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Cubic normal distribution and its significance in structural reliability

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Abstract. Information on the distribution of the basic random variable is essential for the accurate analysis of structural reliability. The usual method for determining the distributions is to fit a candidate distribution to the histogram of available statistical data of the variable and perform approximate goodness-of-fit tests. Generally, such candidate distribution would have parameters that may be evaluated from the statistical moments of the statistical data. In the present paper, a cubic normal distribution, whose parameters are determined using the first four moments of available sample data, is investigated. A parameter table based on the first four moments, which simplifies parameter estimation, is given. The simplicity, generality, flexibility and advantages of this distribution in statistical data analysis and its significance in structural reliability evaluation are discussed. Numerical examples are presented to demonstrate these advantages.

Keywords: structural reliability; probability distributions; statistical moments; data fitting, fourth-moment reliability index.

1. Introduction

In structural reliability evaluation, basic random variables representing uncertain quantities, such as loads, environmental factors, material properties, structural dimensions, and variables introduced to account for modeling and prediction errors, are assumed to have known cumulative distribution functions (CDFs) or probability density functions (PDFs). Determination of the probability distributions of these basic random variables is essential for accurate evaluation of the reliability of a structure.

Usually, the method for determining the required distribution is to fit the histogram of the statistical data of a variable with a candidate distribution (Ang and Tang 1975), and apply statistical goodness-of-fit tests. More recently, the idea of determining a distribution as a weighted sum of common "basis" distribution was introduced (Lind and Chen 1986, Lind and Nowak 1987). Generally, a sum of weighted functions or positive kernels subject to some constraints can be used to approximate a distribution. A method of estimating complex distributions using the B-spline functions has been proposed, and the method is useful to identify an appropriate PDF for a

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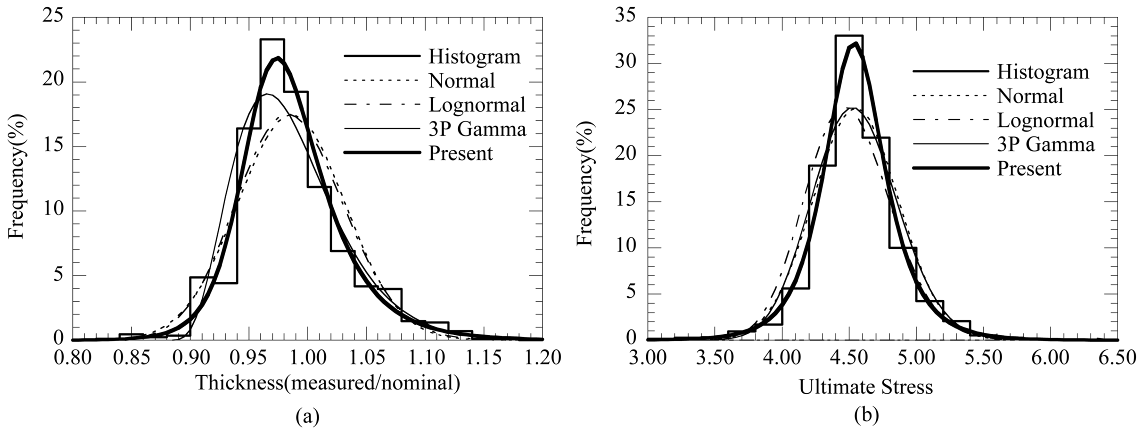

Fig. 1 Two histogram examples and data fitting results of practical data

continuous random variable directly from a sample without using any prior knowledge of the distribution form (Zong and Lam 1998, Zong and Lam 2000, Zong and Lam 2001). The Bayesian parameter estimation method, which provides a framework for processing of information and analysis uncertainties, is proposed by Der Kiureghian (2001) to derive the posterior distribution of model parameters reflecting epistemic uncertainties. Other approaches can be found in recent reports (Xie *et al.* 2002, Nadarajah and Kotz 2006, Schueller 2007).

It has been reported (Zhao and Ang 2002) that the two-parameter (2P) distributions such as the normal, lognormal, Gumbel, and Weibull distributions may not be appropriate when the skewness of the statistical data is important and must be reflected in the distribution. Three-parameter (3P) distributions such as 3P lognormal (Tichy 1993) and 3P Gamma distribution (Zhao and Ang 2002) have been suggested as the candidate distribution. The 3P distributions, which can effectively reflect the information of skewness as well as the mean value and standard deviation of statistical data, have more flexibility for fitting statistical data of basic random variables, and can more effectively fit the histograms of available data than 2P distributions.

If the 3P distributions are selected as the candidate distribution and the three parameters are determined, the distribution form and higher-order moments, such as kurtosis, generally can be evaluated. However, because the kurtosis of the 3P distributions is dependent on the skewness, it may not be consistent with those of the available data. This is illustrated with the following: The two histograms shown in Fig. 1 represent the observed variability in the properties of H-shape structure steel (Ono *et al.* 1986). Fig. 1(a) shows the histogram of the thickness, in which the number of the data is 885 and the first four moments of the data are obtained as the mean value μ = 0.986, the standard deviation σ = 0.0457, the skewness α_3 = 0.883 and the kurtosis α_4 = 5.991. Fig. 1(b) shows the histogram of the ultimate stress, in which the number of the data is 1932 and the first four moments of the data are μ = 4.549, σ = 0.317, α_3 = 0.153, and α_4 = 6.037. The kurtosis of the 3P Gamma distribution that has the same mean value, standard deviation, and skewness of the data in Fig. 1(a) and Fig. 1(b) can be obtained as 4.17 and 3.035, respectively. Apparently, the kurtosis of the 3P distributions is too small to match those of the data for the two illustrated cases.

This is to say, the 3P distributions are not flexible enough to reflect the kurtosis of statistical data of a random variable, and distributions that can be determined by effectively using the information of kurtosis as well as the mean value, standard deviation, and skewness of the statistical data are required.

For the above purpose, four-parameter (4P) distributions are required, such as Lambda distribution (Ramberg and Schmeiser 1974), the Pearson, Johnson, and the Burr systems (Stuart and Ord 1987). However, the Pearson and Johnson systems incorporate a number of functional forms, the Burr system does not include symmetric distributions, and the Lambda distribution does not include the normal distribution.

In the present paper, the cubic normal distribution is investigated, in which the parameters can be determined in terms of the mean value, standard deviation, the skewness, and the kurtosis of the sample data. A table based on the first four moments, which simplifies parameter estimation, is given. From the investigation of this distribution, one can see that this distribution, having characteristics of simplicity, generality and flexibility, can be applied as a candidate distribution in fitting statistical data of basic random variables and can be used to represent or approximate the most commonly used two- and three-parameter distributions. A fourth-moment reliability index based on this distribution is derived and its application in structural reliability assessment is discussed.

2. The cubic normal distribution

2.1 Definition of the distribution

The distribution is defined on the base of the following polynomial normal transformation (Fleishman 1978, Hong and Lind 1996, Zhao *et al.* 2002, Chen and Tung 2003)

$$\frac{X-\mu}{\sigma} = S_U(U) = a_1 + a_2 U + a_3 U^2 + a_4 U^3 \tag{1a}$$

The CDF and PDF corresponding to Eq. (1a) are expressed as

$$F(X) = \Phi(U) \tag{1b}$$

$$f(X) = \frac{\phi(U)}{\sigma(a_2 + 2a_3 U + 3a_4 U^2)} \tag{1c}$$

in which F, f, μ, and σ are the CDF, PDF, mean value, and standard deviation of X, respectively; Φ and ϕ are the CDF and PDF of a standard normal random variable U; and a_1, a_2, a_3, and a_4 are deterministic coefficients. Since the distribution is defined by the third order polynomial of standard normal random variables, hereafter, we call it cubic normal distribution.

Apparently, μ, σ, a_1, a_2, a_3, and a_4 are the parameters of the distribution, and the method for estimating the parameters will be discussed in the next section.

2.2 Parameter estimation and table construction

In the present paper, the approach for estimating the parameters is based on matching the first four product moments of the data. This will be illustrated in detail as follows.

For a random variable, if the first four moments (mean value μ, standard deviation σ, skewness

α_3, and kurtosis α_4) are known, the parameters a_1, a_2, a_3, and a_4 are obtained by making the first four central moment of $S_U(U)$ equal to those of $X_s = (X - \mu)/\sigma$ with the aid of Eq. (1a), i.e.

$$a_1 + a_3 = 0 \tag{2a}$$

$$a_2^2 + 2a_3^2 + 6a_2a_4 + 15a_4^2 = 1 \tag{2b}$$

$$6a_2^2a_3 + 8a_3^3 + 72a_2a_3a_4 + 270a_3a_4^2 = \alpha_3 \tag{2c}$$

$$3(a_2^4 + 20a_2^3a_4 + 210a_2^2a_4^2 + 1260a_2a_4^3 + 3465a_4^4) + 12a_3^2(5a_2^2 + 5a_3^2 + 78a_2a_4 + 375a_4^2) = \alpha_4 \tag{2d}$$

Simplifying Eq. (2), the following equations of parameters a_2 and a_4 can be obtained

$$2A_1A_2 = \alpha_3^2 \tag{3a}$$

$$3A_1A_3 + 3A_4 = \alpha_4 \tag{3b}$$

where

$$A_1 = 1 - a_2^2 - 6a_2a_4 - 15a_4^2 \tag{4a}$$

$$A_2 = 2 + a_2^2 + 24a_2a_4 + 105a_4^2 \tag{4b}$$

$$A_3 = 5 + 5a_2^2 + 126a_2a_4 + 675a_4^2 \tag{4c}$$

$$A_4 = a_2^4 + 20a_2^3a_4 + 210a_2^2a_4^2 + 1260a_2a_4^3 + 3465a_4^4 \tag{4d}$$

Since the values α_3 and α_4 are known, the parameters a_2 and a_4 can be obtained form Eq. (3), which can be solved by a proper nonlinear equations solver, such as the "FindRoot" function in "Mathematica" software (Wolfram 1999). After the parameters a_2 and a_4 have been determined, the parameters a_1 and a_3 can be readily given as

$$a_3 = -a_1 = \frac{\alpha_3}{2A_2} \tag{5}$$

From the above description, one can clearly see that the four parameters a_1, a_2, a_3, and a_4 are functions of α_3 and α_4, but do not dependent upon μ and σ. For convenience, a table used to approximate the four parameters of a_1, a_2, a_3, and a_4 is given (see Table A). The values of a_1, a_2, a_3, and a_4 are given in Table A for selected values of α_3 and α_4. If the values of α_3 and α_4 are known, the four parameters a_1, a_2, a_3, and a_4 can be determined from Table A using the α_3 and α_4 as entry points values. One simply picks the values of a_1, a_2, a_3, and a_4 for which the α_3 and α_4 are closest to the desired values. If α_3 is negative, one uses its absolute value, and after finding the values of a_1, a_2, a_3, and a_4, changes the sign of a_1 and a_3 (The density with a skewness of $-\alpha_3$ is the mirror image of the density with a skewness of α_3). For relatively small values of α_3 and α_4, empirical equations for each polynomial coefficient have been developed (Zhao and Lu 2007). In particular, if $\alpha_3 = 0$ and $\alpha_4 = 3$, then the parameters are obtained as $a_1 = a_3 = a_4 = 0$, $a_2 = 1$, and Eq. (1a) reduces to $(X - \mu)/\sigma = U$, and the corresponding distribution is the normal distribution.

Table A The four parameters of a_1, a_2, a_3 and a_4 for given values of skewness (α_3) and kurtosis (α_4)

$\alpha_3=0.00$				$\alpha_3=0.10$				$\alpha_3=0.20$				$\alpha_3=0.30$			
α_4	$a_1=-a_3$	a_2	a_4	α_4	$a_1=-a_3$	a_2	a_4	α_4	$a_1=-a_3$	a_2	a_4	α_4	$a_1=-a_3$	a_2	a_4
2.0	0.0	1.2210	−0.0802	2.0	−0.0280	1.2268	−0.0828	2.0	−0.0590	1.2459	−0.0917	2.2	−0.0766	1.1836	−0.0678
2.2	0.0	1.1478	−0.0520	2.2	−0.0233	1.1514	−0.0535	2.2	−0.0481	1.1627	−0.0585	2.4	−0.0660	1.1225	−0.0443
2.4	0.0	1.0972	−0.0335	2.4	−0.0206	1.0998	−0.0346	2.4	−0.0422	1.1079	−0.0380	2.6	−0.0595	1.0778	−0.0279
2.6	0.0	1.0585	−0.0199	2.6	−0.0189	1.0605	−0.0207	2.6	−0.0385	1.0668	−0.0233	2.8	−0.0550	1.0425	−0.0154
2.8	0.0	1.0269	−0.0090	2.8	−0.0177	1.0285	−0.0097	2.8	−0.0358	1.0336	−0.0118	3.0	−0.0517	1.0131	−0.0053
3.0	0.0	1.0000	0.0000	3.0	−0.0167	1.0014	−0.0006	3.0	−0.0338	1.0057	−0.0023	3.2	−0.0491	0.9878	0.0033
3.2	0.0	0.9765	0.0078	3.2	−0.0160	0.9777	0.0073	3.2	−0.0322	0.9814	0.0058	3.4	−0.0470	0.9654	0.0107
3.4	0.0	0.9555	0.0146	3.4	−0.0153	0.9566	0.0142	3.4	−0.0309	0.9599	0.0129	3.6	−0.0452	0.9454	0.0172
3.6	0.0	0.9365	0.0207	3.6	−0.0148	0.9375	0.0203	3.6	−0.0298	0.9404	0.0192	3.8	−0.0437	0.9271	0.0231
3.8	0.0	0.9191	0.0263	3.8	−0.0143	0.9200	0.0259	3.8	−0.0288	0.9226	0.0249	4.0	−0.0424	0.9102	0.0285
4.0	0.0	0.9030	0.0314	4.0	−0.0139	0.9038	0.0310	4.0	−0.0280	0.9062	0.0301	4.2	−0.0413	0.8946	0.0335
4.2	0.0	0.8879	0.0361	4.2	−0.0136	0.8886	0.0358	4.2	−0.0273	0.8908	0.0349	4.4	−0.0402	0.8799	0.0380
4.4	0.0	0.8738	0.0404	4.4	−0.0132	0.8744	0.0402	4.4	−0.0266	0.8765	0.0394	4.6	−0.0393	0.8661	0.0423
4.6	0.0	0.8604	0.0445	4.6	−0.0129	0.8610	0.0443	4.6	−0.0260	0.8629	0.0436	4.8	−0.0384	0.8531	0.0463
4.8	0.0	0.8477	0.0484	4.8	−0.0127	0.8483	0.0482	4.8	−0.0255	0.8501	0.0475	5.0	−0.0377	0.8407	0.0501
5.0	0.0	0.8357	0.0521	5.0	−0.0124	0.8362	0.0518	5.0	−0.0250	0.8379	0.0512	5.2	−0.0370	0.8289	0.0537
5.2	0.0	0.8241	0.0555	5.2	−0.0122	0.8247	0.0553	5.2	−0.0245	0.8262	0.0547	5.4	−0.0363	0.8176	0.0571
5.4	0.0	0.8131	0.0588	5.4	−0.0120	0.8136	0.0586	5.4	−0.0241	0.8151	0.0581	5.6	−0.0357	0.8067	0.0603
5.6	0.0	0.8025	0.0620	5.6	−0.0118	0.8029	0.0618	5.6	−0.0237	0.8043	0.0613	5.8	−0.0351	0.7963	0.0634
5.8	0.0	0.7922	0.0650	5.8	−0.0116	0.7927	0.0648	5.8	−0.0233	0.7940	0.0643	6.0	−0.0346	0.7862	0.0664
6.0	0.0	0.7824	0.0679	6.0	−0.0114	0.7828	0.0677	6.0	−0.0230	0.7841	0.0672	6.2	−0.0341	0.7765	0.0693
6.2	0.0	0.7728	0.0707	6.2	−0.0113	0.7732	0.0705	6.2	−0.0226	0.7744	0.0701	6.4	−0.0336	0.7671	0.0720
6.4	0.0	0.7636	0.0734	6.4	−0.0111	0.7639	0.0732	6.4	−0.0223	0.7651	0.0728	6.6	−0.0332	0.7580	0.0747
6.6	0.0	0.7546	0.0760	6.6	−0.0110	0.7550	0.0758	6.6	−0.0220	0.7561	0.0754	6.8	−0.0328	0.7491	0.0772
6.8	0.0	0.7459	0.0785	6.8	−0.0109	0.7462	0.0784	6.8	−0.0218	0.7473	0.0779	7.0	−0.0324	0.7405	0.0797
7.0	0.0	0.7374	0.0809	7.0	−0.0107	0.7377	0.0808	7.0	−0.0215	0.7388	0.0804	7.2	−0.0320	0.7322	0.0821
7.2	0.0	0.7291	0.0833	7.2	−0.0106	0.7295	0.0832	7.2	−0.0213	0.7305	0.0828	7.4	−0.0316	0.7240	0.0844
7.4	0.0	0.7211	0.0856	7.4	−0.0105	0.7214	0.0854	7.4	−0.0210	0.7224	0.0851	7.6	−0.0313	0.7160	0.0867
7.6	0.0	0.7132	0.0878	7.6	−0.0104	0.7135	0.0877	7.6	−0.0208	0.7145	0.0873	7.8	−0.0310	0.7083	0.0889
7.8	0.0	0.7055	0.0900	7.8	−0.0103	0.7058	0.0899	7.8	−0.0206	0.7067	0.0895	8.0	−0.0307	0.7007	0.0911
8.0	0.0	0.6980	0.0921	8.0	−0.0102	0.6983	0.0920	8.0	−0.0204	0.6992	0.0916	8.2	−0.0304	0.6932	0.0931
8.2	0.0	0.6907	0.0941	8.2	−0.0101	0.6909	0.0940	8.2	−0.0202	0.6918	0.0937	8.4	−0.0301	0.6860	0.0952
8.4	0.0	0.6835	0.0961	8.4	−0.0100	0.6837	0.0960	8.4	−0.0200	0.6846	0.0957	8.6	−0.0298	0.6788	0.0972
8.6	0.0	0.6764	0.0981	8.6	−0.0099	0.6767	0.0980	8.6	−0.0198	0.6775	0.0977	8.8	−0.0295	0.6719	0.0991
8.8	0.0	0.6695	0.1000	8.8	−0.0098	0.6697	0.0999	8.8	−0.0196	0.6705	0.0996	9.0	−0.0293	0.6650	0.1010
9.0	0.0	0.6627	0.1019	9.0	−0.0097	0.6629	0.1018	9.0	−0.0195	0.6637	0.1015	9.2	−0.0290	0.6583	0.1029

$\alpha_3=0.40$				$\alpha_3=0.50$				$\alpha_3=0.60$				$\alpha_3=0.70$			
α_4	$a_1=-a_3$	a_2	a_4	α_4	$a_1=-a_3$	a_2	a_4	α_4	$a_1=-a_3$	a_2	a_4	α_4	$a_1=-a_3$	a_2	a_4
2.2	−0.1136	1.2192	−0.0848	2.4	−0.1311	1.1811	−0.0714	2.6	−0.1512	1.1576	−0.0645	2.6	−0.2462	1.2756	−0.1300
2.4	−0.0941	1.1455	−0.0545	2.6	−0.1120	1.1202	−0.0465	2.8	−0.1310	1.1032	−0.0420	2.8	−0.1744	1.1436	−0.0622
2.6	−0.0832	1.0949	−0.0352	2.8	−0.1005	1.0755	−0.0294	3.0	−0.1183	1.0619	−0.0260	3.0	−0.1517	1.0926	−0.0403
2.8	−0.0761	1.0560	−0.0210	3.0	−0.0926	1.0400	−0.0165	3.2	−0.1095	1.0284	−0.0137	3.2	−0.1372	1.0530	−0.0246
3.0	−0.0710	1.0242	−0.0099	3.2	−0.0868	1.0104	−0.0060	3.4	−0.1029	1.0002	−0.0036	3.4	−0.1271	1.0206	−0.0125
3.2	−0.0671	0.9972	−0.0006	3.4	−0.0824	0.9850	0.0027	3.6	−0.0978	0.9757	0.0049	3.6	−0.1196	0.9931	−0.0025
3.4	−0.0640	0.9736	0.0074	3.6	−0.0787	0.9625	0.0103	3.8	−0.0936	0.9540	0.0122	3.8	−0.1136	0.9692	0.0059
3.6	−0.0614	0.9526	0.0143	3.8	−0.0757	0.9424	0.0170	4.0	−0.0901	0.9345	0.0188	4.0	−0.1088	0.9479	0.0132
3.8	−0.0592	0.9336	0.0205	4.0	−0.0731	0.9241	0.0230	4.2	−0.0871	0.9166	0.0246	4.2	−0.1047	0.9286	0.0197
4.0	−0.0574	0.9161	0.0262	4.2	−0.0709	0.9072	0.0284	4.4	−0.0844	0.9002	0.0300	4.4	−0.1012	0.9110	0.0256
4.2	−0.0557	0.9000	0.0313	4.4	−0.0689	0.8916	0.0334	4.6	−0.0821	0.8848	0.0349	4.6	−0.0982	0.8948	0.0309
4.4	−0.0543	0.8849	0.0361	4.6	−0.0672	0.8769	0.0381	4.8	−0.0801	0.8705	0.0395	4.8	−0.0955	0.8796	0.0358
4.6	−0.0530	0.8708	0.0405	4.8	−0.0656	0.8631	0.0424	5.0	−0.0782	0.8569	0.0437	5.0	−0.0932	0.8654	0.0403
4.8	−0.0518	0.8574	0.0446	5.0	−0.0642	0.8501	0.0464	5.2	−0.0766	0.8441	0.0477	5.2	−0.0910	0.8521	0.0445
5.0	−0.0507	0.8447	0.0485	5.2	−0.0629	0.8377	0.0502	5.4	−0.0750	0.8319	0.0515	5.4	−0.0891	0.8394	0.0485

Table A Continued

$\alpha_3=0.40$				$\alpha_3=0.50$				$\alpha_3=0.60$				$\alpha_3=0.70$			
α_4	$a_1=-a_3$	a_2	a_4	α_4	$a_1=-a_3$	a_2	a_4	α_4	$a_1=-a_3$	a_2	a_4	α_4	$a_1=-a_3$	a_2	a_4
5.2	−0.0497	0.8327	0.0522	5.4	−0.0617	0.8259	0.0538	5.6	−0.0736	0.8203	0.0550	5.6	−0.0873	0.8273	0.0523
5.4	−0.0488	0.8212	0.0557	5.6	−0.0606	0.8146	0.0573	5.8	−0.0723	0.8092	0.0584	5.8	−0.0857	0.8158	0.0558
5.6	−0.0480	0.8101	0.0590	5.8	−0.0595	0.8038	0.0605	6.0	−0.0711	0.7985	0.0616	6.0	−0.0842	0.8048	0.0592
5.8	−0.0472	0.7995	0.0622	6.0	−0.0586	0.7934	0.0636	6.2	−0.0700	0.7882	0.0647	6.2	−0.0828	0.7942	0.0624
6.0	−0.0465	0.7893	0.0652	6.2	−0.0577	0.7833	0.0666	6.4	−0.0690	0.7783	0.0677	6.4	−0.0815	0.7840	0.0655
6.2	−0.0458	0.7795	0.0681	6.4	−0.0569	0.7736	0.0695	6.6	−0.0680	0.7687	0.0705	6.6	−0.0803	0.7741	0.0684
6.4	−0.0451	0.7699	0.0709	6.6	−0.0561	0.7643	0.0723	6.8	−0.0670	0.7594	0.0733	6.8	−0.0791	0.7646	0.0712
6.6	−0.0445	0.7607	0.0736	6.8	−0.0553	0.7552	0.0749	7.0	−0.0662	0.7504	0.0759	7.0	−0.0780	0.7554	0.0740
6.8	−0.0439	0.7517	0.0762	7.0	−0.0546	0.7463	0.0775	7.2	−0.0653	0.7417	0.0784	7.2	−0.0770	0.7465	0.0766
7.0	−0.0434	0.7430	0.0788	7.2	−0.0540	0.7377	0.0800	7.4	−0.0645	0.7332	0.0809	7.4	−0.0760	0.7378	0.0791
7.2	−0.0429	0.7346	0.0812	7.4	−0.0533	0.7294	0.0824	7.6	−0.0638	0.7249	0.0833	7.6	−0.0751	0.7293	0.0816
7.4	−0.0424	0.7263	0.0836	7.6	−0.0527	0.7212	0.0847	7.8	−0.0631	0.7168	0.0856	7.8	−0.0743	0.7211	0.0840
7.6	−0.0419	0.7183	0.0858	7.8	−0.0522	0.7133	0.0870	8.0	−0.0624	0.7090	0.0879	8.0	−0.0734	0.7131	0.0863
7.8	−0.0415	0.7104	0.0881	8.0	−0.0516	0.7055	0.0892	8.2	−0.0617	0.7013	0.0901	8.2	−0.0726	0.7053	0.0885
8.0	−0.0411	0.7028	0.0902	8.2	−0.0511	0.6980	0.0913	8.4	−0.0611	0.6937	0.0922	8.4	−0.0719	0.6976	0.0907
8.2	−0.0406	0.6953	0.0924	8.4	−0.0506	0.6905	0.0934	8.6	−0.0605	0.6864	0.0943	8.6	−0.0712	0.6902	0.0928
8.4	−0.0403	0.6880	0.0944	8.6	−0.0501	0.6833	0.0955	8.8	−0.0599	0.6792	0.0963	8.8	−0.0705	0.6828	0.0949
8.6	−0.0399	0.6808	0.0964	8.8	−0.0496	0.6762	0.0975	9.0	−0.0594	0.6721	0.0983	9.0	−0.0698	0.6757	0.0969
8.8	−0.0395	0.6737	0.0984	9.0	−0.0492	0.6692	0.0994	9.2	−0.0589	0.6652	0.1002	9.2	−0.0692	0.6687	0.0989
9.0	−0.0392	0.6668	0.1003	9.2	−0.0488	0.6624	0.1013	9.4	−0.0584	0.6584	0.1021	9.4	−0.0685	0.6618	0.1008
9.2	−0.0388	0.6600	0.1022	9.4	−0.0483	0.6556	0.1032	9.6	−0.0579	0.6518	0.1039	9.6	−0.0680	0.6550	0.1027

$\alpha_3=0.80$				$\alpha_3=0.9$				$\alpha_3=1.0$				$\alpha_3=1.2$			
α_4	$a_1=-a_3$	a_2	a_4	α_4	$a_1=-a_3$	a_2	a_4	α_4	$a_1=-a_3$	a_2	a_4	α_4	$a_1=-a_3$	a_2	a_4
3.0	−0.2021	1.1369	−0.0638	3.2	−0.2380	1.1372	−0.0705	3.6	−0.2375	1.0904	−0.0524	4.2	−0.2960	1.0670	−0.0565
3.2	−0.1750	1.0872	−0.0412	3.4	−0.2027	1.0866	−0.0450	3.8	−0.2096	1.0507	−0.0330	4.4	−0.2578	1.0322	−0.0350
3.4	−0.1580	1.0483	−0.0252	3.6	−0.1815	1.0476	−0.0278	4.0	−0.1910	1.0175	−0.0186	4.6	−0.2333	1.0016	−0.0196
3.6	−0.1461	1.0162	−0.0128	3.8	−0.1670	1.0152	−0.0147	4.2	−0.1776	0.9891	−0.0071	4.8	−0.2158	0.9746	−0.0075
3.8	−0.1372	0.9889	−0.0026	4.0	−0.1563	0.9875	−0.0041	4.4	−0.1674	0.9643	0.0024	5.0	−0.2027	0.9506	0.0025
4.0	−0.1302	0.9651	0.0059	4.2	−0.1480	0.9634	0.0048	4.6	−0.1594	0.9424	0.0105	5.2	−0.1924	0.9293	0.0109
4.2	−0.1246	0.9439	0.0133	4.4	−0.1413	0.9420	0.0124	4.8	−0.1528	0.9227	0.0176	5.4	−0.1841	0.9100	0.0182
4.4	−0.1198	0.9247	0.0199	4.6	−0.1358	0.9227	0.0192	5.0	−0.1472	0.9048	0.0239	5.6	−0.1771	0.8923	0.0246
4.6	−0.1158	0.9072	0.0258	4.8	−0.1311	0.9051	0.0252	5.2	−0.1424	0.8882	0.0295	5.8	−0.1712	0.8761	0.0304
4.8	−0.1123	0.8910	0.0311	5.0	−0.1270	0.8888	0.0307	5.4	−0.1383	0.8729	0.0347	6.0	−0.1661	0.8610	0.0357
5.0	−0.1093	0.8759	0.0360	5.2	−0.1234	0.8737	0.0357	5.6	−0.1346	0.8585	0.0395	6.2	−0.1616	0.8469	0.0405
5.2	−0.1065	0.8618	0.0406	5.4	−0.1203	0.8595	0.0403	5.8	−0.1313	0.8450	0.0439	6.4	−0.1576	0.8335	0.0450
5.4	−0.1040	0.8484	0.0448	5.6	−0.1174	0.8461	0.0446	6.0	−0.1284	0.8322	0.0480	6.6	−0.1540	0.8210	0.0492
5.6	−0.1018	0.8358	0.0488	5.8	−0.1149	0.8334	0.0487	6.2	−0.1257	0.8201	0.0519	6.8	−0.1507	0.8090	0.0531
5.8	−0.0998	0.8238	0.0526	6.0	−0.1125	0.8214	0.0525	6.4	−0.1233	0.8085	0.0556	7.0	−0.1477	0.7976	0.0568
6.0	−0.0979	0.8123	0.0562	6.2	−0.1104	0.8099	0.0561	6.6	−0.1210	0.7974	0.0590	7.2	−0.1450	0.7867	0.0603
6.2	−0.0962	0.8013	0.0595	6.4	−0.1084	0.7989	0.0595	6.8	−0.1189	0.7868	0.0624	7.4	−0.1424	0.7762	0.0636
6.4	−0.0946	0.7908	0.0628	6.6	−0.1066	0.7883	0.0627	7.0	−0.1170	0.7766	0.0655	7.6	−0.1401	0.7661	0.0668
6.6	−0.0931	0.7806	0.0658	6.8	−0.1049	0.7782	0.0658	7.2	−0.1152	0.7667	0.0685	7.8	−0.1379	0.7564	0.0698
6.8	−0.0917	0.7708	0.0688	7.0	−0.1033	0.7683	0.0688	7.4	−0.1135	0.7572	0.0714	8.0	−0.1359	0.7470	0.0727
7.0	−0.090	0.7613	0.0716	7.2	−0.1018	0.7589	0.0717	7.6	−0.1119	0.7480	0.0742	8.2	−0.1340	0.7379	0.0755
7.2	−0.0891	0.7521	0.0743	7.4	−0.1004	0.7497	0.0744	7.8	−0.1104	0.7391	0.0769	8.4	−0.1322	0.7291	0.0781
7.4	−0.0880	0.7432	0.0770	7.6	−0.0991	0.7408	0.0771	8.0	−0.1090	0.7304	0.0794	8.6	−0.1304	0.7206	0.0807
7.6	−0.0869	0.7346	0.0795	7.8	−0.0978	0.7321	0.0796	8.2	−0.1076	0.7220	0.0819	8.8	−0.1288	0.7122	0.0832
7.8	−0.0858	0.7262	0.0820	8.0	−0.0966	0.7237	0.0821	8.4	−0.1064	0.7138	0.0843	9.0	−0.1273	0.7041	0.0856
8.0	−0.0848	0.7180	0.0844	8.2	−0.0955	0.7155	0.0845	8.6	−0.1051	0.7058	0.0867	9.2	−0.1259	0.6962	0.0880
8.2	−0.0839	0.7100	0.0867	8.4	−0.0944	0.7076	0.0868	8.8	−0.1040	0.6980	0.0890	9.4	−0.1245	0.6885	0.0902
8.4	−0.0830	0.7022	0.0889	8.6	−0.0934	0.6998	0.0891	9.0	−0.1029	0.6903	0.0912	9.6	−0.1231	0.6810	0.0925
8.6	−0.0821	0.6946	0.0911	8.8	−0.0924	0.6922	0.0912	9.2	−0.1018	0.6829	0.0933	9.8	−0.1219	0.6736	0.0946
8.8	−0.0813	0.6871	0.0932	9.0	−0.0915	0.6847	0.0934	9.4	−0.1008	0.6756	0.0954	10.0	−0.1207	0.6664	0.0967
9.0	−0.0805	0.6799	0.0953	9.2	−0.0906	0.6774	0.0955	9.6	−0.0999	0.6685	0.0974	10.2	−0.1195	0.6593	0.0987
9.2	−0.0797	0.6727	0.0973	9.4	−0.0897	0.6703	0.0975	9.8	−0.0989	0.6615	0.0994	10.4	−0.1184	0.6524	0.1007
9.4	−0.0790	0.6657	0.0993	9.6	−0.0889	0.6633	0.0995	10.0	−0.0980	0.6546	0.1014	10.6	−0.1173	0.6456	0.1026
9.6	−0.0783	0.6589	0.1012	9.8	−0.0881	0.6565	0.1014	10.2	−0.0972	0.6479	0.1033	10.8	−0.1163	0.6390	0.1045
9.8	−0.0776	0.6521	0.1031	10.0	−0.0873	0.6497	0.1033	10.4	−0.0964	0.6413	0.1051	11.0	−0.1153	0.6324	0.1064
10.0	−0.0770	0.6455	0.1049	10.2	−0.0866	0.6431	0.1051	10.6	−0.0956	0.6348	0.1069	11.2	−0.1144	0.6260	0.1082

Table A Continued

α_3=1.4				α_3=1.6				α_3=1.8				α_3=2.0			
α_4	a_1=$-a_3$	a_2	a_4	α_4	a_1=$-a_3$	a_2	a_4	α_4	a_1=$-a_3$	a_2	a_4	α_4	a_1=$-a_3$	a_2	a_4
5.0	−0.3386	1.0218	−0.0509	2.6	−0.1793	1.1848	−0.0790	6.0	−0.3603	0.9665	−0.0370	8.4	−0.4001	0.8616	−0.0126
5.2	−0.2959	0.9958	−0.0302	2.8	−0.1503	1.1214	−0.0507	6.2	−0.3202	0.9484	−0.0194	8.6	−0.3609	0.8539	0.0020
5.4	−0.2681	0.9703	−0.0152	3.0	−0.1337	1.0759	−0.0324	6.4	−0.2926	0.9279	−0.0059	8.8	−0.3337	0.8407	0.0134
5.6	−0.2482	0.9464	−0.0034	3.2	−0.1225	1.0397	−0.0186	6.6	−0.2726	0.9077	0.0050	9.0	−0.3137	0.8263	0.0227
5.8	−0.2332	0.9247	0.0064	3.4	−0.1144	1.0096	−0.0077	6.8	−0.2573	0.8888	0.0140	9.2	−0.2985	0.8121	0.0305
6.0	−0.2215	0.9049	0.0147	3.6	−0.1082	0.9838	0.0015	7.0	−0.2452	0.8713	0.0217	9.4	−0.2862	0.7985	0.0371
6.2	−0.2120	0.8869	0.0219	3.8	−0.1033	0.9611	0.0093	7.2	−0.2353	0.8551	0.0284	9.6	−0.2761	0.7855	0.0430
6.4	−0.2041	0.8703	0.0282	4.0	−0.0991	0.9408	0.0162	7.4	−0.2270	0.8400	0.0344	9.8	−0.2675	0.7733	0.0483
6.6	−0.1973	0.8550	0.0339	4.2	−0.0957	0.9223	0.0223	7.6	−0.2199	0.8259	0.0398	10.0	−0.2600	0.7616	0.0531
6.8	−0.1915	0.8406	0.0391	4.4	−0.0927	0.9053	0.0279	7.8	−0.2137	0.8127	0.0447	10.2	−0.2535	0.7505	0.0575
7.0	−0.1864	0.8271	0.0438	4.6	−0.0900	0.8895	0.0330	8.0	−0.2083	0.8002	0.0493	10.4	−0.2477	0.7398	0.0615
7.2	−0.1818	0.8144	0.0482	4.8	−0.0877	0.8748	0.0377	8.2	−0.2034	0.7883	0.0535	10.6	−0.2424	0.7296	0.0653
7.4	−0.1777	0.8023	0.0524	5.0	−0.0856	0.8610	0.0421	8.4	−0.1990	0.7770	0.0574	10.8	−0.2377	0.7199	0.0689
7.6	−0.1740	0.7908	0.0562	5.2	−0.0837	0.8479	0.0462	8.6	−0.1950	0.7662	0.0611	11.0	−0.2334	0.7104	0.0723
7.8	−0.1706	0.7798	0.0598	5.4	−0.0820	0.8355	0.0501	8.8	−0.1913	0.7559	0.0646	11.2	−0.2294	0.7013	0.0755
8.0	−0.1675	0.7693	0.0633	5.6	−0.0804	0.8236	0.0537	9.0	−0.1880	0.7459	0.0679	11.4	−0.2257	0.6925	0.0785
8.2	−0.1646	0.7592	0.0665	5.8	−0.0789	0.8123	0.0572	9.2	−0.1849	0.7363	0.0710	11.6	−0.2223	0.6840	0.0814
8.4	−0.1619	0.7495	0.0696	6.0	−0.0776	0.8015	0.0605	9.4	−0.1820	0.7270	0.0740	11.8	−0.2191	0.6757	0.0841
8.6	−0.1594	0.7400	0.0726	6.2	−0.0763	0.7910	0.0636	9.6	−0.1793	0.7181	0.0769	12.0	−0.2161	0.6677	0.0868
8.8	−0.1571	0.7309	0.0755	6.4	−0.0751	0.7810	0.0666	9.8	−0.1768	0.7094	0.0797	12.2	−0.2133	0.6599	0.0894
9.0	−0.1549	0.7221	0.0782	6.6	−0.0740	0.7713	0.0695	10.0	−0.1744	0.7009	0.0823	12.4	−0.2106	0.6523	0.0918
9.2	−0.1528	0.7135	0.0808	6.8	−0.0730	0.7619	0.0723	10.2	−0.1721	0.6927	0.0849	12.6	−0.2081	0.6448	0.0942
9.4	−0.1509	0.7052	0.0834	7.0	−0.0720	0.7528	0.0750	10.4	−0.1700	0.6847	0.0873	12.8	−0.2057	0.6375	0.0965
9.6	−0.1491	0.6971	0.0858	7.2	−0.0711	0.7440	0.0776	10.6	−0.1680	0.6770	0.0897	13.0	−0.2034	0.6304	0.0987
9.8	−0.1473	0.6892	0.0882	7.4	−0.0702	0.7354	0.0801	10.8	−0.1661	0.6694	0.0920	13.2	−0.2013	0.6235	0.1009
10.0	−0.1457	0.6815	0.0905	7.6	−0.0694	0.7270	0.0825	11.0	−0.1643	0.6619	0.0943	13.4	−0.1992	0.6167	0.1030
10.2	−0.1441	0.6740	0.0928	7.8	−0.0686	0.7189	0.0848	11.2	−0.1626	0.6547	0.0965	13.6	−0.1973	0.6100	0.1051
10.4	−0.1426	0.6666	0.0949	8.0	−0.0679	0.7109	0.0871	11.4	−0.1609	0.6476	0.0986	13.8	−0.1954	0.6035	0.1070
10.6	−0.1411	0.6594	0.0971	8.2	−0.0672	0.7032	0.0893	11.6	−0.1593	0.6407	0.1006	14.0	−0.1936	0.5971	0.1090
10.8	−0.1397	0.6524	0.0991	8.4	−0.0665	0.6956	0.0915	11.8	−0.1578	0.6338	0.1027	14.2	−0.1919	0.5907	0.1109
11.0	−0.1384	0.6455	0.1011	8.6	−0.0658	0.6882	0.0936	12.0	−0.1563	0.6272	0.1046	14.4	−0.1902	0.5845	0.1127
11.2	−0.1372	0.6387	0.1031	8.8	−0.0652	0.6809	0.0956	12.2	−0.1549	0.6206	0.1065	14.6	−0.1886	0.5785	0.1145
11.4	−0.1359	0.6321	0.1050	9.0	−0.0646	0.6738	0.0976	12.4	−0.1536	0.6142	0.1084	14.8	−0.1871	0.5725	0.1163
11.6	−0.1348	0.6256	0.1069	9.2	−0.0640	0.6669	0.0996	12.6	−0.1523	0.6078	0.1102	15.0	−0.1856	0.5666	0.1180
11.8	−0.1336	0.6192	0.1087	9.4	−0.0634	0.6600	0.1015	12.8	−0.1511	0.6016	0.1120	15.2	−0.1842	0.5607	0.1197
12.0	−0.1325	0.6129	0.1105	9.6	−0.0629	0.6533	0.1033	13.0	−0.1499	0.5955	0.1137	15.4	−0.1828	0.5550	0.1214

2.3 Representative PDFs of the distribution

Once the parameters are determined, the probability density curves can be plotted with the aid of Eq. (1a) and Eq. (1c).

The representative standard PDFs of this distribution (Each has a mean of zero and standard deviation of one) include a wide range of curve shapes as illustrated by the density plots in Fig. 2. The densities are indexed by the values of the skewness α_3 and kurtosis α_4. In Figs. 2(a)-(c), the skewness is fixed and three values of kurtosis are illustrated, while in Figs. 2(d)-(f), the kurtosis is fixed and three values of skewness are illustrated. From Fig. 2, one can see that the distribution reflects the characteristics of the skewness and kurtosis quite well. And one can also see that the left tail of PDF is long for negative α_3 and the right tail is long for positive α_3. This characteristic is especially important when the distribution is used for defining a fourth-moment reliability index to be described later.

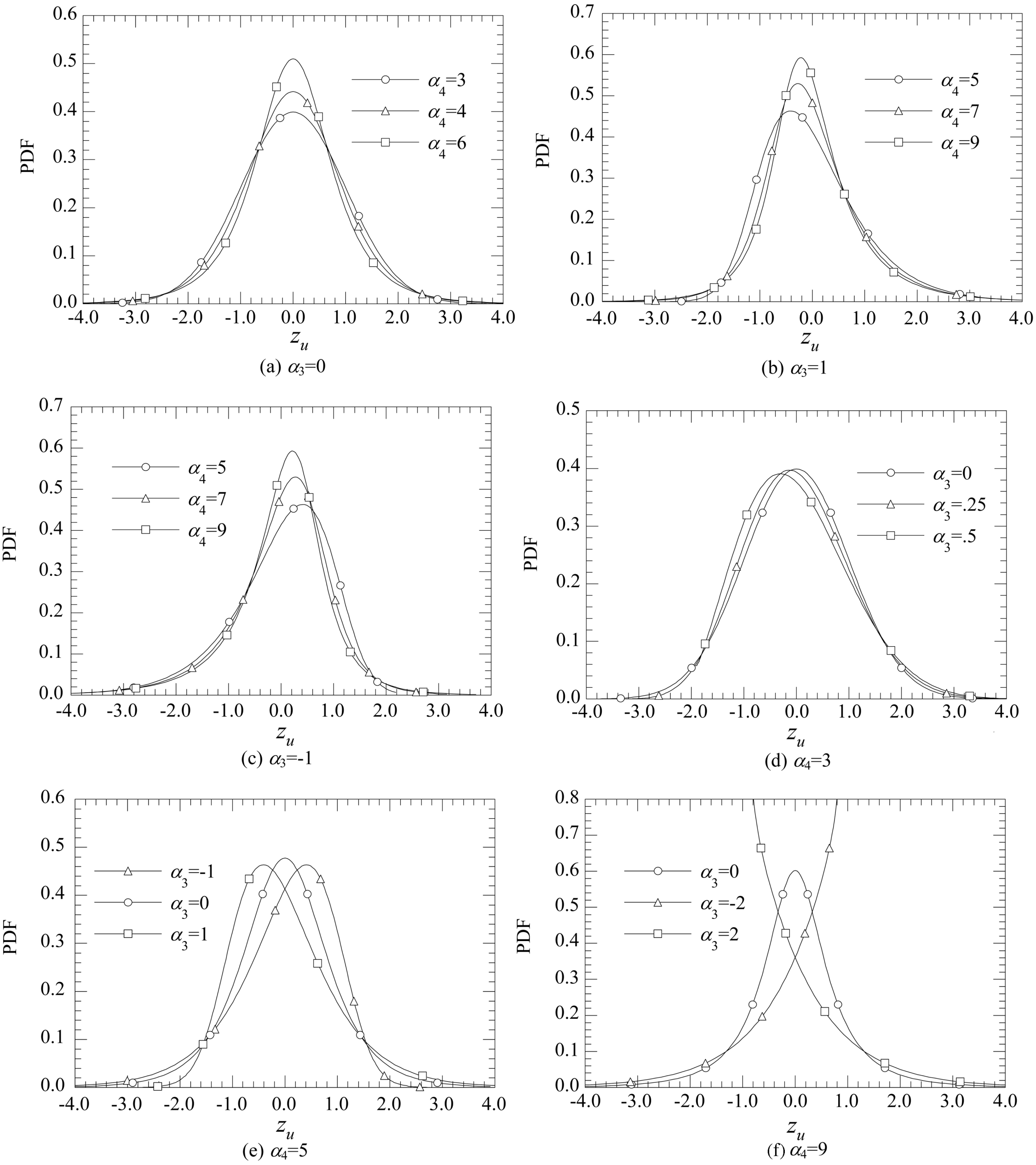

Fig. 2 The representative PDFs for specified α_3 and α_4

2.4 Operable area of the distribution in the α_3^2-α_4 plane

As described previously, the values of parameters a_2 and a_4 are obtained using the "FindRoot" function in "Mathematica" software. For a specified value of α_3, when the values of α_4 are below a limit value, the "FindRoot" function will become inoperable. Using the limit value of α_4 for which

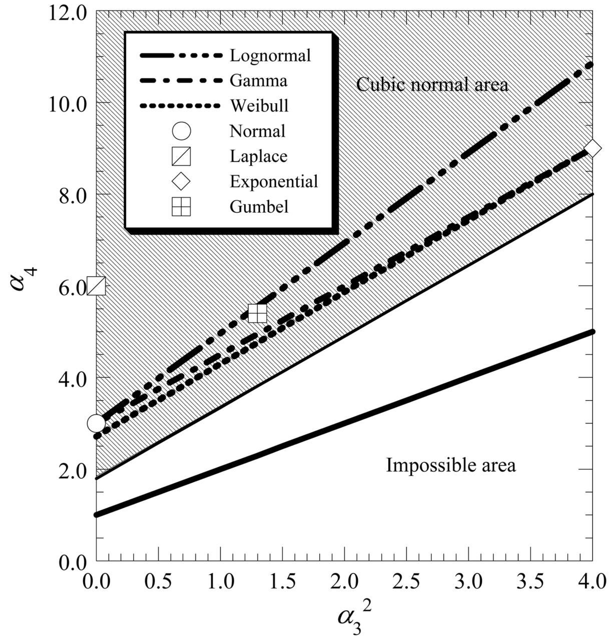

Fig. 3 Operable area of the cubic normal distribution

Eq. (3) is inoperable corresponding to the selected α_3, a lower boundary line in the α_3^2-α_4 plane can be depicted as shown in Fig. 3, in which the operable area of the distribution are indicated by the shade region. The lower boundary line for which Eq. (3) is operable is found to be nearly a straight line that can be approximately by

$$\alpha_4 = 1.88 + 1.55\alpha_3^2 \tag{6}$$

In Fig. 3, the limit for all distributions expressed as $\alpha_4 = 1 + \alpha_3^2$ (Johnson and Kotz 1970) is also depicted, along with α_3^2-α_4 relationship for some commonly used distributions. One can see that the operable area of the cubic normal distribution covers a large area in the α_3^2-α_4 plane, and the α_3^2-α_4 relationships for commonly used distributions are in the operable area of this distribution. This implies that the cubic normal distribution is generally operable for common engineering use.

3. Application in data analysis

3.1 Statistical data analysis

The cubic normal distribution is often appropriate for fitting statistical data of a random variable. Consider the measured data of H-shape structural steel described earlier. The fitting results of the histogram of the ratio between measured values and nominal values of the thickness for some selected distribution modes are also shown in Fig. 1(a), which reveals the following:

Table 1 Results of test for ultimate stress

Intervals	Freq.	Predicted frequency				Goodness of fit			
		Nor.	Log.	3P.	Cub.	Nor.	Log.	3P.	Cub.
<4.0	64	80.5	67.5	71.1	73.1	3.38	0.18	0.71	1.13
4.0-4.2	108	181.3	189.7	187.7	130.8	29.64	35.19	33.84	3.97
4.2-4.4	365	354.9	377.8	371.6	345.4	0.29	0.43	0.12	1.11
4.4-4.6	638	472.8	480.4	477.9	589.5	57.72	51.70	53.63	3.99
4.6-4.8	424	428.6	410.9	415.3	459.1	0.05	0.42	0.18	2.68
4.8-5.0	193	264.4	247.6	252.2	204.8	19.28	12.04	13.90	0.68
5.0-5.2	82	110.9	109.4	110.1	78.1	7.53	6.86	7.17	0.19
>5.2	58	38.6	48.7	46.1	51.2	9.75	1.78	3.07	0.9
Sum	1932	1932	1932	1932	1932	127.64	108.60	112.62	14.67

Note: Freq.=Frequency, Nor.=Normal, Log.=Lognormal, 3P.=3P Gamma, Cub.=Cubic normal

(1) The PDFs of the normal distribution and lognormal distribution have the greatest differences from the histogram of the statistical data among the four distributions.
(2) Since the first three moments of the 3P Gamma distribution are equal to those of the data, it fits the histogram much better than the normal and lognormal distributions.
(3) The first four moments of the cubic normal distribution can be equal to those of the data, and thus can fit the histogram much better than the normal, lognormal and 3P Gamma distributions.

Similarlily, the fitting results of the histogram of the ultimate stress are also shown in Fig. 1(b). From the figure, one can see that since the skewness of the data is quite small, the 3P Gamma distribution cannot show significant improvement upon the normal and lognormal distributions, whereas the cubic normal can effectively fit the histograms of the available data.

Results of the Chi-square tests (Ang and Tang 1975) of the four distributions for data of the ultimate stress are listed in Table 1. From the table one can see that the goodness-of-fit of the cubic normal distribution is T = 27.81 which is much smaller than those of other distributions. Similar results are obtained for the data of thickness.

From the examples above, one can clearly see that since the first four moments of the cubic normal distribution are equal to those of the statistical data, it fits the histogram much better than the normal, the lognormal and the 3P Gamma distributions.

3.2 Approximation for two- and three-parameter distributions

The cubic normal distribution, as defined in Eq. (1), can be used to represent or approximate two- or three-parameter distributions by equating the respective four moments. This is illustrated with the two parameters distributions including normal, Gumbel, lognormal, Gamma, Weibull and three-parameter Gamma distributions. Fig. 4 shows the PDFs of the above selected distributions in solid lines against those by the cubic normal distribution having the same first four moments in thick dash lines. In these figures, all the selected distributions are shown with the mean values of μ = 25, 30, 35 and 40, and coefficient of variations (COVs) V = 0.1, 0.2, 0.3 and 0.4. Fig. 4 shows that two lines coincide closely, demonstrating the flexibility of the cubic normal distribution for representing two- and three-parameter distributions considered.

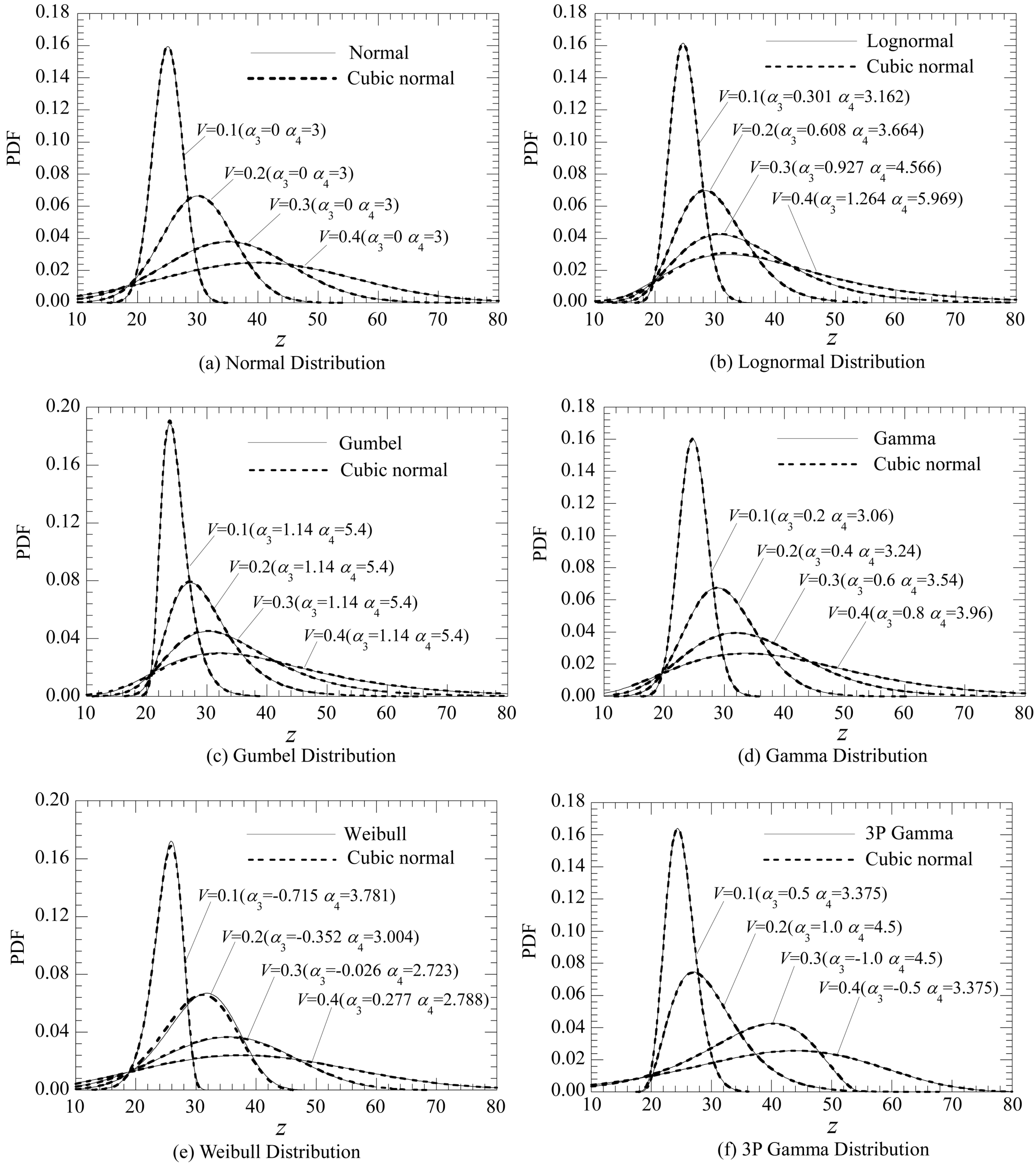

Fig. 4 PDF comparisons with some two- and three-parameter distributions

4. Application in structural reliability assessment as a fourth-moment reliability index

The cubic normal distribution was first suggested by Fleishman (1978) to generate random numbers for Monte Carlo Simulation (MCS). Hong and Lind (1996) presented an approximate method to calculate the probability of failure of a structural system with the aid of this distribution. It has been used as a third-order polynomial normal transformation technique by Chen and Tung (2003). In this paper, a fourth-moment reliability index based on this distribution is derived and its application in structural reliability assessment is discussed.

Consider a performance function $Z = G(\mathbf{X})$ of a structural system, where **X** is the vector of basic random variables. If the first four moments of $G(\mathbf{X})$ can be obtained, the probability of failure, $P(G \le 0)$, can be readily obtained by assuming $G(\mathbf{X})$ obey the cubic normal distribution.

For the standardized random variable Z_u

$$Z_u = \frac{Z - \mu_G}{\sigma_G} \tag{7}$$

since

$$P_f = P[Z \le 0] = P\left[Z_u \le -\frac{\mu_G}{\sigma_G}\right] = P[Z_u \le -\beta_{2M}] \tag{8}$$

where μ_G and σ_G are the mean value and standard deviation of $Z = G(\mathbf{X})$, respectively; $\beta_{2M} = \mu_G/\sigma_G$ is the 2nd-moment (2M) reliability index; and P_f is the probability of failure.

According to Eq. (1a), the standardized random variable Z_u can be expressed as

$$Z_u = S_U(U) = a_1 + a_2 U + a_3 U^2 + a_4 U^3 \tag{9}$$

The fourth-moment (4M) reliability index based on the cubic normal distribution can be given as

$$\beta_{4M} = -S_U^{-1}(-\beta_{2M}) \tag{10a}$$

$$P_f = \Phi(-\beta_{4M}) \tag{10b}$$

where β_{4M} is the fourth-moment (4M) reliability index; S^{-1} is the inverse function of S.

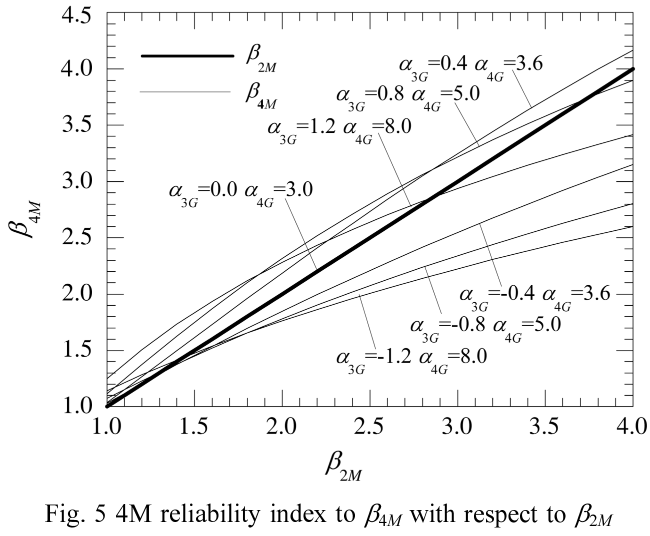

Fig. 5 4M reliability index to β_{4M} with respect to β_{2M}

As described earlier, the left tail of PDF is long for negative α_{3G} and the right tail is long for positive α_{3G}. Since the failure probability is integrated in left tail according to Eq. (8), it is easy to understand that the fourth moment method is more suitable for negative α_{3G} than positive α_{3G}.

When Eq. (9) is applied to reliability analysis, an important problem is the monotonicity of the transformation because Eq. (9) is not a monotonical function as indicated by Cheng and Tung (2003). Partially because of this, the 4M reliability method should not be applied to a problem with extremely strong non-normality (Zhao and Ono 2004), a further study is necessary to determine the range in which Eq. (9) is monotonical and the applicable range of 4M reliability method.

β_{4M} changes with respect to β_{2M} are depicted in Fig. 5. From Fig. 5, one can see that for positive α_{3G}, generally, β_{4M} is larger than β_{2M}. While for negative α_{3G}, generally, β_{4M} is less than β_{2M}. One also can see from Fig. 5 that the 4M reliability index is monotonically increased with the increase of β_{2M}.

5. Numerical examples

In order to investigate the efficiency of the suggested fourth-moment reliability index, several examples are examined under different conditions.

Example 1. Reliability of a two-story two-bay frame

The first example considers an elasto-plastic frame structure with two stories and two bays as shown in Fig. 6. The most likely failure model of this structure is also shown in Fig. 6. The corresponding performance function is

$$G(X) = 2M_1 + 2M_2 + 2M_3 - 15S_1 - 15S_2 \tag{11}$$

where M_i and S_i are independent lognormal random variables with means of $\mu_{M1} = \mu_{M2} = \mu_{M3} =$ 70K-ft, $\mu_{S1} = 5$K, and $\mu_{S2} = 10$K; and COVs of $V_{M1} = V_{M2} = V_{M3} = 0.15$, and $V_{S1} = V_{S2} = 0.25$.

Because all of the random variables in the above function are assumed to be lognormal, the reliability index can be readily obtained using the method of FORM. The FORM reliability index is $\beta_{FORM} = 3.099$, which corresponds to a failure probability of $P_f = 0.000971$.

Since the performance function is a linear sum of dependent random variables, the first four

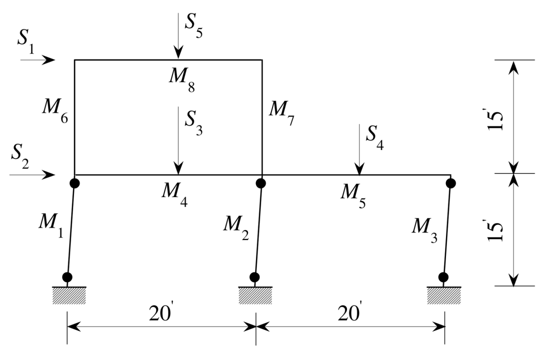

Fig. 6 Most likely failure mode of a two-story two-bay frame

moments of $G(\mathbf{X})$ are readily obtained as μ_G = 195, σ_G = 55.505, α_{3G} = −0.192, and α_{4G} = 3.257 (See Appendix). The 2M reliability index is readily obtained as β_{2M} = 3.513.

Using the suggested formula in the present paper, the 4M reliability index is readily obtained as β_{4M} = 3.0916. The corresponding probability of failure is equal to 0.0009953.

Using the method of MCS with 500,000 samples, the probability of failure for this performance function is obtained as 0.001002 with corresponding reliability index of β = 3.0896. One can see that the probability of failure obtained using the proposed method is closer to the result of MCS than that of FORM for this example.

Example 2. Reliability analysis involving variables with unknown probability distribution

In the first- or second- order reliability method, the probability distributions of the basic random variables are necessary to perform the normal transformations (the *X-U* transformation and its inverse the *U-X* transformation). Usually, in practical applications, the probability distributions of the random variables are unknown, and the probabilistic information may be defined only in terms of the respective first few statistical moments. With the cubic normal distribution, first- or second-order reliability analysis can be conveniently performed using the first four moments μ, σ, α_3, and α in the *X-U* and *U-X* transformations with the aid of Eq. (1).

Furthermore, random samples of the variables can easily generated using Eq. (1) for MCS.

For illustrations, consider the following performance function of a simple structural column subjected to axial compressive loading

$$G(\mathbf{X}) = aX_1X_2 - X_3 \tag{12}$$

where a is the nominal section area; X_1 is a random variable representing the uncertainty in a; X_2 is the yield stress; X_3 is the compressive; and X_1, X_2, and X_3 are independent random variables. Assume the column is made of H-shape structural steel with an area a = 72.38 cm^2. The CDFs of X_1 and X_2 are unknown, the only information about them are their first four moments (Ono *et al.* 1986), i.e., μ_1 = 0.990, σ_1 = 0.051, α_{31} = 0.709, α_{41} = 3.692; μ_2 = 3.055 t/cm^2, σ_2 = 0.364, α_{32} = 0.512, α_{42} = 3.957. X_3 is assumed as a lognormal variable with mean value μ_3 = 100t and standard deviation σ_3 = 40t.

The skewnees and kurtosis of X_3 can be soon obtained as α_{33} = 1.264 and α_{43} = 5.969. Since the performance function is the linear combination of product random variables, the first four moments of $G(\mathbf{X})$ can be analytically obtained as μ_G = 118.910, σ_G = 49.085, α_{3G} = −0.578, α_{4G} = 4.41 (See Appendix). The 2M reliability index is readily obtained as β_{2M} = 2.423. Using the presented formula in the present paper, the 4M reliability index is readily obtained as β_{4M} = 2.085. The corresponding probability of failure is equal to 0.01854.

Although the CDFs of X_1 and X_2 are unknown, since the first four moments are known, the *X-U* and *U-X* transformations can be easily realized using Eq. (1) instead of Rosenblatt transformation and FORM can be readily conducted with results of β_{FORM} = 2.082 and P_f = 0.01867. Furthermore, using Eq. (1), the random sampling of X_1 and X_2 can be easily generated without using their CDFs and MCS can be thus easily conducted. By the MCS with 10,000 samples, the probability of failure of this performance function is P_f = 0.0188 and the corresponding reliability index is 2.079. One can see the present 4M method almost provides the same results with those obtained by MCS and FORM.

Table 2 Random Variables in Example 3

Variables	Mean	Coefficient of variation	Distribution
B_m	1.01	0.06	Normal
F_y	400 MPa	0.10	Lognormal
F_c'	20 MPa	0.18	Normal
D	95.87 kNm	0.10	Normal
L	67.11 kNm	0.25	Gumbel

Example 3. Flexure capacity of concrete beam

Consider a single rectangular reinforced concrete beam with width b = 250 mm, distance from extreme compression fiber to the centroid of tension reinforcement d = 500 mm, and area of tension reinforcement a_s = 1529 mm^2. The flexure capacity of the beam, M_f, is (MacGregor 1988)

$$M_f = a_s F_y d(1 - 0.59 a_s F_y / (F_c' bd)) \tag{13}$$

where F_y is the yield strength of reinforcement and F_c' is the compressive strength of concrete.

The limit state function under dead and live loads, $G(\mathbf{X})$, is then given by

$$G(X) = B_m a_s F_y d(1 - 0.59 a_s F_y / (F_c' bd)) - D - L \tag{14}$$

where B_m is the modeling uncertainty factor for flexure; D is the dead load effect; and L is the maximum live load effect during 50 years. Assumptions of the uncertain variables are shown in Table 2 (Hong and Lind 1996).

Because all of the random variables in the above function have a known PDF (or CDF), the reliability index can be readily obtained using the method of FORM. The FORM reliability index is β_{FORM} = 2.886, which corresponds to a failure probability of $P_f = 1.95\times10^{-3}$.

Using the seven-point estimate (Zhao and Ono 2000), the first four moments of G(X) are approximately μ_G = 99.221, σ_G = 34.347, $\alpha_{3G} = 9.740\times10^{-3}$, and α_{4G} = 3.209. Using the proposed method, the fourth-moment reliability index is obtained as β_{4M} = 2.792 with $P_f = 2.62\times10^{-3}$. Using MCS with 1,000,000 samples, the probability of failure for this system is 2.76×10^{-3} with a corresponding reliability index of β = 2.775 (Hong and Lind 1996). One can see that the results obtained using the proposed method are closer to the results of MCS than those of FORM for this example.

Example 4. A performance function with correlative random variables

Consider the following performance function

$$G(\mathbf{X}) = X_1 + X_2 \tag{15}$$

Assume that the basic variables follow the Gumbel's bivariate distribution

$$F(X_1, X_2) = 1 - e^{-X_1} - e^{-X_2} - e^{-(X_1 + X_2 - X_1 X_2)}$$

The first four moments of G(X) are obtained as μ_G = 2.0152, σ_G = 1.1030, α_{3G} = 1.2429, and α_{4G} = 8.7665 (Grigoriu 1983). The 4M reliability index is obtained as β_{4M} = 2.122 with $P_f = 1.69\times10^{-2}$

with the aid of Eq. (10). The exact value of the probability of failure is obtained as $P_f = 1.73\times10^{-2}$ with the corresponding reliability index of β = 2.113 (Grigoriu 1983). Apparently, the results obtained by the proposed method agree well with the exact ones.

6. Conclusions

The cubic normal distribution is investigated and a table for determining the parameters is given, and its applications are emphasized including statistical data analysis and structural reliability assessment. It is found that

(1) The cubic normal distribution has a single expression and it is generally operable for common engineering use.
(2) The distribution has more flexibility for fitting statistical data of basic random variables, and can more effectively fit the histograms of available data than two-parameter or three-parameter distributions.
(3) The cubic normal distribution can be used to approximate some popular distributions, such as two-parameter distributions including normal, Gumbel, lognormal, Gamma, Weibull distributions and three-parameter Gamma distribution.
(4) For some performance functions, if the first four moments can be obtained, the distribution can be conveniently applied to obtain a fourth-moment reliability index.
(5) The structural reliability assessment can be conducted using the suggested 4M reliability index even when the CDFs or PDFs of random variables are unknown.

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Appendix. Computation of the first four moments of some simple functions

(A) When the function is a linear sum of independent random variables

For this case, the performance function can be expressed as

$$G(\mathbf{X}) = \sum_{i=1}^{n} a_i X_i \tag{A-1}$$

where X_i, $i = 1, ..., n$ are mutually independent random variables and a_i, $i = 1, ..., n$ are coefficients.
The first four moments of Eq. (A-1) are as follows

$$\mu_G = \sum_{i=1}^{n} a_i \mu_i \tag{A-2a}$$

$$\sigma_G^2 = \sum_{i=1}^{n} a_i^2 \sigma_i^2 \tag{A-2b}$$

$$\alpha_{3G}\sigma_G^3 = \sum_{i=1}^{n} \alpha_{3i} a_i^3 \sigma_i^3 \tag{A-2c}$$

$$\alpha_{4G}\sigma_G^4 = \sum_{i=1}^{n}\alpha_{4i}a_i^4\sigma_i^4 + 6\sum_{i=1}^{n-1}\sum_{j>i}^{n}a_i^2a_j^2\sigma_i^2\sigma_j^2 \tag{A-2d}$$

where μ_i (μ_G), σ_i (σ_G), α_{3i} (α_{3G}), and α_{4i} (α_{4G}) are the mean value, standard deviation, skewness, and kurtosis of X_i ($G(\mathbf{X})$), respectively.

(B) When the function is the product of independent random variables

For this case, the performance function can be expressed as

$$G(\mathbf{X}) = \prod_{i=1}^{n}X_i \tag{B-1}$$

The first four moments of Eq. (B-1) are given as

$$\mu_G = \prod_{i=1}^{n}\mu_i \tag{B-2a}$$

$$\sigma_G^2 = \mu_G^2\left[\prod_{i=1}^{n}(1+V_i^2) - 1\right] \tag{B-2b}$$

$$\alpha_{3G} = \left[\prod_{i=1}^{n}(\alpha_{3i}V_i^3 + 3V_i^2 + 1) - 3\prod_{i=1}^{n}(1+V_i^2) + 2\right]\Big/V_G^3 \tag{B-2c}$$

$$\alpha_{4G} = \left[\prod_{i=1}^{n}(\alpha_{4i}V_i^4 + 4\alpha_{3i}V_i^3 + 6V_i^2 + 1) - 4\prod_{i=1}^{n}(\alpha_{3i}V_i^3 + 3V_i^2 + 1) + 6\prod_{i=1}^{n}(1+V_i^2) - 3\right]\Big/V_G^4 \tag{B-2d}$$

where V_i and V_G are the COVs of X_i and $G(\mathbf{X})$ respectively.