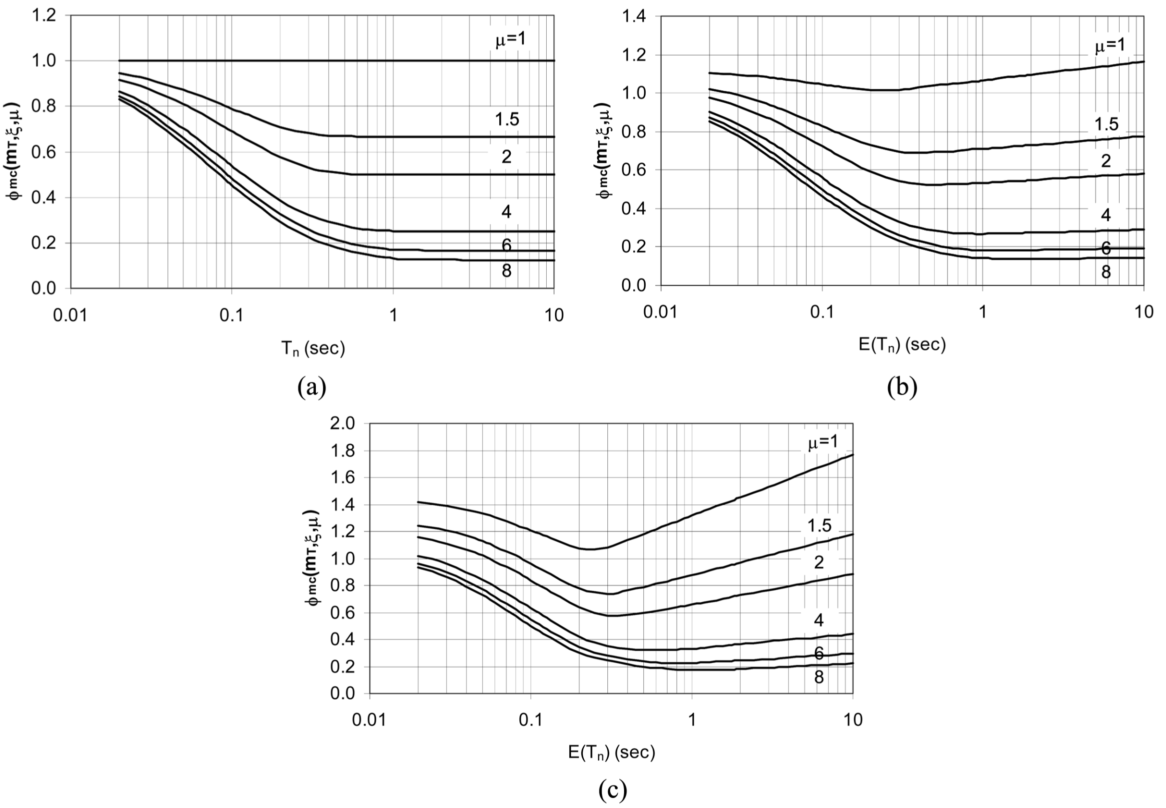


Fig. 9 Multiplication factor $\phi_{mc}(m_T, v_T, \mu)$ used to evaluate the approximate peak responses; (a) $v_T = 0$, (b) $v_T = 0.2$, (c) $v_T = 0.4$

$m_R(m_T, 0, \mu)$, the use of the acceleration-response seismic demand criterion is equivalent to the use of the displacement-response seismic demand criterion, and $\phi_{mc}(m_T, 0, \mu)$ equals $m_R(m_T, 0, \mu)/\mu$. The results shown in Figs. 9(b) and 9(c) are for $v_T = 0.2$ and $v_T = 0.4$, respectively.

Comparison of the results shown in these figures indicates that $\phi_{mc}(m_T, v_T, \mu)$ increases as v_T increases. Further, for flexible and not very ductile systems the increase in $\phi_{mc}(m_T, v_T, \mu)$ is very significant if there is uncertainty in the predicted natural vibration period. This increase is governed by the mean of $\psi_\mu(\mu)$, $m_\psi(m_T, v_T, \mu)$, which is given by Eqs. (10) and (11). Therefore, for flexible structures with uncertainty in T_n, the adequacy of the structure is controlled by the acceleration-response seismic demand. However, for rigid structures with a low ductility factor, the value of $\phi_{mc}(m_T, v_T, \mu)$ is governed by the mean of $R_\mu(\mu)$, $m_R(m_T, v_T, \mu)$, which is given by Eq. (7). Therefore, for stiff structures with uncertainty in T_n one should use the ratio $R_\mu(\mu)$, that is assessed based on the displacement responses, to evaluate the seismic demand.

By following the numerical values obtained in each of the calculation steps leading to the results shown in Fig. 9, it was identified that $\phi_{mc}(m_T, v_T, \mu)$ equals $m_\psi(m_T, v_T, \mu)/\mu$ if m_T is greater than about 0.3 to 0.5, and it equals $m_R(m_T, v_T, \mu)/\mu$ otherwise. Therefore, as an approximation one may consider that

$$\phi_{mc}(m_T, v_T, \mu) = \begin{cases} m_R(m_T, v_T, \mu)/\mu & \text{for} \quad 0.05 \le m_T \le 0.3 \\ m_\psi(m_T, v_T, \mu)/\mu & \text{for} \quad 0.3 \le m_T \le 10 \end{cases} \tag{24}$$

Note that since depending on the value of v_T, $m_\psi(m_T, v_T, \mu)/m_\psi(m_T, 0, \mu)$ and $m_R(m_T, v_T, \mu)/m_R(m_T, 0, \mu)$ can be significantly larger than unit, $\phi_{mc}(m_T, v_T, \mu)/\phi_{mc}(m_T, 0, \mu)$ can be much greater than unit as well. Use of Eq. (24) leads to $\phi_{mc}(m_T, v_T, \mu)/\phi_{mc}(m_T, 0, \mu)$ up to about 1.2 for v_T equal to 0.2, and up to 1.9 for v_T equal to 0.4. Therefore, ignoring the uncertainty in T_n may significantly underestimate the critical seismic demand.

Consider now that one is interested in defining probability-consistent seismic demand, i.e., $(D_y, A_y)_{qc}$. Based on Eq. (23a) this demand can be calculated directly using the non-dimensional multiplication factor $\phi_{qc}(m_T, v_T, \mu, q)$ and the quantile of the peak linear elastic acceleration response, A_{Eq}.

For a probability of exceedance of 10% in 50 years, (i.e., $1 - q = 0.021$ or $T = 475$ years, $\beta_q = 2.86$), the obtained values of $\phi_{qc}(m_T, v_T, \mu, q)$ are shown in Fig. 10 for a few sets of values of v_A and v_T. Comparison of the results shown in Figs. 9 and 10 indicates that $\phi_{qc}(m_T, v_T, \mu, q)$ is greater than that of $\phi_{mc}(m_T, v_T, \mu)$, implying that the quantile of the critical seismic demand $(D_y, A_y)_{qc}$ is larger than its approximation $(D_y, A_y)_{mc}$. The observations made about the results shown in Fig. 9 are equally true for the results shown in Fig. 10. Again, it was identified that $\phi_{qc}(m_T, v_T, \mu, q)$ equals $h_\psi(m_T, v_T, \mu, q)/\mu$ if m_T is greater than about 0.3 to 0.5, and it equals $h_R(m_T, v_T, \mu, q)/\mu$ otherwise. Therefore, one may consider the following approximation for evaluating $\phi_{qc}(m_T, v_T, \mu, q)$,

$$\phi_{qc}(m_T, v_T, \mu, q) = \begin{cases} h_R(m_T, v_T, \mu, q)/\mu & \text{for} \quad 0.05 \le m_T \le 0.3 \\ h_\psi(m_T, v_T, \mu, q)/\mu & \text{for} \quad 0.3 \le m_T \le 10 \end{cases} \tag{25}$$

Note that $h_\psi(m_T, v_T, \mu, q)$ is equal to $m_\psi(m_T, v_T, \xi)$ times the quantile related factor $\exp(\beta_q(\sqrt{\ln(1+v_{\psi A}^2)} - \sqrt{\ln(1+v_A^2)}))/\sqrt{1+(v_\psi(m_T, v_T, \xi))^2}$, and that $h_R(m_T, v_T, \mu, q)$ is equal to $m_R(m_T, v_T, \xi)$ times the quantile related factor $\exp(\beta_q(\sqrt{\ln(1+v_{RD}^2)} - \sqrt{\ln(1+v_D^2)}))/\sqrt{1+(v_R(m_T, v_T, \xi))^2}$ (see Eq. (22)). For $v_\psi(m_T, v_T, \xi)$ or $v_R(m_T, v_T, \xi)$ ranging from 0 to 2, v_A (which is equal to v_D) varying from

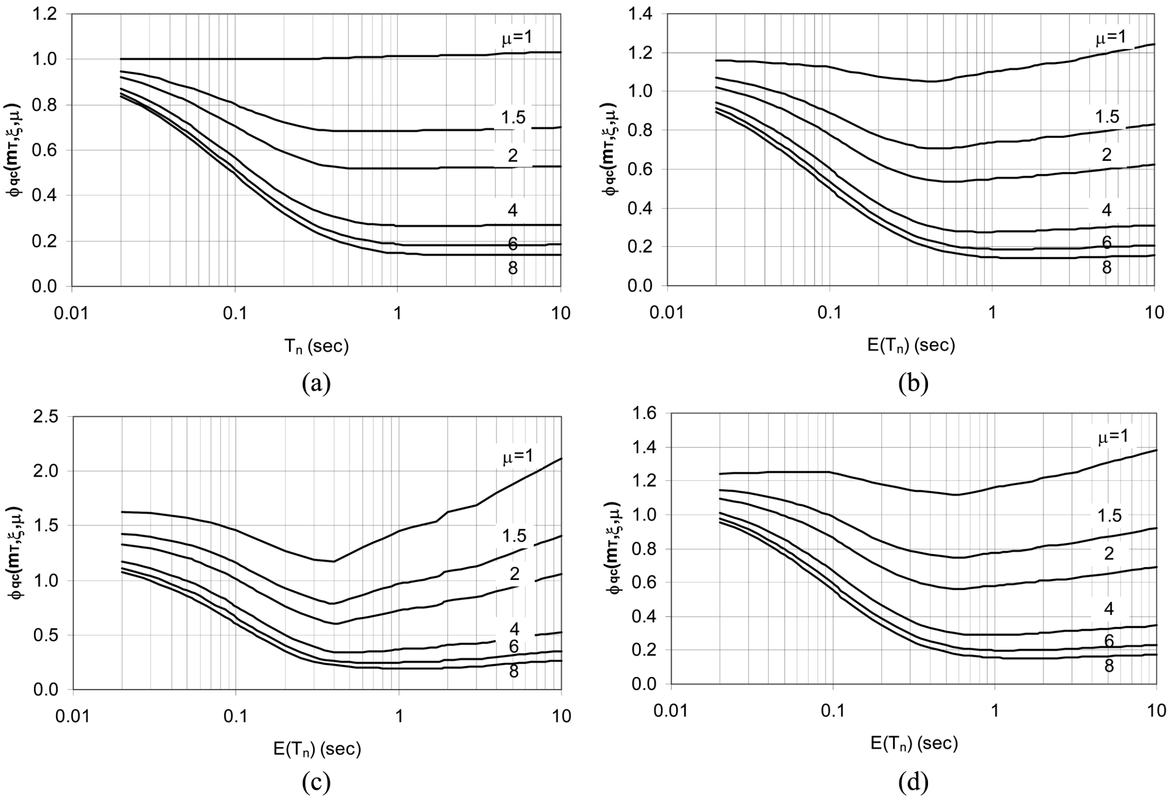


Fig. 10 Multiplication factor $\phi_{mc}(m_T, v_T, \mu, q)$ used to evaluate the peak responses; (a) $v_T = 0$, $v_A = 6$, (b) $v_T = 0.2$, $v_A = 6$, (c) $v_T = 0.4$ $v_A = 6$, (d) $v_T = 0.2$ $v_A = 2$

2 to 12, the average value of these quantile related factor is approximately equal to 1.15 for β_q = 2.86, and 1.3 for β_q = 3.35 which corresponds to a probability of exceedance of 2% in 50 years, (i.e., $1 - q = 0.0004$, $T = 2475$). Therefore, as a crude guideline, one may consider $\phi_{qc}(m_T, v_T, \mu, q) = 1.15\phi_{mc}(m_T, v_T, \mu)$ for β_q = 2.86 and $\phi_{qc}(m_T, v_T, \mu, q) = 1.3\phi_{mc}(m_T, v_T, \mu)$ for β_q = 3.35. In other words, the approximate seismic demand $(D_y(T_n, \xi, \mu), A_y(T_n, \xi, \mu))_{mc}$ underestimates the quantile of the seismic demand $(D_y(T_n, \xi, \mu), A_y(T_n, \xi, \mu))_{qc}$, on average, by about 15% for 475-year return period value of seismic demand and by about 30% for 2475-year return period value of seismic demand.

4. Conclusions

The study showed that the uncertainty in natural vibration period T_n affects the acceleration and displacement responses differently, and two ratios, one based on peak acceleration responses and the other based on peak displacement responses, are not equal and both must be employed in defining and evaluating the critical seismic demand.

Empirical equations for assessing the statistics of the ratios are recommended. These equations are obtained based on simulation results with more than 200 strong ground motion records. By using the statistics of the ratios, a procedure and sets of simple to use equations are recommended for estimating the quantile of critical seismic demand conditioned on the ductility factor.

The results indicate that:

1) The critical seismic demand by considering uncertainty in natural vibration period with a coefficient of variation of 0.2 and 0.4 can be, respectively, up to 20% and 90% higher than that without considering the uncertainty; and
2) The use of the mean value of the yield reduction factor to scale the design response spectrum (i.e., using $\phi_{mc}(m_T, v_T, \mu)$) is inadequate and can lead to probability-inconsistent seismic demand. The error, on average, can be about 15% to 30% for return period of about 500 to 2500 years.

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