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Symbolic computation and differential quadrature method - A boon to engineering analysis

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Abstract. Nowadays computers can perform symbolic computations in addition to mere number crunching operations for which they were originally designed. Symbolic computation opens up exciting possibilities in Structural Mechanics and engineering. Classical areas have been increasingly neglected due to the advent of computers as well as general purpose finite element software. But now, classical analysis has reemerged as an attractive computer option due to the capabilities of symbolic computation. The repetitive cycles of simultaneous – equation sets required by the finite element technique can be eliminated by solving a single set in symbolic form, thus generating a truly closed-form solution. This consequently saves in data preparation, storage and execution time. The power of Symbolic computation is demonstrated by six examples by applying symbolic computation 1) to solve coupled shear wall 2) to generate beam element matrices 3) to find the natural frequency of a shear frame using transfer matrix method 4) to find the stresses of a plate subjected to in-plane loading using Levy's approach 5) to draw the influence surface for deflection of an isotropic plate simply supported on all sides 6) to get dynamic equilibrium equations from Lagrange equation. This paper also presents yet another computationally efficient and accurate numerical method which is based on the concept of derivative of a function expressed as a weighted linear sum of the function values at all the mesh points. Again this method is applied to solve the problems of 1) coupled shear wall 2) lateral buckling of thin-walled beams due to moment gradient 3) buckling of a column and 4) static and buckling analysis of circular plates of uniform or non-uniform thickness. The numerical results obtained are compared with those available in existing literature in order to verify their accuracy.

Keywords: symbolic computation (Computer Algebra); coupled shear wall; natural frequency; flexural stiffness; geometric stiffness; mass matrix; differential quadrature; mesh points.

1. Introduction

Numerical methods rule supreme in modern structural analysis, thanks to the advances in computing hardware over the last three decades. Nowadays the analysis of even the most complex structures can be performed into relative ease using modern numerical tools such as the finite element method but are also seen as allowing these to be undertaken even by engineers whose background in mechanics is relatively modest or low. Parallel with this increase in capacity for solving large problems, there has also been a gradual but definite shift in emphasis as classical techniques have been totally neglected in favour of numerical techniques. This with few exceptions

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of relatively simple closed form solutions, there is an increasing tendency to assume indiscriminately that Finite Element method is to be preferred even when long established analytical techniques whether approximate or exact are readily available. It is usually the case that new advances in mathematics are first taken up by scientists, and that it is only later that such advances come to the attention of the engineering community. This observation certainly holds in the development of computerized symbolic manipulations where application to structural engineering lagged behind their counterparts in fields such as quantum physics, celestial mechanics etc. Thus about five decades elapsed since the first reported use of symbolic computations (Pavlovic 2003) and serious attempts to harness these to structural mechanics problems. A very important illustration of the potential use of computer algebra concerns those instances where a given set of calculations needs to be solved on a recurring basis. Such calculations may include multiple (partial) differentiation and integration, trigonometric and exponential functions, matrix inversions etc. Now all these operations find important applications in structural analysis, and there are frequent instances where the relevant calculations will be performed a very large number of times during the solution of a particular problem. Then it is clearly advantageous to perform such calculations in symbolic form, especially if the relevant computer package is used regularly.

Closed form solutions would often provide good understanding of the relevant physical phenomena and permits the identification of, and classification into different regimes of behaviour that enables the designer to attain a "feel" for the problem at hand. Economic parametric studies could be undertaken based on newly discovered "laws" which do not demand a vastly increased computational expenditure. The size of the problem which computer can handle algebraically as a rule is very small compared to that in which straight forward numerical computations are performed. There is already considerable scope for the application of computer generated symbolic calculations in the field of structural engineering (Wilkins 1973).

The aim of this present paper is to outline briefly some of the features of application software using Symbolic Computation or Computer Algebra to deal with symbolic data, and to illustrate some of the potential use in structural engineering to specific problems such as 1) coupled shear wall 2) to generate beam element matrices 3) natural frequency of shear frame 4) stress distribution of a rectangular plate 5) influence surface for deflection and moment for a simply supported plate, for which Finite Element method requires modeling and solving large size of algebraic equations 6)getting dynamic equilibrium equations from Lagrange's equations. The paper also presents a new approach using the differential quadrature method (DQM) for analysis of a)coupled shear wall and b) lateral buckling of thin-walled beams due to moment gradient and c) buckling of a column and d) static and buckling analysis of circular plates of uniform or non-uniform thickness.

2. Background

The history of the development of symbolic algebra manipulated by computers is traced by Sammet (1969). The first programs for symbolic differentiation appeared in 1953, but the subject seems to have been dropped from then until the early 1960's. At that time, work commenced on several general programs or languages for algebraic manipulations.

"Symbolic Manipulation" or "Computer Algebra" is essentially "expert systems" incorporating knowledge in the field of Mathematics as given by Beltzer (1990a,b). These programs have the remarkable capability of manipulating not only numbers but also abstract symbols which represent numerical quantities. One can refer to the powerful systems namely earlier ALTRAN (ALgebra TRANslator) and the more recent MATHEMATICA (Wolfram 1991), MATHCAD (Benker 1999) or MAPLE. These systems are more versatile than traditional compute codes such as FORTRAN, BASIC and C which can perform only numerical computations. Pavlovic (2002) has reviewed the development of symbolic computation until 2003. The use of the MACSYMA system in symbolic Finite Element computations can be found in Korchoff and Fenves (1979). Mbakogu and Pavlovic (1998, 2000) applied symbolic computation for the free vibration response of plates other than circular ones. The solutions presented by them are well suited to parametric studies covering arbitrary plane geometries and material properties. Levy *et al.* (2004) developed geometric stiffness of members using Symbolic Algebra. Some of the applications of symbolic computations include the theory of random vibration and stability (Benaroya and Rehak 1987).

MATHEMATICA, a general software system developed by Wolfram is capable of performing numerical, symbolic and graphical computations in a unified manner. It is a very large software system containing hundreds of functions for carrying out various tasks in science and engineering. The symbolic computation capability enables the performance of many kinds of algebraic operations such as expansion, factorization and simplification of polynomials and rational expressions as well as solution of polynomials. The system can handle numbers of any precision and supports linear algebraic operations such as matrix inversion and eigen system computation. This has well defined rules and syntax, which provides a ready platform for creating programs and graphics as well as for extending its capabilities. .

3. Benefits of symbolic processing

Mathematica has two components – an interface and a kernel viz: -a) Interface – Type in equations to solve and commands b) Kernel – Evaluates commands and stores data. They are somewhat independent! Like any programming language,

Mathematica has syntax
Learning the syntax is the key to proficiency
Almost all errors are syntax mistakes
Error messages in Mathematica can be cryptic
Input Interface has cells
Different types of cells for different types of text
Palettes too
MATHEMATICA has additional packages such as

- LAPLACE TRANSFORMS
- VECTOR ANALYSIS
- SYMBOLIC SUM
- DESCRIPTIVESTATISTICS
- PADE
- MISCELLANEOUS CHEMICAL ELEMENT
- GRAPHICS POLYHEDRA

Using Mathematica one can expand the equation, differentiate a function, solve the algebraic and differential equations, integrate a function, plot a function and do matrix operations with ease.

The symbolic calculation

- *gives much more visibility to the analyst*
- *the engineer can more easily grasp the inter-relationship of the problem variables recognize simplification to be made*
- *he can do more accurate job*
- *the easiness with which these codes provide analytical results allow to focus on the ideas rather than on overcoming computational difficulties.*
- *it encourages appreciation of the qualitative aspects of investigations .*
- *it will have greater impact on engineering education.*

4. Limitations

Memory space is the most common limiting factor in MATHEMATICA calculation. Time can also however be a limiting factor. The internal code of MATHEMATICA typically uses polynomial time algorithm whenever feasible. In case no polynomial time algorithm is available, Mathematica can use non-polynomial algorithm. The usefulness of computer based symbolic operations is provided in the cases of expansions containing functions such as for instances, the exponential Sinh and Cosh. Care must be taken to maintain valid results and avoid undue contamination by range and rounding errors. More specific applications in the field of structural engineering will follow.

5. Examples

5.1 Coupled shear wall

Frame action is obtained by the interaction of slabs and columns and this is not adequate to give the required lateral stiffness for buildings taller than 15-20 stories. Shear walls must be strategically located. Shear walls can be of open or closed sections. They are provided around elevators and

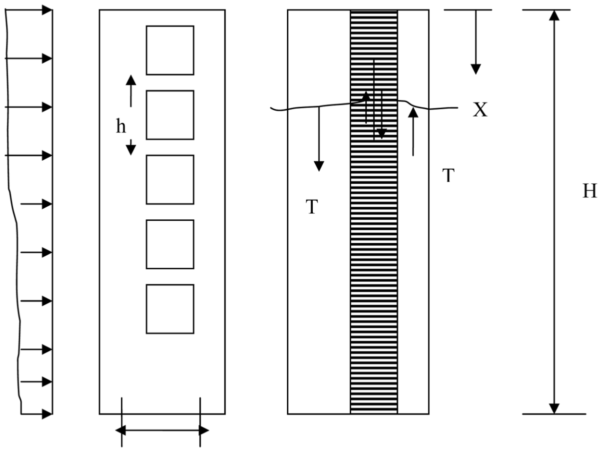

Fig. 1 Coupled shear wall

staircases. Shear walls possess adequate lateral stiffness to reduce inter storey distortions due to earthquake induced motions. They are better both with respect to life safety and damage control. When two or more shear walls are connected by a system of beams or slabs, total stiffness exceed the summation of individual stiffnesses. Openings normally occur in vertical rows as shown in Fig. 1 throughout the height of the wall and the connection between wall cross section is provided by a connecting beam which forms part of the wall or floor slab or a combination of both. The degree of coupling between the two walls separated by a row is expressed in the form of geometric parameter α. Coupled shear walls may be analyzed by using Rosman's method (Rosman 1966) considering it as a continuum. The individual connecting beams of finite stiffness I_b are replaced by an imaginary continuous connection of lamina as shown in Fig. 1. The equivalent stiffness of the lamina for a height of $dx = \frac{I_b}{h}dx$ where 'h' is the height of the storey.

When the walls deflect inducing vertical shear force in the lamina and the system is made statically determinate by introducing a cut in the lamina which is assumed to lie at the point of contra flexure as shown in Fig. 1. The displacement at each wall is determined by the addition of a) wall rotation b) link beam bending c) beam shear deflection d) due to axial compression and tension in the wall. Governing differential equation is given by

$$\frac{d^2T}{dx^2} - \alpha^2 T = -\psi M_e \tag{1}$$

where T is the tension (compression) in the wall and the geometric parameter α^2 and ψ are given by Eqs. (2a), (2b).

$$\alpha^2 = \left(\frac{l^2}{I_1 + I_2} + \frac{1}{A_1} + \frac{1}{A_2}\right)\frac{12}{h}\frac{I_b}{b^3} \tag{2a}$$

$$\psi = \frac{12 I_b l}{b^3 h(I_1 + I_2)} \tag{2b}$$

H : Total height of the shear wall
h : storey height
l : centre to centre distance of the wall
I_1, I_2 : moments of Inertia of wall 1 and 2 respectively
A_1, A_2 : areas of wall 1 and 2 respectively

$$I_b = \frac{I_b'}{(1 + 2.4(d/b)^2(1 + \nu))} \tag{3}$$

Where I_b' is the moment of Inertia of the link beam without shear deformation
b, d width of the opening and depth of the link beam
ν-Poisson's ratio of concrete taken as 0.16
I_b-Moment of Inertia of the link beam with shear deformation.

Shear/unit length of the continuum is denoted as

$$q = \frac{dT}{dx} \tag{4}$$

M_e is the moment due to external lateral load (may be earthquake or wind load) and for loading 1) constant load at the top 2) for uniformly distributed load 3) for trapezoidal distribution 4) for parabolic distribution the moment due to external load in general given by

$$M_e = \langle p0 \rangle \{ \underline{c} \}$$

where

$$\begin{aligned} \langle p0 \rangle &= \langle 1 \;\; x \;\; x^2 \;\; x^3 \;\; x^4 \rangle \\ \langle \underline{c} \rangle &= \langle c0 \;\; c1 \;\; c2 \;\; c3 \;\; c4 \rangle \end{aligned} \tag{5}$$

<c> can be obtained for the type of loading specified on the coupled shear wall and T can be solved from the differential Eq. (1) and using the boundary conditions $T[0] = 0$ and $T'[H] = 0$ where H is the height of the shear wall.

Total moments resisted by two walls is given by

$$m = M_e - Tl = \langle p0 \rangle \{ \underline{c} \} - Tl \tag{6}$$

Moments resisted by each wall is according to their stiffness as

$$\begin{Bmatrix} m(wall-1) \\ m(wall-2) \end{Bmatrix} = \begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix} m/(I_1 + I_2) \tag{7}$$

Moment in the link beam is given by

$$M(LB) = Qb/2 = qhb/2 \tag{8}$$

For earthquake resistant design of shear walls, one needs to know the drift and the governing equation for deflection may be written as

$$EI\frac{d^2y}{dx^2} = M_e - Tl \tag{9}$$

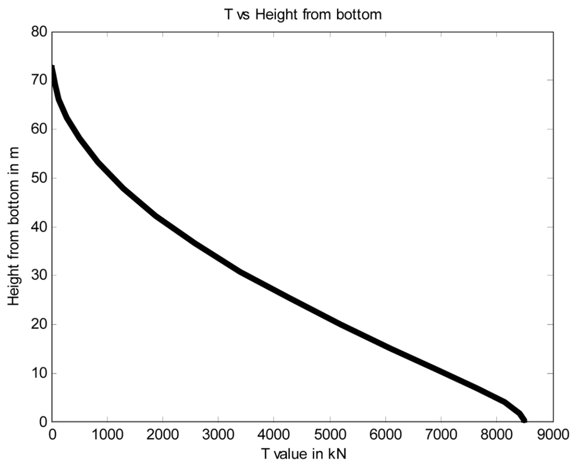

Fig. 2 Variation of T along the height of the wall (x = 73.15 base)

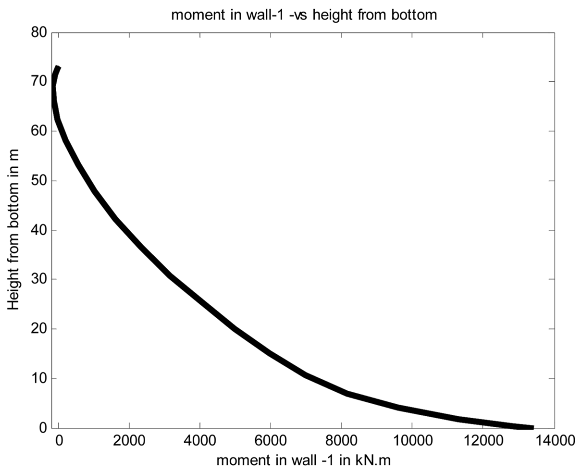

Fig. 3 Variation of Moment in wall-1 along the depth

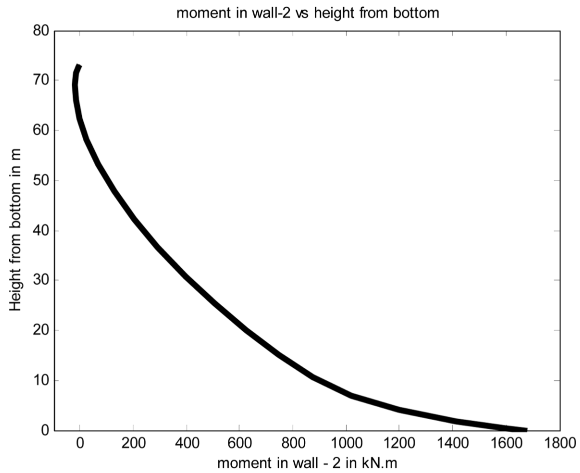

Fig. 4 Variation of Moment in wall-2 along the depth

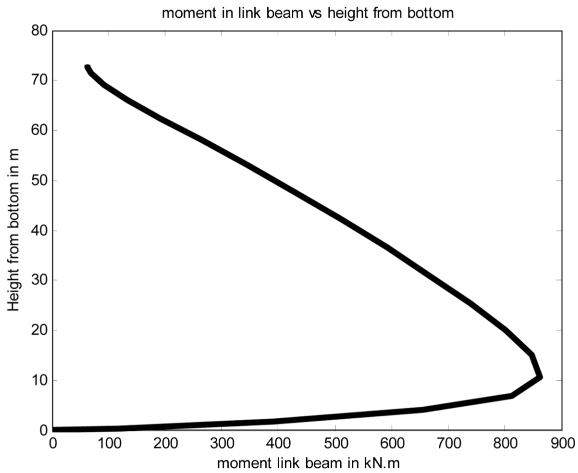

Fig. 5 Variation of Moment in Link Beam along the depth

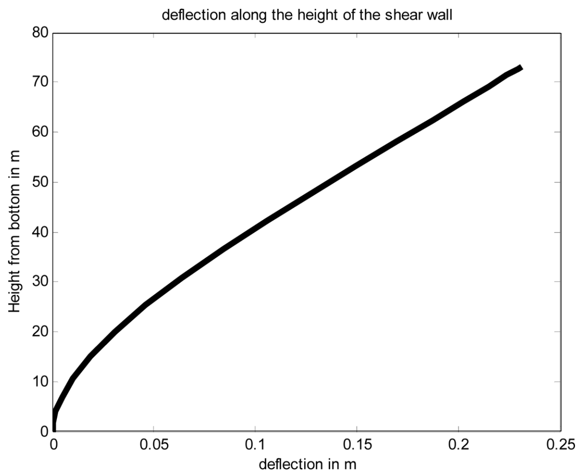

Fig. 6 Deflection of the coupled shear wall

Integrating the above equation (or solving the differential equation) with respect to x and substituting the boundary conditions $y=0$ and $y'=0$ at $x=H$ one will be able to arrive at the deflection of the coupled shear wall due to wind or earthquake loading.

The program is executed in MATHEMATICA and the variation of T, moment in wall-1, moment in wall-2, moment in link beam and deflection of the wall are shown in Figs. 2-5 and 6 respectively for the coupled shear wall of height 73.15 m with width of the walls 4.88 and 2.44 m and thickness of the wall 0.3 m, width and depth of opening as 2.44 m, depth of link beam as 1.22 m and centre to centre of the wall as 6.1 m subjected to trapezoidal load of 30 kN/m at the top and 15 kN/m at the bottom. Young's modulus and Poisson's ratio may be assumed as 20 GPa and 0.2. For the problem considered $\alpha^2 = 0.08237$; $\psi = 0.0114$; $c_2 = 15$; $c_3 = -0.0346$. The program in Mathematica is shown in below.

DSolve[{T''[x] – 0.08237 * T[x] == –0.0114 * (15 * x^2 – 0.0346 * x^3),
T[0] == T'[73.15] == 0}, T[x], x]
.
$T = T[x]$

$$t = -6.01903\times 10^{-7}e^{-0.287002x}(8.37456\times 10^{7} - 8.37456\times 10^{7}e^{0.287002x} + 1\cdot e^{0.574003x} - 579519\cdot e^{0.287002x}x^{2} + 7955.83e^{0.287002x}x^{3})$$

Plot[t, {x,0,73.15}]
m = 15 * x^2 – 0.0346 * x^3 – t * 6.1
$m1$ = m * 2.9557/3.2688
$m2$ = m – $m1$
Plot[$m1$,{x,0,73.15}]
Plot[$m2$,{x,0,73.15}]
q = D[t, x]
mlb = q * 2.44 * 3.66/2
Plot[mlb,{x,0,73.15}]
Dsolove[y''[x] == m/(20000000 * 3.2688), y[73.15] == y'[73.15] == 0}, y[x], x]
$z = y[x]$

$$z = 6.81818\times 10^{-14}e^{-0.287002x}(8.37456\times 10^{7} + 3.38539\times 10^{11}e^{0.287002x}x - 3.44906\times 10^{6}e^{0.287002x}x^{3} + 4367.98e^{0.287002x}x^{4} - 6.04528e^{0.287002x}x^{4} - 6.04525e^{0.287002x}x^{5})$$

Plot[x,{x,0,73.15}]
Do[Print[x," ",t," ",m1," ", m1," ",mlb," ",z," "], {x,0,73.15,7.315}]

If the same problem has to be carried out by Finite element packages, the coupled shear wall has to be modeled into 4 node finite elements. Wilson (2002) recommends that four node element cannot model linear bending if fine mesh is used and produces infinite stresses. Hence the coupled shear wall has to be modeled into beam, column and rigid zones otherwise the results are not reliable. Parametric studies for the coupled shear considered could be easily made by changing α, ψ in Symbolic Programming whereas it cannot be done so easily with Finite element method.

5.2 Automatic generation of stiffness matrix in finite element method

One of the first investigations on automatic generation of stiffness matrix was performed by Cecchi and Lami (1977). The flexural stiffness matrix, geometric stiffness matrix and mass matrix of a beam element can be generated using MATHAMATICA as

$$n = \begin{pmatrix} (1-3\cdot x^2/L^2 + 2x^3/L^3) \\ L*(x/L - 2\cdot x^2/L^2 + x^3/L^3) \\ (3\cdot x^2/L^2 - 2\cdot x^3/L^3) \\ L*(x^3/L^3 - x^2/L^2) \end{pmatrix}$$

$$g = \begin{pmatrix} D[n[[1, 1]], x] \\ D[n[[2, 1]], x] \\ D[n[[3, 1]], x] \\ D[n[[4, 1]], x] \end{pmatrix}$$

$$f = \begin{pmatrix} D[g[[1, 1]], x] \\ D[g[[2, 1]], x] \\ D[g[[3, 1]], x] \\ D[g[[4, 1]], x] \end{pmatrix}$$

where n is the shape function vector and g and f are the first and second derivatives of n.

$$\begin{aligned} nt &= \text{Transpose}[n] \\ gt &= \text{Transpose}[g] \\ ft &= \text{Transpose}[f] \\ k &= f \cdot ft \\ ei &= ym * i \\ s &= ei \int_0^L k dlx \end{aligned}$$

Flexural Stiffness matrix [s] is given by

$$\begin{bmatrix} \frac{12iym}{L^3} & \frac{6iym}{L^2} & -\frac{12iym}{L^3} & \frac{6iym}{L^2} \\ \frac{6iym}{L^2} & \frac{4iym}{L} & -\frac{6iym}{L^2} & \frac{2iym}{L} \\ -\frac{12iym}{L^3} & -\frac{6iym}{L^2} & \frac{12iym}{L^3} & -\frac{6iym}{L^2} \\ \frac{6iym}{L^2} & \frac{2iym}{L} & -\frac{6iym}{L^2} & \frac{4iym}{L} \end{bmatrix}$$

$$kg = q \int_0^L g \cdot gt\, dlx$$

Geometric Stiffness matrix [kg] is given by

$$\begin{bmatrix} \frac{1.2q}{L} & 0.1q & -\frac{1.2q}{L} & 0.1q \\ 0.1q & 0.1333333333Lq & -0.1q & -0.03333333333Lq \\ -\frac{1.2q}{L} & -0.1q & \frac{1.2q}{L} & -0.1q \\ 0.1q & -0.03333333333Lq & -0.1q & 0.1333333333Lq \end{bmatrix}$$

$$rf = r * ev$$

$$ma = rf\int_{n}^{L} n \cdot ntdlx$$

where 'r', 'ev' and [ma] represent mass density, area of the cross section and the mass matrix respectively.

Mass matrix

$$\begin{bmatrix} 0.37142evLr & 0.0523evL^2r & 0.1285evLr & -0.0309evL^2r \\ 0.0523evL^2r & 0.0095evL^3r & 0.0309evL^2r & -0.0071evL^3r \\ 0.12857evLr & 0.0309evL^2r & 0.37142evLr & -0.05238evL^2r \\ -0.03095evL^2r & -0.00714evL^3r & -0.05238evL^2r & 0.0095evL^3r \end{bmatrix}$$

5.3 Natural frequency using transfer matrix method for a shear wall frame

Consider a shear frame shown in Fig. 7 consisting of 'n' stories. The free body diagram of the i^{th} storey is shown in Fig. 8. Here 'm' is the mass, 'k' is the stiffness, 'V' the shear, 'w' is the natural frequency and 'v' the displacement.

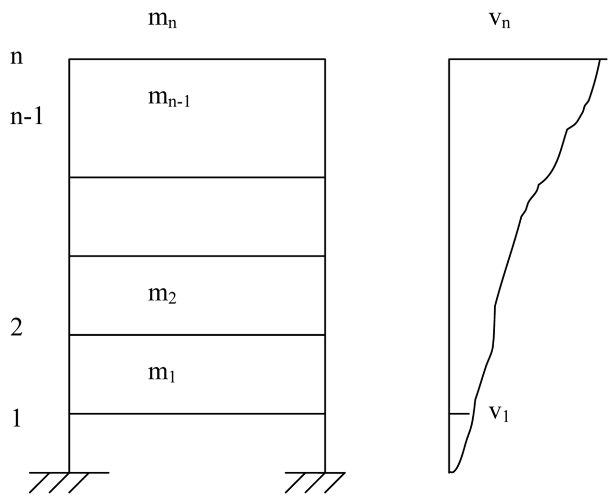

Fig. 7 Shear frame of 'n' stories

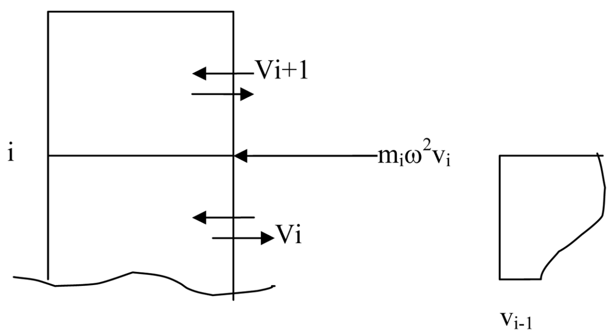

Fig. 8 Free body diagram of i^{th} storey

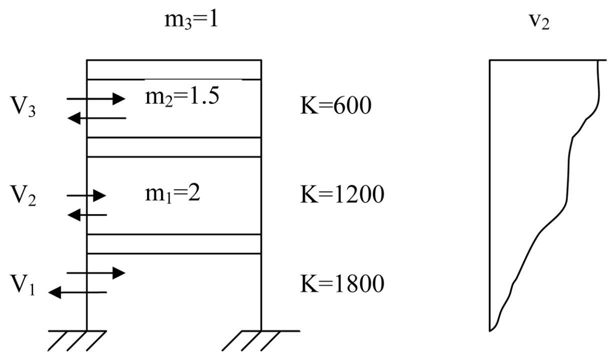

Fig. 9 3 storeyed shear frame

Considering inertia force, the shear in the i^{th} storey can be written in terms of shear of $(i+1)^{th}$ storey.

$$V_i = V_{i+1} + m_1 \omega^2 v_1 \tag{10}$$

$$v_i - v_{i-1} = \frac{V_i}{k_i} = \frac{V_{i+1}}{k_i} + \frac{m_i \omega^2 v_1}{k_i} \tag{11}$$

or

$$\begin{Bmatrix} V_i \\ v_{i-1} \end{Bmatrix} = \begin{bmatrix} 1 & m_1 \omega^2 \\ -\dfrac{1}{k_i} & 1 - \dfrac{m_i \omega^2}{k_i} \end{bmatrix} \begin{Bmatrix} V_{i+1} \\ v_i \end{Bmatrix} \tag{12}$$

Initially the displacement at the top storey level and the natural frequency are both assumed. By using the transfer matrix method, it is possible to find the displacement at the base as well as shear in the base storey. If the support displacement is not zero, a new value for the natural frequency is assumed and the procedure is repeated till we get the value of the base displacement as zero. To find the natural frequency of the three storeyed shear building as shown in Fig. 9 given the mass and the stiffness MATHCAD (Benker 1999) is used.

Since 'MATHCAD' can solve the problem using symbolic processing, we will write in symbolic form

Transfer matrix at top frame (assume $p = \omega^2$)

$$\begin{Bmatrix} V_3 \\ v_2 \end{Bmatrix} = \begin{bmatrix} 1 & m_3 p \\ -\dfrac{1}{k_i} & 1 - \dfrac{m_3 p}{k_3} \end{bmatrix} \begin{Bmatrix} V_{4=0} \\ v_3 \end{Bmatrix} = [a] \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{13}$$

$$\begin{Bmatrix} V_2 \\ v_1 \end{Bmatrix} = \begin{bmatrix} 1 & m_2 p \\ -\dfrac{1}{k_2} & 1 - \dfrac{m_2 p}{k_2} \end{bmatrix} \begin{Bmatrix} V_3 \\ v_2 \end{Bmatrix} = [b] \begin{Bmatrix} V_3 \\ v_2 \end{Bmatrix} = [b][a] \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{14}$$

$$\begin{Bmatrix} V_1 \\ v_0 \end{Bmatrix} = \begin{bmatrix} 1 & m_1 p \\ -\dfrac{1}{k_1} & 1-\dfrac{m_1 p}{k_1} \end{bmatrix} \begin{Bmatrix} V_2 \\ v_1 \end{Bmatrix} = [c]\begin{Bmatrix} V_2 \\ v_1 \end{Bmatrix} = [c][b][a]\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{15}$$

$$(c)[b][a]\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \{d\} = \begin{Bmatrix} V_1 \\ v_0 \end{Bmatrix} \tag{16}$$

where d is a function of (m_1, m_2, m_3, k_1, k_2, k_3, p) and using few trials p can be found out as

$$d(2,1.5,1,1800,1200,600,211) = \begin{Bmatrix} 543.43 \\ -4504 \times 10^{-4} \end{Bmatrix} \tag{17}$$

$$p = 211 \quad v_0 = 0 \tag{18}$$

One can also get the mode shape as <1 0.6428 0.301, 0>. Hence $\omega = \sqrt{211} = 14.520$ radians/sec. A program ‘frtr.ncd’ is written in MATHCAD to calculate the fundamental frequency of 3 storey shear frame.

5.4 Application of Levy’s method using MATHCAD

The Levy method (Pavlovic 2003) is a powerful analytical technique which finds wide application in the fields of elasticity, plates and shells. The method consists of reducing a governing differential equation from partial to ordinary form, the latter being amenable to a standard solution process. The main requirement which needs to be fulfilled before the method can be used is that the boundary conditions along a portion of the contour of the system must be of a specific kind. Now, we will consider our discussion to rectangular boundaries shown in Fig. 10 although other contours can also be dealt with. Governing partial differential equation for two dimensional elasticity problems (under zero or constant body forces) is given by

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \tag{19}$$

Where ϕ is the Airy’s stress function and the stresses are given by

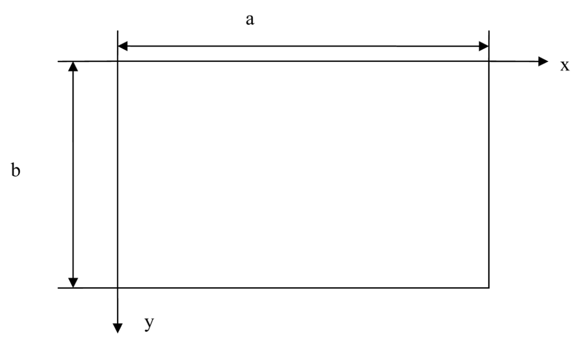

Fig. 10 Rectangular plate subjected to in plane stresses

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}; \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}; \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{20}$$

Assuming zero direct stress action along the edges $x = 0$ and $x = a$, the solution of Eq. (19) can be expressed in the following form as

$$\phi = \sum_{m=1}^{\infty} Y_m(y) \sin \alpha_m; \quad \alpha_m = \frac{m\pi}{a} \tag{21}$$

The function Y_m can be found out by substituting each term of Eq. (21) into Eq. (20) which then becomes ordinary differential equation as

$$\frac{d^4 Y_m}{dy^4} - 2\alpha_m^2 \frac{d^2 Y_m}{dy} + \alpha_m^4 Y_m = 0 \tag{22}$$

The solution to Eq. (22) is easily obtained, so that a typical term Y_m in Eq. (21) takes of the form

$$Y_m = (C_1)_m \cosh \alpha_m y + (C_2)_m \sinh \alpha_m y + (C_3)_m y \cosh \alpha_m y + (C_4)_m y \sinh \alpha_m y \tag{23}$$

The constants *Cs* can be evaluated using the specified boundary conditions a the edges $y = 0$ and $y = b$ as (expressing the loadings in terms of Fourier series)

$$\begin{aligned}
(\sigma_y)_{y=0} &= \sum_{m=1}^{\infty} (D_1)_m \sin \alpha_m x \\
(\tau_{xy})_{y=0} &= \sum_{m=1}^{\infty} (D_2)_m \sin \alpha_m x \\
(\sigma_y)_{y=b} &= \sum_{m=1}^{\infty} (D_3)_m \sin \alpha_m x \\
(\tau_{xy})_{y=b} &= \sum_{m=1}^{\infty} (D_4)_m \sin \alpha_m x
\end{aligned} \tag{24}$$

In this way, for each value of m, the set of *Cs* is solved in terms of the corresponding set of *Ds* and this gives the m^{th} term of the stress function, from which the stress distribution due to the m^{th} term of the applied surface loading can be determined. The full solution can be obtained by superposing the solutions. This means that, although the whole procedure is fairly straight forward, the calculation of each series term requires the solution of a system of four simultaneous equations. One can reduce the effort considerably if instead of solving each set of equations numerically one could obtain a single, general solution of a typical set symbolically. Hence the matrix for the m^{th} term of the Fourier series as

$$\begin{bmatrix} \alpha^2 & 0 & 0 & 0 \\ 0 & \alpha^2 & \alpha & 0 \\ \alpha^2\beta & \alpha^2\gamma & \alpha^2 b\beta & \alpha^2 b\gamma \\ \alpha^2\gamma & \alpha^2\beta & (\alpha\beta + \alpha^2 b\gamma) & (\alpha\gamma + \alpha^2 b\beta) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = -\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D \end{Bmatrix} \tag{25}$$

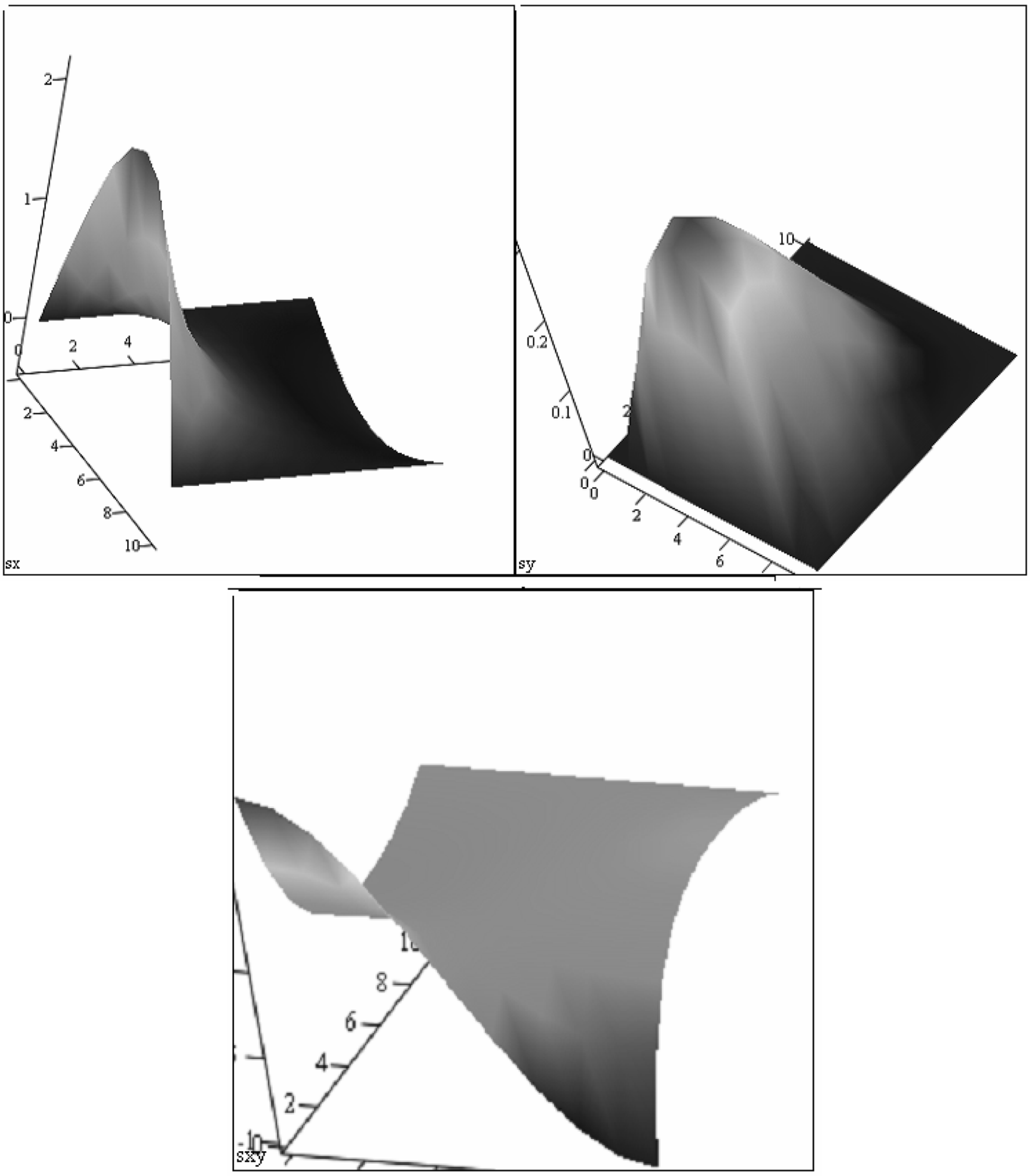

Fig. 11 Stress distributions in a plate

where

$$\beta = \cosh \alpha b; \quad \gamma = \sinh \alpha b \tag{26}$$

The above is programmed in MATHCAD and the σ_x, σ_y, τ_{xy} stress distribution due to cosine variation of shear stress distribution at the edge $y = 0$ are shown in Fig. 11.

5.5 Influence surface for deflection and moments in a simply supported isotropic plate.

Consider a simply supported isotropic plate of sides ax and by subjected to unit load at a point (x, y). The deflection at a point (ξ, η) is represented by $K(x, y, \xi, \eta)$. Regarding x, y as the variables, $w = K(x, y, \xi, \eta)$ is the equation of the elastic surface of the plate submitted to a unit load at a fixed point (x, y). Now considering (x, y) as variable, the deflection equation given by Eq. (27) defines the influence surface for the deflection of the plate at a fixed point (ξ, η) the position of the traveling unit load being given by (x, y). If, therefore, some load of intensity $f(x, y)$ distributed over an area A is given, the corresponding deflection at any point of the plate may easily be obtained. For Navier's solution the influence surface for the deflection is given by Timoshenko and Krieger (1959)

$$w = K(x, y, \xi, \eta) = \frac{4}{\pi^4 abD}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi\xi}{a}\sin\frac{m\pi\eta}{b}\sin\frac{m\pi x}{a}\sin\frac{m\pi y}{b}}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \tag{27}$$

The corresponding MATHEMATICA program is given below.

$C[xi, et, xc, yc, ax, by, m, n] = \sin[m*Pi*xc/ax]*\sin[n*Pi*yc/by]*$
$\sin[m*Pi*xi/ax]*\sin[n*Pi*et/by]$
$f[m, n.ax, by] = 4/((\pi^4*ax*by)*((m/ax)^2 + (n/by)^2))$
$h = C[xi, et, xc, yc, ax, by, m, n]*f[m, n.ax, by]$
$g = sum[h, \{m, 1, 5\}, \{n, 1, 5\}]$
$Plot3D[g, \{xc, 0, 1\}, \{yc, 0, 1\}]$

The influence surface for deflection and moment in the direction of X axis are shown in Figs. 12 and 13 respectively for a square plate ($ax = by = 1$)

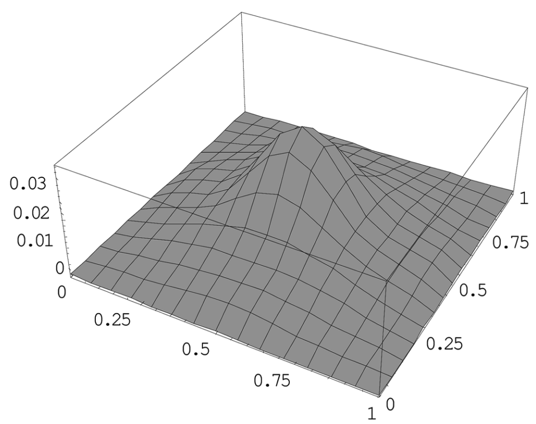

Fig. 12 Influence surface for deflection at centre (for a simply supported isotropic square plate)

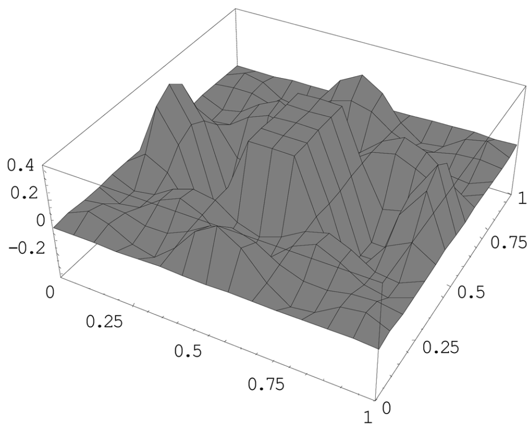

Fig. 13 Influence surface for Moment in the direction of x-axis (at centre of the plate)

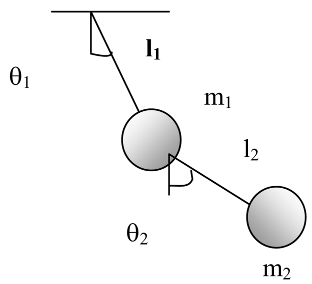

Fig. 14 Two bar pendulum

5.6 To arrive at Euler lagrange equilibrium equations

Consider a two bar pendulum shown in Fig. 14. The resultant velocities $\dot{R}_1, \dot{R}_2$ of the first and second mass may be written as

$$\dot{R}_1 = l_1\dot{\theta}_1 ; \quad \dot{R}_2^2 = l_2^2\dot{\theta}_2^2 + l_1^2\dot{\theta}_1^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1) \tag{28}$$

Lagrange is written as

$$L = T - V = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)$$
$$- m_2gl_2\cos\theta_2 - (m_1 + m_2)gl_1\cos\theta_1 \tag{29}$$

Substituting $$\theta_1 = h_1; \ \theta_2 = h_2; \dot{\theta}_1 = h_1t; \dot{\theta}_2 = h_2t \tag{30}$$

By differentiating one will be able to get two Euler Lagrange equilibrium equations as obtained from Mathematica.

```
"Section D19, TWO DEGREES OF FREEDOM"
"state the lagrangian of the two bar pendulum -Lagrangian is"
la=(m1+m2)*L1^2*h1t^2/2+m2*L2^2*h2t^2/2+m2*L1*L2*h1t*h2t*Cos[h1-h2]+
m2*g*L2*Cos[h2]+(m1+m2)*g*L1*Cos[h1];
"compute the derivatives and get two Euler-Lagrange equations"
q1=D[la,h1t];q2=D[la,h2t];f1=D[la,h1];f2=D[la,h2];
d={h1t->D[h1[t],t],h2t->D[h2[t],t],h1->h1[t],h2->h2[t]};
s1=q1/.d;c1=f1/.d;s2=q2/.d;c2=f2/.d;
equation1==Simplify[c1-D[s1,t]]
equation2==Simplify[c2-D[s2,t]]

Section D19, TWO DEGREES OF FREEDOM
state the lagrangian of the two bar pendulum -Lagrangian is
compute the derivatives and get two Euler-Lagrange equations
equation1 == -(L1 (g m1 Sin[h1[t]] + g m2 Sin[h1[t]] + L2 m2 Sin[h1[t]-h2[t]] h2'[t] + L1 (m1 +
 m2) h1''[t]+L2 m2 Cos[h1[t] - h2[t]] h2''[t]))
equation2 == -(L2 m2 (g Sin[h2[t]] - L1 Sin[h1[t] - h2[t]] h1'[t]  + L1 Cos[h1[t] - h2[t]] h1''[t] +
 L2 h2''[t]))
```

6. Differential quadrature method

It is well known that analysis of engineering systems include two main steps; construction of mathematical models for a given physical phenomenon and the solution of mathematical equation. As seen in the examples before, most engineering problems are governed by a set of differential equations. Because of finite difference and finite element methods, the solution of static and dynamic equations of complicated structures is now possible. There is yet another efficient method called Differential Quadrature Method (DQM) which can be used quite easily for solving differential equations. DQM was introduced by Bellman and Casti (1971). The basic concept of the method is that derivative of a function at a given point can be approximated as a weighted sum of the functional values at all of the sample points in the domain of that variable. Hence it is possible to reduce differential equations into a set of algebraic equations using the above approximation with boundary conditions applied. The accuracy of the method depends on the number of sampling points used. Since the introduction of Quadrature method, application to various engineering problems has been investigated and their success has shown potential of the method as attractive numerical techniques (Bert and Malik 1996, Sherbourne and Pandy 1991). Tomasiello (1998) analysed static and free vibration of beam and rectangular plates by the differential quadrature method. Shu *et al.* (2000) applied Differential quadrature method for curvilinear quadrilateral plates. Li and Huang (2004) applied moving Least squares differential quadrature method for free vibration of anti-symmetric laminates. Civalek (2004, 2005) and (Civalek and Ulker 2004) combined harmonic differential quadrature method with finite difference method for the geometrically non-linear analysis of double curved isotropic shells resting on elastic foundation.

6.1 DQM Method explained

For a function $f(x)$, DQM approximation of the m^{th} order derivative at the i^{th} sampling point is given by

$$\frac{d^m}{dx^m}\begin{Bmatrix} f(x_1) \\ f(x_2) \\ \cdot \\ f(x_n) \end{Bmatrix} = H_{ij}^{(m)}\begin{Bmatrix} f(x_1) \\ f(x_2) \\ \cdot \\ f(x_n) \end{Bmatrix} \quad \text{for} \quad i, j = 1, 2, \ldots, n \tag{31}$$

where 'n' denotes the number of sampling points in the domain. Assuming Lagrangian interpolation polynomial

$$f(x) = \frac{M(x)}{(x - x_i)M_1(x_i)} \quad \text{for} \quad i = 1, 2, \ldots, n \tag{32}$$

where

$$M(x) = \Pi_{j=1}^{n}(x - x_j)$$

$$M_1(x_i) = \Pi_{j=1, j \neq i}^{n}(x_i - x_j) \quad \text{for} \quad i = 1, 2, \ldots, n \tag{33}$$

The differential of above function given by Eq. (31) or first derivative leads to

$$H_{ij}^{(1)} = \frac{M_1(x_i)}{(x_i - x_j)M_1(x_i)} \quad \text{for} \quad i = j = 1, \ldots n \ldots i \neq j$$

$$H_{ij}^{(1)} = -\sum_{j=1, j \neq i}^{n} H_{ij}^{(1)} \tag{34}$$

The second and third and higher derivative can be calculated as

$$H_{ij}^{(2)} = \sum_{k=1}^{n} H_{ik}^{(1)} H_{kj}^{(1)} \ldots i = j = 1m2 \ldots n$$

$$H_{ij}^{(m)} = \sum_{k=1}^{n} H_{ik}^{(1)} H_{kj}^{(m-1)} \quad \text{for} \quad i = j = 1, 2, \ldots n \tag{35}$$

and the number of sampling points $n > m$.

For the sampling point, we adopt well accepted Chebyshev-Gauss-Lobatto (CGL) mesh distribution given by Shu (2000) in a non-dimensional form as

$$x_i = \frac{1}{2}\left[1 - \cos\frac{(i-1)}{(n-1)}\pi\right] \tag{36}$$

For some cases, uniform distribution may also be adopted.

6.2 Treatment of boundary and initial conditions

6.2.1 Boundary conditions

Consider a column pinned at both ends. The boundary conditions may be written as $v = 0$; at $x = 0$ and $x = 1$ and $v'' = 0$ at $x = 0$ and $x = 1$. These boundary conditions may be incorporated in DQM as

$$\begin{aligned} D(n+1,\ 1)v_1 &= 0 \\ D(n+2,\ n)v_n &= 0 \\ D(n+3,1{:}n)\{v\} = c(1,\ 1{:}n,\ 2)\{v\} &= 0 \\ D(n+4,1{:}n)\{v\} = c(n,\ 1{:}n,\ 2)\{v\} &= 0 \end{aligned} \tag{37}$$

Similarly other types of boundary conditions such as fixed and free ends may be incorporated.

6.2.2 Initial conditions

In time coordinate problems one can use inverse node numbering as in space rocket launching. That is one can use initial time point as the N^{th} point and the time domain end point as the first point. Usually time coordinate is normalized between 0 and 1. If we assume initial displacement and velocity as 1 accordingly shape functions could be determined. For the initial displacement the shape function will have the following properties.

$$h_N(\tau_n) = 1;\ h_N^{(1)}(\tau_N) = 0;\ h_N(\tau_i) = 0\ (i = 1, 2, N-1) \tag{38}$$

$$h_N(\tau_n) = 0;\ h_N^{(1)}(\tau_N) = 1;\ h_N(\tau_i) = 0\ (i = 1, 2, N-1) \tag{39}$$

For details, one can refer to the paper by (Wu and Liu 1999).

6.3 Coupled shear wall

The governing differential equation for coupled shear wall as discussed in section 5.1 with $\alpha^2 = 0.08237$, $\psi = -0.0114$, $c_0 = 0$, $c_1 = 0$, $c_2 = 15$, $c_3 = -0.0346$, $c_4 = 0$ is given by

$$\frac{d^2T}{dx^2} - 0.08237T = M_e = c_0 + c_1 * x + c_2 * x^2 + c_3 * x^3 + c_4 * x^4 \tag{40}$$

where M_e is the moment due to external lateral load of 30 kN/m at the top and 15 kN/m at the bottom. Assume $C(: , : , 2)$ = is the 2nd differential of the function, the left hand side of Eq. (40) is written as

$$\begin{aligned} Z(1:n, 1:n) &= C(:, :, 2)/H^2 \\ \text{with } Z(i, i) &= Z(i, i) - 0.08237 \end{aligned} \tag{41a}$$

and the right hand side can be written as

$$e(i, 1) = -0.0114 M_e(x_i) \tag{41b}$$

The boundary condition $T[0] = 0$ and $T'[H] = 73.15]$ can be incorporated as

BC1--------$Z(n+1, 1) = 1.0$ and
BC2--------$Z(n+2, 1{:}n) = \mathrm{C}(n, 1 : n, 1)/H$

The boundary conditions are given by

$$\begin{aligned} &[Z]_1 \{T\} = \{0\} \\ &2 \times n \;\; n \times 1 \;\; n \times 1 \end{aligned} \tag{42}$$

Combining governing equations and boundary conditions we get

$$\begin{bmatrix} [Z]_0 \\ n \times n \\ [Z]_1 \\ 2 \times n \end{bmatrix} \{T\} = \begin{Bmatrix} \{e\} \\ n \times 1 \\ \{0\} \\ 2 \times 1 \end{Bmatrix} \tag{43}$$

Using Lagrange multiplier approach as recommended by Wilson (2002), Eq. (43) can be modified to square matrix as

$$\begin{bmatrix} [Z]_0 & [Z]_1^T \\ [Z]_1 & [0] \end{bmatrix} \begin{Bmatrix} \{T\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{e\} \\ \{0\} \end{Bmatrix} \tag{44}$$

The above equation has both equilibrium and equation of geometry. This matrix is also symmetric and is not positive definite. Therefore pivoting may be required during the solution process. Hence penalty method is preferably be applied. Once T is solved, one can fine the moments in the walls as M_1, M_2 and in the link beam M(LB).

To solve for the deflection the governing equation is (see Eq. (6))

$$\frac{d^2y}{dx^2} = M_e - 6.1 * T \tag{45}$$

with boundary conditions as $y[H = 73.15] = y'[H = 73.15] = 0$, using DQM we get

$$[Z]_0 = Z(1{:}n,\ 1{:}n) = C(:,:,2)/H^2$$
$$[Z]_1 = \begin{bmatrix} Z(n+1, n) = 1 \\ Z(n+2, 1{:}n) = c(:,:,1)/H \end{bmatrix} \tag{46}$$

and the right hand side is

$$e(i, 1) = (M_e i) - 6.1 * T(i) \tag{47}$$

Again using Lagrange multiplier approach we get

$$\begin{bmatrix} [Z]_0 & [Z]_1^T \\ [Z]_1 & [0] \end{bmatrix} \begin{Bmatrix} \{y\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{e\} \\ \{0\} \end{Bmatrix} \tag{48}$$

Solving Eq. (48), the deflection along the height of coupled shear wall can be found out. The values are exactly same as those obtained by MATHEMATICA.

6.4 Lateral Buckling of I beam subjected to moment gradient (Chen and Atsuta 1977)

A simply supported beam with unequal major axis end moments αM_{cr}, M_{cr} is shown in Fig. 15 The governing differential equation can be written as

$$EI_\omega \phi'''' - GK_T \phi' = \frac{M_e^2}{EI_y}\phi \tag{49a}$$

where I_ω and K_T are the warping moment of inertia and torsional constant respectively and E and G are the Youngs modulus and Rigidity modulus respectively. M_e is the externally applied moment given by

$$M_e = M_{cr}\left(\alpha + (1-\alpha)\frac{x}{L}\right) \tag{49b}$$

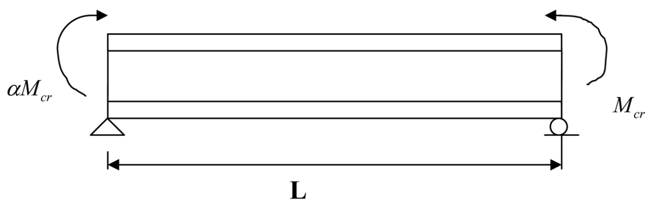

Fig. 15 Lateral buckling of beam due to moment gradient

with boundary conditions

$$\phi[0] = \phi[L] = \phi''[0] = \phi''[L] = 0 \text{ for simply supported ends} \tag{50a}$$

free to warp and

$$\phi[0] = \phi[L] = \phi'[0] = \phi'[L] = 0 \text{ restrained to warp} \tag{50b}$$

Considering simply supported beam free to warp the equilibrium equation of Eq. (48) and boundary condition of Eq. (50a) can be written as

$$[Z]_0 = Z(1{:}n,1{:}n) = EI_\omega{*}c(:,:,4)/L^4 - GK_T{*}c(:,:,2)/L^2$$

$$[Z]_1 = \begin{bmatrix} Z(n+1,1) = 1 \\ Z(n+2,n) = 1 \\ Z(n+3,1{:}n) = c(1,1{:}n,2)/L^2 \\ Z(n+4,1{:}n) = c(n,1{:}n,2)/L^2 \end{bmatrix} \tag{51}$$

The right hand side matrix $[e]$ is written as

$$[e] = e_{i,i} = \frac{1}{EI_y}\left(\alpha + (1-\alpha)\frac{x_i}{L}\right)^2$$
$$n \times n \tag{52}$$

Using Wilson's method of applying boundary conditions, the governing equation is written as

$$\frac{1}{M_{cr}^2}\begin{bmatrix}[Z]_0 & [Z]_1^T \\ [Z]_1 & [0]\end{bmatrix}\begin{Bmatrix}\{\phi\} \\ \{\lambda\}\end{Bmatrix} = \begin{bmatrix}[e] & [0] \\ [0] & [0]\end{bmatrix}\begin{Bmatrix}\{\phi\} \\ \{\lambda\}\end{Bmatrix} \tag{53}$$

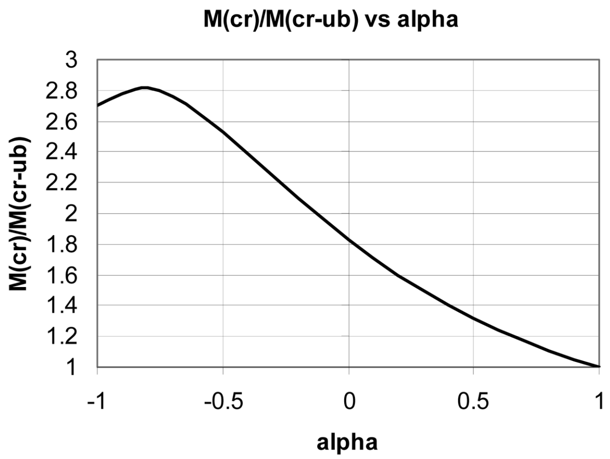

Fig. 16 Variation of critical moment with respect to α

Consider a simply supported beam of span 8.633 m with $I_y = 18.6\text{e-}6$ m^4, $I_\omega = 0.918\text{e-}6$ m^6; $K_T = 0.691\text{e-}6$ m^4, $E = 200$ GPa and $G = 77$ Gpa. Solving as an eigen value problem, one will be able to get the critical moment. When the beam is subjected to uniform moment the critical moment is found to be 195 kN·m. Fig. 16 shows the variation of critical moment with respect to α. Moreover, the value for critical moment for uniform bending agrees with the values given by (Chen and Atsuta 1977).

6.5 Buckling of column (comparison with Finite Element method)

Consider a cantilever column of wide flange section with a width of 0.2 m and depth of 0.3 m and length of 2 m with overall thickness of 0.04 at flanges and at web subjected to axial load. The properties have been obtained from a program "PROP" as,

$$(EA) = 2.8\text{e-}2$$
$$(EIx) = 4.52\text{e-}4$$
$$(EIy) = 5.49\text{e-}5$$
$$(EI_\omega) = 1.2\text{e-}6$$
$$(GJ) = 1.49\text{e-}5$$

The number of sampling points used here is 20 and (CGL) is adopted. The buckling load obtained from DQM is 6777 kN which agrees with the closed form solution $(\pi^2 EI_y/4L^2) = 6773$ kN of (Timoshenko and Gere 1961). The same beam could be idealized with either plate elements or beam elements. When the same problem is solved in FEAST-C (Anonymous 1995), using plate elements, the buckling load obtained is 7429 kN with 9% error. Because of the kinks between plates, when a thin-walled beam is idealized into plate elements, it is not possible to get the exact value for the buckling load. The buckled shape of the model obtained using FEAST-C is shown in Fig. 17. When the beam is idealized with 1,2 4 and 8 beam elements with three degrees of freedom at each node, the buckling loads obtained are 7441, 6900, 6698 and 6618 kN for 1,2,4 and 8 elements as compared to the closed form solution of 6773 kN.

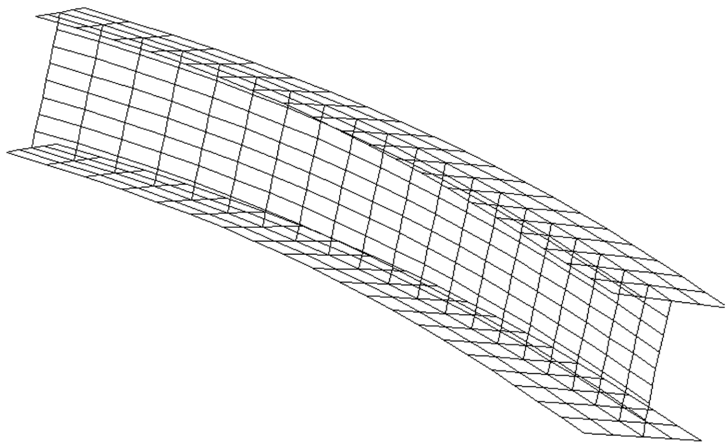

Fig. 17 Buckled shape of a column obtained using FEAST-C

6.6 Bending of circular plates subjected to uniformly distributed load.

Consider a simply supported thin circular plate of uniform thickness (flexural rigidity, D-uniform) subjected to uniformly distributed load q. There are two governing differential equations in the literature both the third and fourth order differential equations can be used to obtain the deflection of the plate. In this study, fourth order equation is derived from the equilibrium equation of the plate as

$$2\frac{dM_r}{dr} + 2\frac{d^2M_r}{dr^2} - \frac{dM_t}{dr} = -q;\ M_r = -D\left(\frac{d^2w}{dr^2} + \frac{\upsilon dw}{r\,dr}\right);\ M_t = -D\left(\frac{1}{r}\frac{dw}{dr} + \upsilon\frac{d^2w}{dr^2}\right) \quad (54)$$

where M_r, M_t are the radial and tangential moments in a circular plate and r is the radius and w is the deflection and υ is the Poisson's ratio.

If the plate is of uniform thickness (D-constant) the governing equation reduces to a fourth order differential equation as

$$\frac{d^4w}{dr^4} + \frac{2}{r}\frac{d^3w}{dr^3} - \frac{1}{r^2}\frac{d^2w}{dr^2} + \frac{1}{r^3}\frac{dw}{dr} = \frac{q}{D} \quad (55)$$

Using Differential quadrature method with 20 grid points of CGL distribution, the above equation reduces to

$$\begin{aligned} Z(1{:}n-3, 1{:}n) = {} & c(2{:}n-2, 1{:}n, 4)/l^4 + 2*inx1(2{:}n-2, 2{:}n-2)*c(2{:}n-2, 1{:}n, 3)/l^3 \\ & - inx2(2{:}n-2, 2{:}n-2)*c(2{:}n-2, 1{:}n, 2)/l^2 + \\ & inx3(2{:}n-2, 2{:}n-2)*c(2{:}n-2, 1{:}n, 1)/l \end{aligned} \quad (56)$$

where 'l' is the radius of the plate and $inx1$, $inx2$, $inx3$ are the inverse of r, r^2 and r^3 respectively. The boundary conditions of the simply supported plate may be written as

$$\begin{aligned} Z(n-2, 1{:}n) &= c(1, 1{:}n, 1)/l{:} \quad Z(n-1, n) = 1.0; \\ Z(n, 1{:}n) &= c(n, 1{:}n, 2)/l^2 + \upsilon*c(n, 1{:}2, 1)/l^2 \end{aligned} \quad (57)$$

The right hand side $e(1{:}\ n-3) = 1$; Z is made symmetric using the Wilson's procedure of Lagrangian multiplier and solving the following equation $[Z]\{\delta\} = \{e\}$, the deflection δ is obtained.

Numerical example:

For a circular plate of 3 m radius subjected to lateral load of 1 kN/sq·m with Poisson's ratio of 0.3 the deflection at centre is obtained as $5.15895/D$ whereas the closed form solution gives the value of $\left(\frac{(5+\upsilon)}{64(1+\upsilon)}\frac{ql^4}{D} = 5.15985/D\right)$.

If D is assumed to vary along the radius, the variation of D may considered in differential quadrature and the governing differential equation is

$$D\left(\frac{d^4w}{dr^4}+\frac{2}{r}\frac{d^3w}{dr^3}-\frac{1}{r^2}\frac{d^2w}{dr^2}+\frac{1}{r^3}\frac{dw}{dr}\right)+\frac{dD}{dr}\left(\frac{1}{r}\frac{d^2w}{dr^2}(2+\upsilon)+2\frac{d^3w}{dr^3}-\frac{1}{r^2}\frac{dw}{dr}\right)+\frac{d^2D}{dr^2}\left(\frac{\upsilon dw}{r\,dr}+\frac{d^2w}{dr^2}\right)=q \tag{58}$$

where $\frac{dD}{dr}$; and $\frac{d^2D}{dr^2}$ may be written as

$$\frac{dD}{dr}=(c(i,j,1)/l)*D(j,j);\quad \frac{d^2D}{dr^2}=(c(i,j,2)/l^2)*D(j,j) \tag{59}$$

and using the similar procedure, the maximum deflection may be computed.

Numerical example

Assume D varies as $D(2-\xi^2)$ where $\xi = r/l$ measured from centre of the plate. The maximum deflection at the centre of the plate due to lateral load of 1kN/sq·m is $3.045/D$ and there is no literature available to compare this value.

6.7 Buckling of circular plate subjected to radial edge loads.

For buckling calculation, the right hand side of Eq .(53) may be replaced by $-N_r\left(\frac{1}{r}\frac{dw}{dr}+\frac{d^2w}{dr^2}\right)$, one can solve as an eigenvalue problem as $[Z]\{\delta\} = N_{cr}[F]\{\delta\}$ to find the critical radial buckling load.

Numerical example

For a plate of uniform thickness, the buckling load N_{cr} is obtained as 0.46649 and $\overline{N}_{cr} = N_{cr}l^2/D = 4.19$ which agrees with the value of 4.20 obtained by closed form solution.(Iyengar 1988). If D varies as $D(2-\xi^2)$ the buckling load obtained as 0.7750 and $\overline{N}_{cr} = N_{cr}l^2/D = 6.775$ and no closed form result is available for comparison.

7. Conclusions

The present article has attempted to bring attention to some of the potential uses of computer languages capable of performing algebraic instead of merely numerical operations by considering the examples of a) Coupled Shear wall b) automatic generation of stiffness matrix in finite element method c) natural frequency by transfer matrix method d) application of Levy's method e)influence surface of a simply supported plate and f) writing equilibrium equations from Lagrange. Thus in discussing MATHEMATICA it was made clear that the range of these symbolic operations is considerable, encompassing for example, differentiation, integration, equation solving, matrix inversion and other procedures which constitute the basic ingredients of most structural analyses. Direct use was made of one such facility for solving linear system of equations in order to obtain a truly closed form solution of the Levy method for plane stress problems. Large number of repetitive numerical computations is eliminated as in the case of coupled shear wall problem. One of the more obvious advantages in performing operations such as equation solving or matrix inversion symbolically is the drastic savings in computing time. Parametric studies could easily be carried out in symbolic computation whereas in Finite element method it is time consuming to perform parametric studies. Series solutions can be applied for the study of the elastic buckling or the natural response of thin-walled members made of thin plates. This reduces to an eigen value problem in which the order of the determinant is four times number of plates. Such determinant may be computed symbolically so that the search for the critical condition reduces on the trivial task of numerical iteration of a simple expression instead of much more time consuming problem of iterating actual determinant. The use of symbolic algebra gives to the analyst much more visibility with respect to the solution methods. The symbolic algebra manipulation languages are important steps in the trend of reducing the drudgery of an analysis, thereby freeing the analyst to concentrate on solution techniques. The engineer can then more easily grasp the interrelationships of the problem variables, recognize simplifications to be made, and do a better and more accurate job. Unfortunately some of the codes do not yet support Greek letters, which may require the use of two notation systems, one with Latin letters for symbolic computation and the other for the usual ones for the rest. This inconvenience may disappear in near future with further developments in software and hardware. All areas of structural design and analysis are tending towards more rigorous physical arguments for behavioural predictions. The replacement of empirical techniques by analyses founded upon sound principles of elasticity, vibration, statistics, etc was fostered by the use of computers to handle the detailed numerical problems that arose. It is only logical to further exploit the computer by using it to formulate the difficult problems it has let us attack. There are also some problems and limitations. The most unfavorable condition is its relative slow sped and large storage requirements.

The DQM is applied to solve a) coupled shear wall b) lateral buckling of thin-walled beams c) static and buckling analysis of uniform and non-uniform circular plates subjected to uniformly distributed lateral load and uniform radial edge loads. Accurate results are obtained for the problems. This approach is convenient for solving problems governed by the higher order differential equations. It is also easy to write algebraic equations in the place of differential equations and application of boundary conditions is also an easy task. It is also explained in the paper how Lagrange multiplier method is used to convert rectangular matrix to square matrix for the solution of equations. Results with high accuracy are obtained and fast convergent trend is observed in all study cases.

It is expected that the symbolic computation and DQM will find a wide range of applications in structural engineering.

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