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Spectral SFEM analysis of structures with stochastic parameters under stochastic excitation

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Abstract. In this paper, linear elastic isotropic structures under the effects of both stochastic operators and stochastic excitations are studied. The analysis utilizes the spectral stochastic finite elements (SSFEM) with its two main expansions namely; Neumann and Homogeneous Chaos expansions. The random excitation and the random operator fields are assumed to be second order stochastic processes. The formulations are obtained for the system solution of the two dimensional problems of plane strain and plate bending structures under stochastic loading and relevant rigidity using the previously mentioned expansions. Two finite element programs were developed to incorporate such formulations. Two illustrative examples are introduced: the first is a reinforced concrete culvert with stochastic rigidity subjected to a stochastic load where the culvert is modeled as plane strain problem. The second example is a simply supported square reinforced concrete slab subjected to out of plane loading in which the slab flexural rigidity and the applied load are considered stochastic. In each of the two examples, the first two statistical moments of displacement are evaluated using both expansions. The probability density function of the structure response of each problem is obtained using Homogeneous Chaos expansion.

Keywords: stochastic structures; stochastic finite elements method (SFEM); stochastic excitation; stochastic operators; neumann expansion; Homogeneous Chaos expansion.

1. Introduction

In structural problems, uncertainty arises for many reasons such as the geometric shape imperfection due to the loss of accuracy during dimensioning and casting, cracking, the uncertainty

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in Young's modulus and Poisson's ratio, and material strength due to possible non-homogeneity of the material. These types of uncertainties are all included in the operator of the structure differential equation. The excitation itself could be uncertain; as it is well known that some types of loads have stochastic nature such as wind, earthquake, and traffic loads. Besides, the modeling process to applied loads could be uncertain. For the sake of realistic modeling of structures, uncertainties of both the operator and the excitation functions have to be included in the formulation.

To asses the structural response due to these uncertainties, many numerical methods were developed such as stochastic mesh less methods (Rahman and Xu 2005), stochastic finite difference method (Kaminski 2002) and stochastic finite elements method such as Monte-Carlo simulation (MCS), perturbation and spectral approaches. Among these methods, the spectral stochastic finite elements method (SSFEM) appears to be a good candidate for solving uncertain structural systems.

Neumann expansion was first introduced by Neumann and was further investigated by Fredholm (1903). It witnessed successive developments until Shinozuka and Nomoto (1980) introduced this expansion to the field of structural mechanics. Later, Ghanem and Spanos (1991) introduced this expansion for the case of random operator under the effect of deterministic excitation. On the other hand, the concept of Homogeneous Chaos was first introduced by Wiener (1938). Some refinements were evaluated to this expansion till Ghanem and Spanos (1990) employed it in the context of the finite element method to solve random operator problems. Finally, both of Neumann expansion and Homogeneous Chaos were extended to include the effect of random excitation along with the random operator by Galal *et al.* (2005).

Using Neumann expansion, the solution of the stochastic system is evaluated to get an explicit expression for the solution process. This solution is obtained in a set of uncorrelated random variables where only the mathematical expressions for the first two statistical moments can be practically obtained. Homogenous Chaos expansion (H.C.) is proven to have relative advantages in evaluating the statistical moments of any order in addition to the probability distribution function (p.d.f.) of the system response. Also, it can handle these systems that have high level of variability when Neumann expansion fails. By using the H.C. expansion, the p.d.f. of the solution process is obtained as a multiplication of random functionals with deterministic constants.

Ghanem and Spanos (1990, 1991) utilized the SSFEM to evaluate the stochastic response of rectangular and non-rectangular plates under deterministic in-plane loads. The plate rigidity is considered to be stochastic field with zero mean. Consequently, Galal *et al.* (2005) solved a rectangular plate with stochastic operator under the effect of stochastic in-plane excitation as a validation problem for the derived expression in their work.

Using other techniques, the case of random excitation applied to plates with deterministic material properties is investigated in some works. Chang (1994) proposed an equivalent stochastic linearization method to develop a finite element formulation for the dynamic response analysis associated with hysteretic plates. These plates are assumed to be subjected to stochastic excitation, and the plate material is assumed to have deterministic and non-linear properties. The proposed method generates the variance and covariance functions of the nodal displacements and velocity. Young *et al.* (2002) studied the dynamic stability of cantilever skew plates subjected simultaneously to an aerodynamic force in the chord-wise direction and a random in-plane force in the span-wise direction. The aerodynamic force is modeled by the piston theory, and the random in-plane force is assumed as a physical noise with zero mean. The perturbation method with second-moment calculations is employed to determine the stability boundary of the system.

The case of a plate resting on random foundation was also considered. Przewlocki and Gorski (2001) introduced a stochastic description for a strip foundation resting on two and three dimensional soil medium which is modeled under plane stress condition. The three dimensional problem was reduced to a two dimensional one. It is assumed that the soil medium is statistically homogeneous and its mechanical behavior is governed by the linear elasticity theory. It is also assumed that the elastic parameters can be modeled as multi-dimensional random fields. A stochastic finite elements method based on MCS is used to perform the stochastic response of the strip foundation. Another work considering random material properties is introduced by Noh (2004) who proposed a new SFEM formulation for plates with random Poisson's ratio. This formulation depends on decomposing the constitutive matrix into sub-matrices based on a binomial expansion correlation. Using the perturbation method, Benoit (2003) has developed a new technique to handle material and shape uncertainties for elastodynamical problems. As a case study, a rectangular plate with random material properties subjected to deterministic dynamic loading was solved. Another example for plate with random flatness defaults involving the simultaneous occurrence of membrane and bending behavior is treated under the same loading condition. In the context of the meshless methods, Rahman and Roa (2001) used the perturbation technique to solve two examples of plates. The first is a cantilever plate under deterministic tension force and the second is a rectangular plate with a circular hole at its center. In both of these two examples, the plate material is assumed to be linear elastic random field subjected to deterministic in plane load. Again, Rahman and Xu (2005) solved the latter example, with the same assumption, using the element free Galerkin's method (EFGM). In this work, K-L expansion was employed to expand the stochastic field of the random material instead of the perturbation technique.

A more complicated case was considered by Lin (2000). His work aimed to investigate the reliability of the random laminated composite plate with the consideration of multiple buckling failure modes corresponding to different random loads. The uncertainties were considered to be in the material and in the stacking sequences. Numerical examples were presented to demonstrate the feasibility and the applications of the proposed procedure and investigate the dependence of the structural lengths corresponding to different failure modes and the random loads on reliability analysis.

The present work focuses on the application of the SSFEM to solve stochastic plane-strain as well as stochastic plate bending problems. The case of stochastic excitation with deterministic operator and the case of both stochastic excitation and stochastic operator are considered.

2. Basic concepts

2.1 Karhunen - loeve expansion

It is practical to represent the random parameters as random fields (second order stochastic process) $\alpha(x, \theta)$. Where x represents the spatial coordinates in R^n, n is the physical dimension of the problem. Also, the argument θ indicated the random nature of the corresponding quantity. This process is defined by its mean and covariance function, $C_{\alpha\alpha}(x_1, x_2)$, where x_1, x_2 represent two points in the spatial domain. In this work, random fields are discretized using Karhunen-Loeve (K-L) expansion which is based on a spectral decomposition of the covariance kernel (Ghanem and Spanos 1991)

$$\alpha(x, \theta) = \overline{\alpha}(x) + \sum_{n=1}^{\infty} \sqrt{\lambda_n} f_n(x) \xi_n(\theta) \tag{1}$$

where $\overline{\alpha}(x)$ is the mean of the process, $\{\xi_n(\theta)\}_{n=1}^{\infty}$ is a set of uncorrelated random variables, $\lambda_n, f_n(x)$ are the eigen values and the eigen functions of the covariance kernels, and they can be obtained by solving the Fredholm integral equation of the second kind

$$\int_D C_{\alpha\alpha}(x_1, x_2) f_n(x_1) dx_1 = \lambda_n f_n(x_2) \tag{2}$$

where D is the spatial domain over which the process is defined. Hence, the random field is represented by a set of deterministic functions in the spatial variables $\{f_n(x)\}_{n=1}^{\infty}$ multiplied by random coefficients $\{\xi_n(\theta)\}_{n=1}^{\infty}$ which are independent of them.

2.2 The solution of fredholm integral equation

The applicability of K-L expansion is hinged on the solution of the Fredholm second kind integral equation which is described by Eq. (2). For one dimensional domain, some analytic solutions for this equation are reported in Van Tree (1968). In case of multidimensional domains or complicated covariance kernels, other than the reported ones, a numerical technique must be employed to solve this equation. Using Galerkin's finite elements method, the eigen functions $f_k(x)$ are expanded in terms of $h_i(x)$, which are piecewise polynomials forming a complete set of functions in the Hilbert space. These functions are multiplied by some real constants c_i, and the summation will be truncated at the N^{th} term where N is the number of nodes of the element. Then the eigen-functions can be expressed as

$$f_k(x) = \sum_{i=1}^{N} c_i^{(k)} h_i(x) \tag{3}$$

The error ε_N due to this approximation is

$$\varepsilon_N = \sum_{i=1}^{N} c_i^{(k)} \left[\int_D Cov(x_1, x_2) h_i(x_2) dx_2 - \lambda_k h_i(x_2) \right] \tag{4}$$

According to Galerkin's method, this error is required to be orthogonal to the approximating space and this leads to a system of linear equations in the form

$$C^{(e)} \overline{F}^{(e)} = A^{(e)} B^{(e)} \overline{F}^{(e)} \tag{5}$$

where

$$C^{(e)}_{ij} = \int_D \int_D Cov(x_1, x_2) h_i(x_2) h_j(x_1) dx_1 dx_2, \quad B^{(e)}_{ij} = \int_D h_i(x) h_j(x) dx$$
$$\overline{F}^{(e)}_{ij} = c_i^{(j)}, \quad A^{(e)}_{ij} = \delta_{ij} \lambda_i \tag{6}$$

where each of the previous matrices is of the dimension $N \times N$ and defined over the element domain. Assembling the elements of these matrices according to their connectivity leads to a system of equation for the whole structure in the form

$$C\overline{F} = AB\overline{F} \tag{7}$$

Eq. (7) is a generalized eigen value problem for matrices (C, B). Solving this problem leads to the matrices A and $\overline{F}$, where A is a diagonal matrix in which each element is one of the eigen values of the covariance kernel, and $\overline{F}$ is a square matrix in which each column is the nodal values of eigen function corresponding to the eigen value on the same column on matrix A.

2.3 Homogeneous chaos expansion

In this expansion Ghanem and Spanos (1990), the solution process can be expressed as a summation of nonlinear functionals of the set $\{\xi_n(\theta)\}_{n=1}^{\infty}$ multiplied by deterministic constants. These functionals can be expanded in a set of polynomials of second order Gaussian random variables referred as polynomial chaos

$$\alpha(\theta) = a_0\Gamma_0 + \sum_{i_1=1}^{\infty} a_{i_1}\Gamma_1(\xi_{i_1}(\theta)) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty} a_{i_1 i_2}\Gamma_2(\xi_{i_1}(\theta), \xi_{i_2}(\theta)) + \ldots \tag{8}$$

In which $\Gamma_n(\xi_{i_1}(\theta), \ldots, \xi_{i_n}(\theta))$ is the polynomial chaos of order n in the set of variables $(\xi_{i_1}(\theta), \ldots, \xi_{i_n}(\theta))$

$$\Gamma_p(\xi_{i_1}(\theta), \ldots, \xi_{i_n}(\theta)) = \begin{cases} \displaystyle\sum_{\substack{r=n \\ r=even}}^{p} (-1)^r \sum_{\pi(i_1, \ldots, i_p)} \prod_{k=1}^{r} \xi_{i_k} < \prod_{l=r+1}^{n} \xi_{i_l} >, & n \text{ is even} \\ \displaystyle\sum_{\substack{r=n \\ r=even}}^{p} (-1)^{r-1} \sum_{\pi(i_1, \ldots, i_p)} \prod_{k=1}^{r} \xi_{i_k} < \prod_{l=r+1}^{n} \xi_{i_l} >, & n \text{ is odd} \end{cases} \tag{9}$$

where $\pi(.)$ denotes the permutation of its arguments, and the sum is over all such permutations and $< . >$ is the expectation operator. Truncating at the P^{th} order, Eq. (8) may be reduced to

$$\alpha(\theta) = \sum_{i=0}^{P} C_i \Psi_i[\{\xi_n\}] \tag{10}$$

where C_i is a set of deterministic coefficients and $\Psi_i[\{\xi_n\}]$ is a set of polynomials in the set of random variables $(\xi_{i_1}(\theta), \ldots, \xi_{i_n}(\theta))$. Also, $\Psi_i[\{\xi_n\}]$ have the following orthogonality properties

$$<\Psi_i[\{\xi_n\}]> = 1 \text{ for } i = 0 \text{ or otherwise equals zero, and}$$

$$<\Psi_i[\{\xi_n\}]\Psi_j[\{\xi_n\}]> = \delta_{ij} <\Psi_i^2[\{\xi_n\}]> \tag{11}$$

3. Problem formulation

Considering a structural system defined on a domain D in R^n and its boundary is Γ. This system has random properties and subjected to external random excitation where both of them can be modeled as a second order stochastic process. Also, it is assumed to have deterministic boundary conditions. It is convenient to divide the random operator and the random excitation into two parts, the first is deterministic and represents the mean value of the process and the second is a stochastic part with zero mean. The response vector $u(x;\theta)$ should satisfy the equation

$$[L(x) + \Phi(\alpha(x;\theta), x)]u(x;\theta) = \bar{f}(x) + \beta(x;\theta)$$
$$\text{and} \quad \Lambda(x;\theta)u(x;\theta) = 0, \quad x \in \Gamma \tag{12}$$

where

$L(x)$: is the deterministic part of the operator
$\Phi(\alpha(x;\theta), x)$: the stochastic part of the operator with zero mean
$\bar{f}(x)$: mean excitation function
$\beta(x;\theta)$: a second order stochastic process with zero mean and a covariance function
$C_{\beta\beta}(x_1, x_2)$: represents the stochastic part of the excitation function
$\Lambda(x;\theta)$: a random operator applied on the boundary

The stochastic term is considered to be a multiplicative factor, so the operator can be represented as

$$\Phi(\alpha(x;\theta), x) = \alpha(x;\theta)R(x) \text{ in which}$$

$\alpha(x;\theta)$: a stochastic process with zero mean and a covariance function $C_{\alpha\alpha}(x_1, x_2)$
$R(x)$: deterministic operator.

This system can be represented as

$$\left[\bar{K}_{ij} + \sum_{n=1}^{M} \xi_n K_{(n)_{ij}}\right]u = \bar{f}_j + \sum_{n=1}^{M} \xi_n f_{(n)_j} \tag{13}$$

in which

M : The number of terms taken from K-L expansion

$$\bar{K}_{ij} = \int_D [L(x)g_i]g_j dx, \quad K_{(n)_{ij}} = \int_D \sqrt{\lambda_n} a_n(x)[L(x)g_i]g_j dx$$
$$\bar{f}_j = \int_D f(x)g_j dx, \quad f_{(n)_j} = \int_D \sqrt{\lambda_n} b_n(x)g_j dx \tag{14}$$

$a_n(x)$, λ_n, $b_n(x)$ which λ_n appear in Eq. (9) are the eigen pairs of the covariance kernels, and they can be obtained by solving the following two integral equations, respectively

$$\int_D C_{\alpha\alpha}(x_1;x_2)a_n(x_1)dx_1 = \lambda_n a_n(x_2) \tag{15}$$

$$\int_D C_{\beta\beta}(x_1;x_2)b_n(x_1)dx_1 = \gamma_n b_n(x_2) \tag{16}$$

4. The system solutions

4.1 The system solution using neumann expansion

The solution of the described system in Eq. (8) using Neumann expansion was given in Galal

(2005) as

$$u = \left[\sum_{i=0}^{\infty}(-1)^i\left(\sum_{n=1}^{M}\xi_n Q_{(n)}\right)^i\right]\left[g + \sum_{n=1}^{M}\xi_n H_{(n)}\right] \quad (17)$$

with a convergence condition as $\quad \|Q_{(n)}\| < 1 \qquad \forall n$
where

$$u = \left[I - \sum_{n=1}^{M}\xi_n Q_{(n)} + \sum_{n=1}^{M}\sum_{m=1}^{M}\xi_n\xi_m Q_{(n)}Q_{(m)} - \sum_{n=1}^{M}\sum_{m=1}^{M}\sum_{k=1}^{M}\xi_n\xi_m\xi_k Q_{(n)}Q_{(m)}Q_{(k)}\right.$$
$$\left. + \sum_{n=1}^{M}\sum_{m=1}^{M}\sum_{k=1}^{M}\sum_{l=1}^{M}\xi_n\xi_m\xi_k\xi_l Q_{(n)}Q_{(m)}Q_{(k)}Q_{(l)} - \dots\right]g +$$
$$\left[\sum_{n=1}^{M}\xi_n H_{(n)} - \sum_{n=1}^{M}\sum_{n=1}^{M}\xi_n\xi_m Q_{(n)}H_{(m)} + \sum_{n=1}^{M}\sum_{m=1}^{M}\sum_{k=1}^{M}\xi_n\xi_m\xi_k Q_{(n)}Q_{(m)}H_{(k)}\right.$$
$$\left. - \sum_{n=1}^{M}\sum_{m=1}^{M}\sum_{k=1}^{M}\sum_{k=1}^{M}\xi_n\xi_m\xi_k\xi_l Q_{(n)}Q_{(m)}Q_{(k)}H_{(l)} + \dots\right] \quad (18)$$

in which

$$g = \bar{K}^{-1}\bar{f}, \quad Q_{(n)} = \bar{K}^{-1}K_{(n)}, \quad H_{(n)} = \bar{K}^{-1}f_{(n)} \quad (19)$$

The mean value of the structure response is given as

$$<u> = \left[g + \sum_{n=1}^{M}Q_{(n)}\hat{Q}_{(n)} + \sum_{n=1}^{M}\sum_{m=1}^{M}Q_{(n)}Q_{(n)}Q_{(m)}\hat{Q}_{(m)} + Q_{(n)}Q_{(m)}Q_{(m)}\hat{Q}_{(n)} + Q_{(n)}Q_{(m)}Q_{(m)}\hat{Q}_{(n)} + \dots\right] \quad (20)$$

in which, $\quad \hat{Q}_{(n)} = Q_{(n)}g - H_{(n)}$

The covariance matrix can be evaluated using the following

$$R_{uu} = <u.u^T> - <u><u^T>$$
$$\text{or} \quad R_{uu} = R_{uu1} + R_{uu2} + R_{uu3} - <u><u^T> \quad (21)$$

where

$$R_{uu1} = <\left[\sum_{i=0}^{\infty}(-1)^i\left(\sum_{n=1}^{M}\xi_n Q_{(n)}\right)^i\right]G\left[\sum_{j=0}^{\infty}(-1)^j\left(\sum_{n=1}^{M}\xi_n Q_{(n)}^T\right)^j\right]> \quad (22a)$$

$$R_{uu2} = <\left[\sum_{i=0}^{\infty}(-1)^i\left(\sum_{n=1}^{M}\xi_n Q_{(n)}\right)^i\right]\left[\sum_{m=1}^{M}\xi_m\hat{P}_{(n)}\right]\left[\sum_{j=0}^{\infty}(-1)^j\left(\sum_{n=1}^{M}\xi_n Q_{(n)}^T\right)^j\right]> \quad (22b)$$

$$R_{uu3} = <\left[\sum_{i=0}^{\infty}(-1)^i\left(\sum_{n=1}^{M}\xi_n Q_{(n)}\right)^i\right]\left[\sum_{m=1}^{M}\sum_{k=1}^{M}\xi_m\xi_k H_{(m)}H_{(k)}^T\right]\left[\sum_{j=0}^{\infty}(-1)^j\left(\sum_{n=1}^{M}\xi_n Q_{(n)}^T\right)^j\right]> \quad (22c)$$

where $\quad G = gg^T, \quad P_{(n)} = H_{(n)}g^T, \quad \hat{P}_{(n)} = P_{(n)} + P_{(n)}^T$

The case of a structure having a deterministic operator and exposed to a stochastic excitation could be viewed as a sub case from the pervious case and hence

$$u = g + \sum_{n=1}^{M} \xi_n H_{(n)} \tag{23}$$

with a mean value

$$<u> = g \tag{24}$$

and covariance matrix

$$R_{uu} = \sum_{n=1}^{M} H_{(n)} H_{(n)}^T \tag{25}$$

4.2 The system solution using Homogeneous Chaos (H.C.) expansion

Using H. C. expansion, the response function is represented as

$$u(x;\theta) = \sum_{i=0}^{P} d_i \Psi_i[\{\xi_r\}] \tag{26}$$

where d_i is a set of deterministic coefficients, and $\Psi_i[\{\xi_r\}]$ is a set of polynomials that have M-dimensions and order p. Substituting Eq. (26) into the described system in Eq. (13), leads to

$$\sum_{p=0}^{P}\left[b_{pp}\bar{K} + \sum_{n=1}^{M} c_{npq} K^{(n)}\right] d_p = e_p \bar{f} + \sum_{n=1}^{M} a_{np} f^{(n)} \tag{27}$$

where

$$b_{pp} = <\Psi^2{}_p[\{\xi_r\}]>$$

$$c_{npq} = <\xi_n \Psi_p[\{\xi_r\}] \Psi_q[\{\xi_r\}]>$$

$e_p = <\Psi_q[\{\xi_r\}]>$, which is unity for $q = 0$ and zero otherwise. Also,

$$a_{np} = <\xi_n \Psi_p[\{\xi_r\}]>$$

The values of b_{pp}, c_{npq}, e_p and a_{np} can be determined and tabulated. Solving the system of equations in Eq. (27), leads to get $d_p = \{d_o, d_1, \ldots, d_P\}$. Substituting into Eq. (26), One obtains the density functions of the response vector. The first two statistical moments of the response function can be obtained by

$$<u(x;\theta)> = d_o \tag{28}$$

$$R_{uu} = \sum_{i=1}^{P} b_{ii} d_i d_i^T \tag{29}$$

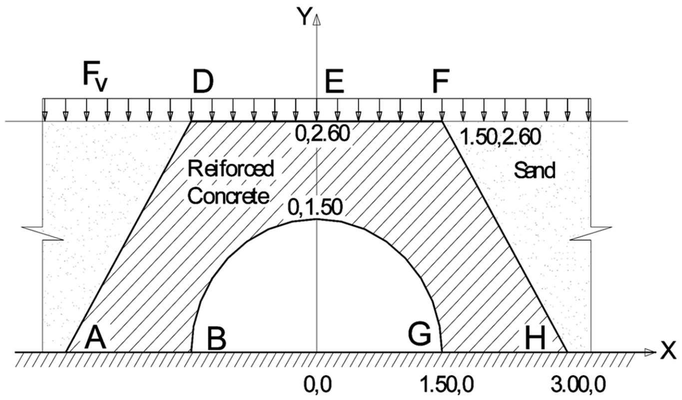

Fig. 1 The concrete culvert

5. Illustrative examples

Reinforced concrete was chosen as the structure material in the two examples, since concrete material properties are uncertain and can be randomly spread over the structure (Lawanwisut *et al.* 2003, Pukl *et al.* 2003, Vouwenvelder 2004).

5.1 Illustrative Example I: Stochastic Plane – strain Problem

5.1.1 Problem description

Let us consider a reinforced concrete culvert with 6.0 m base width and 2.6 m in height as shown in Fig. 1. This culvert is exposed to vertical loads F_v which can be expressed as Fv = $5250(1 + \alpha(x, y; \theta))$ kN/m'. Considering a typical 1.0 m of the normal direction of the culvert, it can be modeled as a plane strain structure. Poisson's ratio is considered to be 0.20 while the Young's modulus (EA) is considered to be random in the form: $EA(x, y, \theta) = \overline{EA}(x, y)(1 + \alpha(x, y; \theta))$, with $\overline{EA}(x, y)$ = 2.2×10^7 (kN/m^2).m^2. $\alpha(x, y; \theta)$ is assumed to be a second-order Gaussian stochastic process with exponential covariance model given in Ghanem and Spanos (1991). The standard deviation (S.D.) is assumed to be 0.60 and the correlation lengths are 3.0 m in both directions.

5.1.2 Results

Since there is no available closed-form solution for Fredholm integral for the present case, the numerical technique described in section 3 is employed. Dividing one half of the structure into 21 nodes and 12 finite elements, 21 eigen-values are obtained as

{0.8791, 0.2290, 0.0709, 0.0496, 0.0317, 0.0275, 0.0218, 0.0132, 0.0125, 0.0079, 0.0071, 0.0053, 0.0038, 0.0032, 0.0029, 0.0028, 0.0023, 0.0021, 0.0018, 0.0013, 0.0008}

Each eigen-function will be obtained through its nodal values at the mesh.

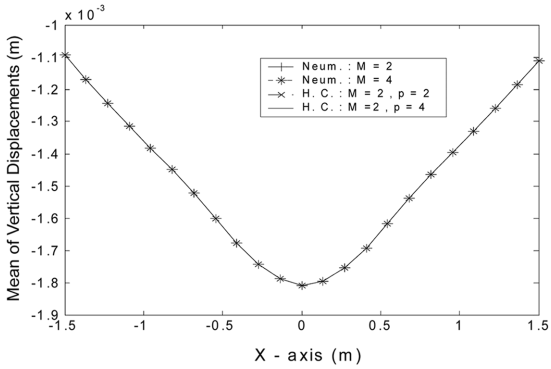

Fig. 2 The Mean of vertical displacements for points of side *DF* in case of stochastic excitation only

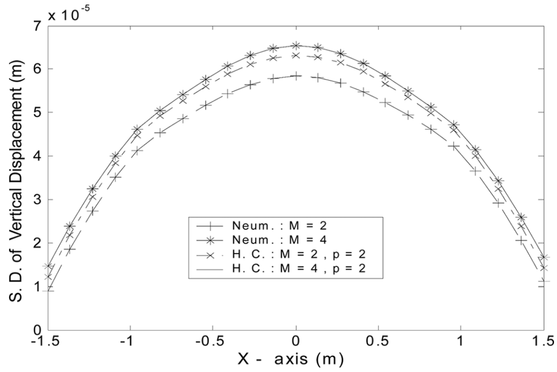

Fig. 3 The standard deviation of vertical displacements for points of side *DF* in case of stochastic excitation only

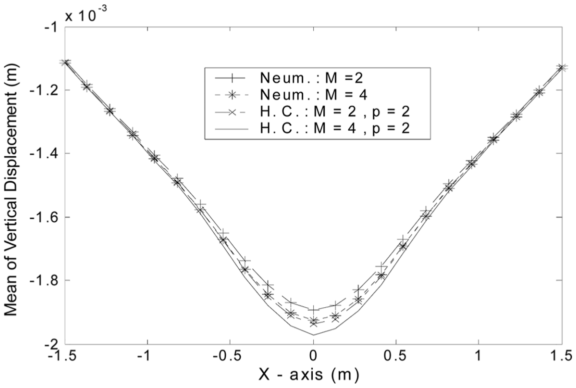

Fig. 4 The Mean of vertical displacements for points of side *DF* in case of stochastic operator and stochastic excitation

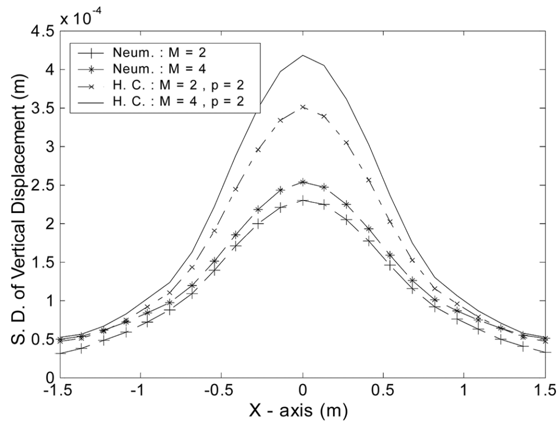

Fig. 5 The standard deviation of vertical displacements for points of side *DF* in case of stochastic operator and stochastic excitation

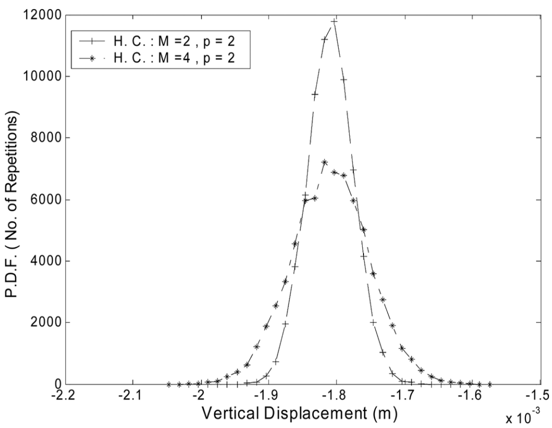

Fig. 6 The probability distribution function of vertical displacement at point *E* in case of stochastic excitation only

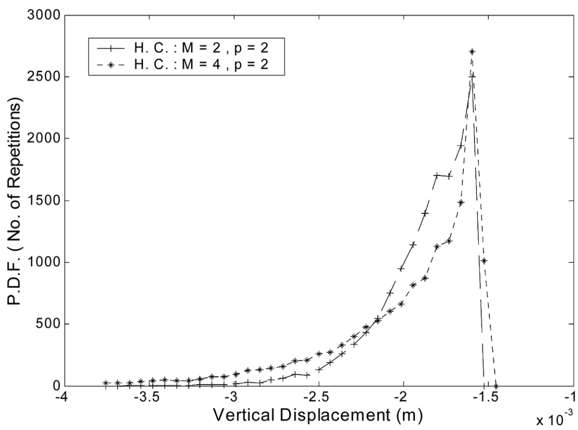

Fig. 7 The probability distribution function of the vertical displacement at point *E* in case of stochastic excitation and stochastic rigidity

5.1.3 Discussion of the results

The previous plots showed the mean and the S.D. of vertical displacements of different points at the culvert upper surface. This exhibition focuses on the case of stochastic excitation (force) with deterministic operator (rigidity) and the case where both operator and excitation are stochastic. In addition, the p.d.f.s of vertical displacement at point E was plotted using H.C. expansion. The main objective of the present problem is to show the ability of the solution methods in dealing with arbitrary shape domains. The most notable comment here is that H.C. expansion has better convergence than Neumann expansion especially for S.D. evaluation. The difference between the results of the two methods is relatively large and can not be neglected. This is due to the successive amount of approximations that made in evaluating the eigen-pairs numerically and the usage of numerical integrations in the evaluation of the stiffness and excitation matrices. Considering these approximations, the convergence of H.C. expansion becomes more evident.

5.2 Illustrative Example II : Stochastic plate bending problem

The current case aims to study plate bending problem that have stochastic Young's modulus and exposed to stochastic transverse loading.

5.2.1 Problem description

Assume a linear elastic isotropic square plate of side length L = 4.0 m, thickness = 0.15 m and simply supported at each side. This plate is fully loaded transversely by a uniformly distributed stochastic load which can be expressed as: $q = 13.25\ (1 + \beta(x, y, \theta))$ kN. Young's modulus is also assumed to be random in the form of: $E(x, y, \theta) = 2.2 \times 10^7(1 + \alpha(x, y, \theta))\mathrm{Kn/m^2}$, where $\alpha(x, y, \theta)$ and $\beta(x, y, \theta)$ are second-order stochastic processes. Poisson's ratio is assumed to be deterministic and assumed as 0.2. $\alpha(x, y, \theta)$ is assumed to be a Gaussian process with exponential covariance function in the form

$$C_\alpha(x_1, y_1; x_2, y_2) = \sigma_m^2 e^{-\frac{|x_1 - x_2|}{l_{x1}} - \frac{|y_1 - y_2|}{l_{y1}}} \tag{27}$$

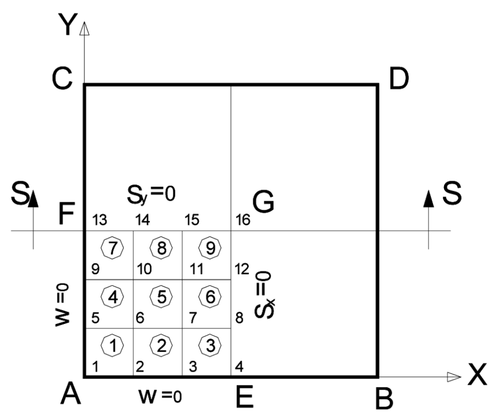

Fig. 8 FE mesh and B.C.s of the plate

The excitation function is also assumed to be Gaussian with triangular covariance function in the following from

$$C_\beta(x_1, y_1; x_2, y_2) = \begin{cases} \sigma_e^2\left(1 - \dfrac{|x_1 - x_2|}{l_{x2}}\right)\left(1 - \dfrac{|y_1 - y_2|}{l_{y2}}\right) & \text{if} \quad |x_1 - x_2| \le l_{x2} \\ & \text{and} \quad |y_1 - y_2| \le l_{y2} \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

The values of σ_m^2, σ_e^2 are initially assumed to be 1.0 each and 2.0 for each of the following parameters l_{x1}, l_{y1}, l_{x2} and l_{y2}.

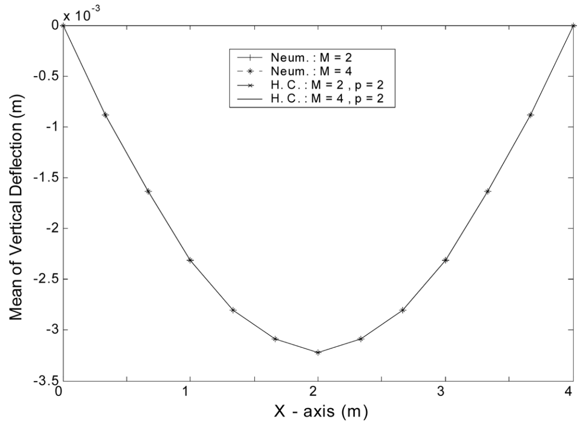

Fig. 9 The mean of vertical deflection of sec. S-S in case of stochastic excitation only

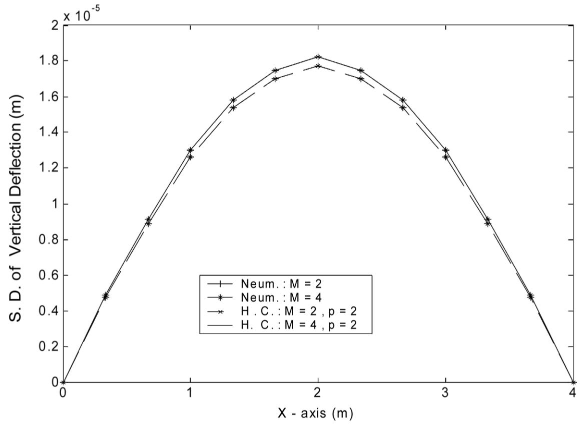

Fig. 10 The standard deviation of vertical deflection of sec. S-S in case of stochastic excitation only

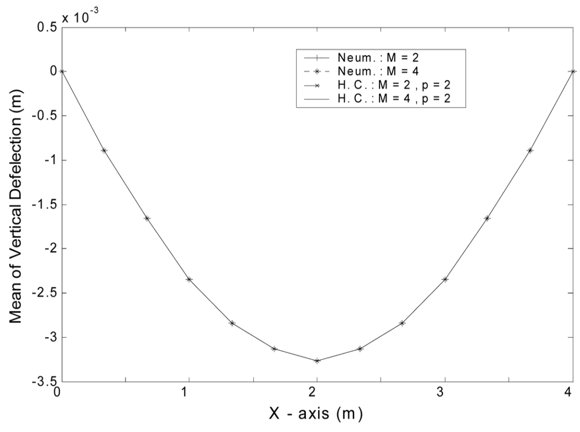

Fig. 11 The mean of vertical deflection of sec. S-S in case of stochastic operator and stochastic excitation

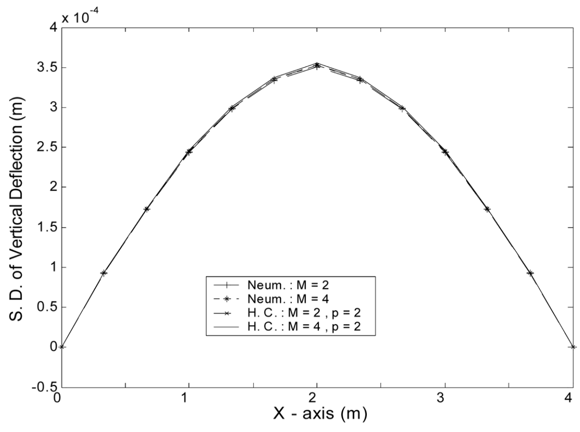

Fig. 12 The standard deviation of vertical deflection of sec. S-S in case of stochastic operator and stochastic excitation

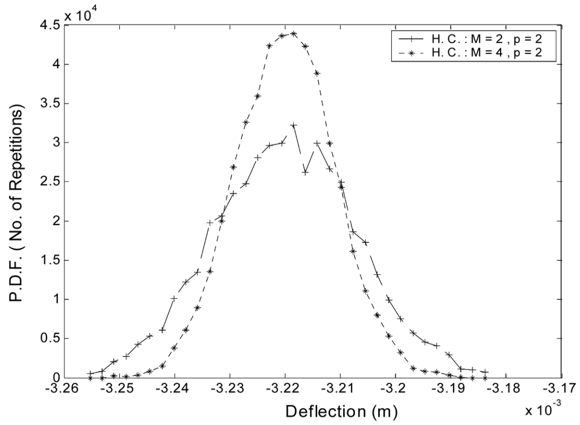

Fig. 13 The probability distribution function of vertical deflection at point G in case of stochastic excitation only

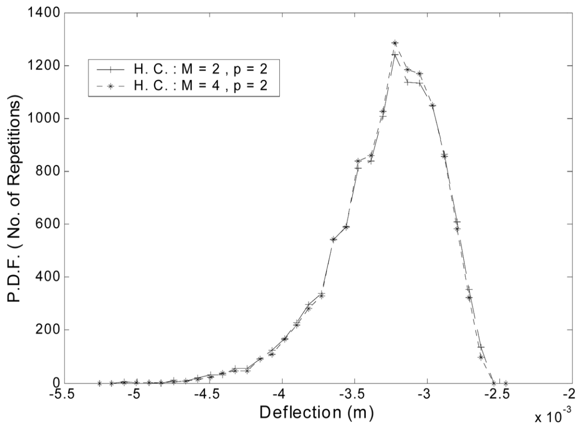

Fig. 14 The probability distribution function of vertical deflection at point G in case of stochastic operator and stochastic excitation

5.2.2 Results

With assumptions mentioned above, the mean and the standard deviation of section S-S are plotted in case of stochastic excitation with deterministic operator and in case of both the operator and excitation are stochastic. Also, the p.d.f. of the plate deflection at point G (max. deflection) can be plotted in these two cases using different methods as follows.

5.2.3 Discussion of the results

In case of stochastic excitation only, the mean values of the deflection in Fig. 9 are typical using different solution methods, this is due to the fact that these mean values are the solution under the deterministic part of the load whatever the used method is. Hence there is no approximation for the solution. The standard deviations of the deflection in this case are changing slightly using the different methods as shown in Fig. 9.

As expected, the best approximation is obtained using four dimensional H.C. with order two. Also, the probability distribution function is nearly Gaussian due to the Gaussian nature of the excitation function with the linearity of the system as shown in Fig. 13. In case of both stochastic excitation and operator, both of the mean value and the standard deviations are changing slightly according to the method used, see Figs. 11, 12. This is due to the small values of the standard deviations σ_m, σ_e. It is expected that a notable difference will be obtained using higher values for both σ_m, σ_e.

6. Conclusions

In this research, linear elastic isotropic plane strain and plate structures that have stochasticity in both the operator and the excitation are studied. This is the most general case of stochasticity can be considered for structural systems. The analysis utilizes the spectral stochastic finite element (SSFEM) with its two main expansions namely; Neumann and Homogeneous Chaos expansions. Both of random excitation and random operator fields are considered to be modeled as second order stochastic processes. The formulations are obtained for the system solution of two dimensional problems of plane strain and plate bending structures under the effect of stochastic loading and relevant rigidity using Neumann and Homogeneous Chaos expansions. Two finite element programs were developed to incorporate such formulations. Two illustrative examples are introduced: the first is a reinforced concrete culvert with stochastic in-plane rigidity and subjected to a stochastic load. The culvert is modeled as a nonrectangular plane strain problem since it is subjected to in-plane loading. The second example considers a simply supported square reinforced concrete slab subjected to uniform transverse loading. The slab modeling incorporates a stochastic flexural rigidity and an stochastic applied load. In each of the two examples, the first two statistical moments are evaluated using both of the two mentioned expansions. The probability density function of the structure response of each problem is obtained using Homogeneous Chaos expansion.

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