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# Natural frequency of laminated composite plate resting on an elastic foundation with uncertain system properties

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**Abstract.** Composite laminated structures supported on elastic foundations are being increasingly used in a great variety of engineering applications. Composites exhibit larger dispersion in their material properties compared to the conventional materials due to large number of parameters associated with their manufacturing and fabrication processes. And also the dispersion in elastic foundation stiffness parameter is inherent due to inaccurate modeling and determination of elastic foundation properties in practice. For a better modeling of the material properties and foundation, these are treated as random variables. This paper deals with effects of randomness in material properties and foundation stiffness parameters on the free vibration response of laminated composite plate resting on an elastic foundation. A $C^0$ finite element method has been used for arriving at an eigen value problem. Higher order shear deformation theory has been used to model the displacement field. A mean centered first order perturbation technique has been employed to handle randomness in system properties for obtaining the stochastic characteristic of frequency response. It is observed that small amount of variations in random material properties and foundation stiffness parameters significantly affect the free vibration response of the laminated composite plate. The results have been compared with those available in the literature and an independent Monte Carlo simulation.

**Keywords:** Composite plates; uncertain system properties; elastic foundation; free vibration; second order statistics; $C^0$ finite element.

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## 1. Introduction

Composite materials permit the designer to uniquely fashion the structural components to achieve specific objectives, such as high strength to weight ratio, stiffness to weight ratio, excellent corrosion resistance, very good fatigue characteristics, etc. However, these advantages may be offset by larger uncertainty in system properties of the composite compared to the conventional materials. The uncertainties in the properties are due to large number of parameters associated with manufacturing, fabrication and modeling processes of the composites. These uncertainties are reflected as random variations in the system properties of composite laminates. Components like plates resting on elastic foundation often find application in the construction of aerospace, civil, mechanical, etc. The modeling and determination of the foundation properties to obtain accurate response is also a matter of concern in the practice. The uncertainty in the foundation stiffness parameters cannot be avoided for accurate design in the constructions. Hence, the scatter in the response is due to variations in material as well as foundation stiffness parameters. In the present study parameters like elastic modulus, shear modulus, Poisson's ratio and foundation stiffness parameters are considered as random.

Mean values of system parameters are used in the conventional structural analysis. This gives only the mean response and misses the deviation caused by the randomness in the system parameters. For accurate analysis required in sensitive applications, it is necessary that the analysis technique incorporate the effect of system parameter randomness. This is of special importance for an accurate analysis of composites, which yield wide dispersion in the structural parameters, compared to conventional materials.

Considerable efforts have been made in the past by researchers and investigators on the prediction of the free vibration response of structures made of laminated composites considering the system properties as deterministic. Notably among them are due to Reddy and Phan (1985), Handian and Nayfeh (1993), Reddy (1996), Shankara and Iyenger (1996), Aiello and Ombres (1999) and Shen *et al.* (2003). Extensive literature is available on the response analysis of the deterministic structures to random excitations (Nigam and Narayanan 1994). However, the analysis of the structures with random system properties is not adequately reported in the literature. Some literature is available on the analysis of the structures made of metallic materials with random system properties. Zhang and Chen (1990) have presented a method to estimate the standard deviation of eigen value and eigen vector of random multiple degree of freedom system. Zhang and Ellingwood (1993) have evaluated the effect of random material field characteristics on the instability of a simply supported beam on elastic foundation and a frame using perturbation technique. Yamin *et al.* (1996) have investigated the stochastic perturbation method to vector-valued and matrix-valued function for response and reliability of uncertain structures. Manohar and Ibrahim (1999) have presented excellent reviews of structural dynamic problems with parameters uncertainities. Limited literature is available on analysis of the composite structures with random material properties. Salim *et al.* (1993) have employed first order perturbation technique for the analysis of composite plates. The problem is formulated using classical laminate theory and energy approach. The material properties have been modeled as random variables. Rayleigh-Ritz technique has been used for the solution. Specially orthotropic composite laminates with all edges simply supported have been analyzed with deterministic loading to obtain the standard deviation (SD) of deflections. In another paper, Salim *et al.* (1998) have obtained the second order statistics of natural frequencies of the laminate. The results have been compared with that of Monte Carlo simulation (MCS). Naveenthraj *et al.* (1998)

have obtained the static response statistics of graphite – epoxy composite laminates with randomness in material properties under deterministic loading by using combination of finite element method (FEM) and MCS. Singh *et al.* (2001) have investigated the natural frequencies of composite plate using exact solution approach in conjunction with higher order shear deformation theory (HSDT) considering random material properties. They have employed a first order perturbation technique (FOPT) to obtain the second order statistics of the first five natural frequencies. Venini and Mariani (2002) have investigated the eigenproblem associated with the free vibrations of uncertain composite plates. The elastic moduli of the system, the stiffness of the Winkler foundation on which the plate rests and the mass density are considered to be uncertain. Given their random field-based description, a new method is presented for the computation of the second order statistics of the eigen properties of the laminate. Onkar and Yadav (2003) have investigated nonlinear response statistics of composite laminates using classical approach with random material properties under random loading. Onkar *et al.* (2006) have investigated the buckling analysis of laminated composite plates with random material properties using stochastic finite elements based on a generalized layer-wise theory. The statistics of buckling strength has been determined using first order perturbation technique.

However, to the best of authors' knowledge no work dealing with free vibration analysis of the laminated composite plate resting on an elastic foundation using the HSDT with random material properties and random foundation stiffness parameters has been reported in the literature.

In the present study, the second order statistics of the fundamental frequency of laminated composite plates has been investigated. The plates are supported by elastic medium in the presence of small random variations in system parameters. The transverse shear strains are taken into account using the higher order shear deformation theory. The uncertain material properties including Young's modulus, Poisson's ratio, etc. of each constituent material and the stiffness parameters of the foundation are modeled as independent random variables. A $C^0$ finite element method in conjunction with a mean centered first order perturbation technique are employed to determine the second-order statistics (mean and standard deviation) of the natural frequency of laminated composite plate. Numerical results are presented for different boundary conditions. The numerical results showing the effect of uncertain materials properties, uncertain foundation parameters, and plate side to thickness ratio on the fundamental frequency and its dispersion with respect to various random variables are presented.

## 2. Formulation

Consider a rectangular laminated composite plate of length $a$, width $b$, and thickness $h$, which consist of $N$ number of orthotropic layers. All orthotropic layers of the composite plate are of uniform thickness. The mid plane of the plate is considered as the reference plane. The thickness coordinates $z$ of the top and bottom surfaces of any ($k$th) layer are denoted by $z^{(k+1)}$ and $z^k$, respectively. The fibers of $k$th layer are oriented at an angle $\theta_k$ to the $x$-axis.

The plate is supported by the foundation excluding any separation during the process of deformation as shown in Fig. 1. The load displacement relation between the plate and the supporting foundation follows the two- parameters model (Pasternak-type) as

$$P = K_1 w - K_2 \nabla^2 w \tag{1}$$

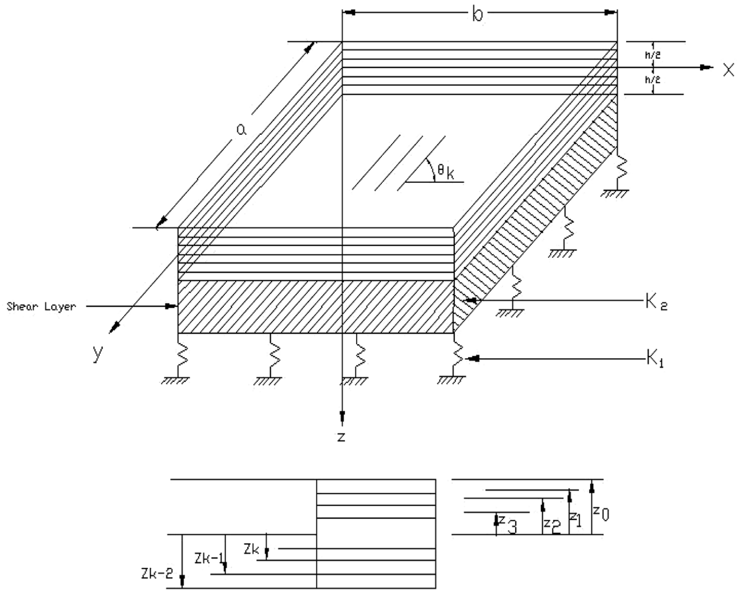



Fig. 1 Geometry of laminated composite plate resting on an elastic foundation

where $P$ is the foundation reaction per unit area, and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is Laplace differential operator; $K_1$ and $K_2$ are normal and shear stiffnesses of the foundation, respectively. This model is simply known as Winkler type when $K_2 = 0$ (e.g., Shen *et al.* 2003, Huang and Zheng 2003).

### *2.1 Displacement field model*

In the present work the higher order shear deformation theory has been used. The following displacement fields are assumed (Reddy 1984, 1996)

$$\begin{aligned} \bar{u}(x, y, z) &= u(x, y) + z\,\psi_x(x, y) + z^2 \varsigma_x(x, y) + z^3 \xi_x(x, y) \\ \bar{v}(x, y, z) &= v(x, y) + z\,\psi_y(x, y) + z^2 \varsigma_y(x, y) + z^3 \xi_y(x, y) \\ \bar{w} &= w \end{aligned} \tag{2}$$

where $\bar{u}$, $\bar{v}$ and $\bar{w}$ denote the displacements of a point along the $(x, y, z)$ coordinates; $u$, $v$, and $w$ are corresponding displacements of a point on the mid plane; $\psi_x$ and $\psi_y$ are the rotations of the normal to the mid plane about the $y$-axis and $x$-axis, respectively. The functions $\varsigma_x$, $\varsigma_y$, $\xi_x$ and $\xi_y$ are the higher-order terms in the Taylor series expansion, also defined in the mid-plane of the plate. These functions are determined using the condition that the transverse shear stresses $\tau_{xz} = \tau_4$ and $\tau_{yz} = \tau_5$ vanish on the plate top and bottom surfaces. Applying boundary conditions, the displacement field becomes

$$\begin{aligned} \bar{u} &= u + z\,\psi_x - z^3(4/3h^2)(\psi_x + \partial w/\partial x); \quad \bar{v} = v + z\,\psi_y - z^3(4/3h^2)(\psi_y + \partial w/\partial y) \\ \bar{w} &= w \end{aligned} \tag{3}$$

To avoid the difficulties associated with $C^1$ elements, the displacement model has been slightly modified, so that a $C^0$ continuous element would be sufficient. In modified form, displacements along the $x$-, $y$-, and $z$-directions for an arbitrary composite laminated plate are

$$\bar{u} = u + f_1(z)\psi_x + f_2(z)\theta_x; \quad \bar{v} = v + f_1(z)\psi_y + f_2(z)\theta_y; \quad \bar{w} = w \tag{4}$$

where, $f_1(z) = C_1 z - C_2 z^3$; $f_2(z) = -C_4 z^3$; $C_1 = 1$; $C_2 = C_4 = 4/3h^2$; $\theta_x = \dfrac{\partial w}{\partial x}$ and $\theta_y = \dfrac{\partial w}{\partial y}$

It can be seen that the number of degrees of freedom (DOFs) per node, by treating $\theta_x$ and $\theta_y$ as separate DOFs, increases from 5 to 7 for the HSDT model (e.g., Shankara and Iyengar 1996, Singh *et al.* 2002).

The displacement vector for the model is

$$\{\Lambda\} = [u \ \ v \ \ w \ \ \theta_y \ \ \theta_x \ \ \psi_y \ \ \psi_x]^T \tag{5}$$

### 2.2 Strain-displacement relations

The strain-displacements relations are obtained by using small deformation theory. The strain vectors corresponding to the displacement field given by Eq. (3) are expressed as

$$\varepsilon_{xx} = \varepsilon_1 = \partial\bar{u}/\partial x = \varepsilon_1^0 + z(k_1^0 + z^2 k_1^2); \quad \varepsilon_{yy} = \varepsilon_2 = \partial\bar{v}/\partial y = \varepsilon_2^0 + z(k_2^0 + z^2 k_2^2)$$
$$\gamma_{xy} = \varepsilon_6 = \partial\bar{u}/\partial y + \partial\bar{v}/\partial x = \varepsilon_6^0 + z(k_6^0 + z^2 k_6^2); \quad \gamma_{yz} = \varepsilon_4 = \partial\bar{v}/\partial z + \partial\bar{w}/\partial y = \varepsilon_4^0 + z^2 k_4^2$$
$$\gamma_{xz} = \varepsilon_5 = \partial\bar{u}/\partial z + \partial\bar{w}/\partial x = \varepsilon_5^0 + z^2 k_5^2 \tag{6a}$$

where

$$\varepsilon_1^0 = \frac{\partial u}{\partial x}; \quad k_1^0 = C_1\frac{\partial\psi_x}{\partial x}; \quad k_1^2 = -C_2\frac{\partial\psi_x}{\partial x} - C_4\frac{\partial\theta_x}{\partial x}$$
$$\varepsilon_2^0 = \frac{\partial v}{\partial y}; \quad k_2^0 = C_1\frac{\partial\psi_y}{\partial y}; \quad k_1^2 = -C_2\frac{\partial\psi_y}{\partial y} - C_4\frac{\partial\theta_y}{\partial y}$$
$$\varepsilon_4^0 = C_1\psi_y + \frac{\partial w}{\partial y}; \quad k_4^2 = -3C_2\psi_y - 3C_4\theta_y$$
$$\varepsilon_5^0 = C_1\psi_x + \frac{\partial w}{\partial x}; \quad k_5^2 = -3C_2\psi_x - 3C_4\theta_x$$
$$\varepsilon_6^0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}; \quad k_6^0 = C_1\left(\frac{\partial\psi_y}{\partial x} + \frac{\partial\psi_x}{\partial y}\right); \quad k_6^2 = -C_2\left(\frac{\partial\psi_y}{\partial x} + \frac{\partial\psi_x}{\partial y}\right) - C_4\left(\frac{\partial\theta_y}{\partial x} + \frac{\partial\theta_x}{\partial y}\right) \tag{6b}$$

### 2.3 Stress-strain relation

The linear constitutive relation for an orthotropic layer is given by

$$\{\sigma\} = [\bar{Q}]\{\varepsilon\} \tag{7}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_u \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} & 0 & 0 \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} & 0 & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} \\ 0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix} \tag{8}$$

where, $\{\sigma\}$, $\{\varepsilon\}$ and $\overline{Q}_{ij}$ are stress vector, strain vector and reduced elastic material constants, respectively (Jones 1975).

### 2.4 Strain energy of the plate

Using the stress-strain relations, the elastic strain energy due to bending of a laminated composite plate can be expressed as

$$U = \frac{1}{2}\int_A \{\bar{\varepsilon}\}^T[D]\{\bar{\varepsilon}\}dA \tag{9}$$

where

$$\{\bar{\varepsilon}\} = (\varepsilon_1^0\ \varepsilon_2^0\ \varepsilon_6^0\ k_1^0\ k_2^0\ k_6^0\ k_1^2\ k_2^2\ k_6^2\ \varepsilon_4^0\ \varepsilon_5^0\ k_4^2\ k_5^2)^T \tag{10}$$

and

$$[D] = \begin{bmatrix} [A1] & [B] & [E] & 0 & 0 \\ [B] & [D1] & [F1] & 0 & 0 \\ [E] & [F1] & [H] & 0 & 0 \\ 0 & 0 & 0 & [A2] & [C2] \\ 0 & 0 & 0 & [C2] & [F2] \end{bmatrix} \tag{11}$$

with

$$(A1_{ij}, B_{ij}, D1_{ij}, E_{ij}, F1_{ij}, H_{ij}) = \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k} \overline{Q}_{ij}^{(k)}(1, z, z^2, z^3, z^4, z^6)dz$$

for $i, j = 1, 2, 6$

$$(A2_{ij}, C2_{ij}, F2_{ij}) = \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k} \overline{Q}_{ij}^{(k)}(1, z^2, z^4)dz \quad \text{for} \quad i, j = 4, 5$$

where $\overline{Q}_{ij}^{(k)}$ are the reduced elastic material constants of the $k$th lamina (layer).

### 2.5 Strain energy due to foundation

The strain energy due to the foundation is expressed as

$$U_f = \frac{1}{2}\int_A \begin{Bmatrix} w \\ \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{Bmatrix}^T \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_2 \end{bmatrix} \begin{Bmatrix} w \\ \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{Bmatrix} dA \tag{12}$$

### *2.6 Kinetic energy of the laminate*

The kinetic energy of the vibrating laminated plate can be expressed as

$$T = \frac{1}{2}\int_V \rho\{\dot{u}\}^T\{\dot{u}\}dV \tag{13}$$

where $\rho$ and $\{u\} = \{\bar{u} \;\; \bar{v} \;\; \bar{w}\}^T$ are the density and global displacement vector of the plate.

For $N$ number of layers of composite plate, the kinetic energy can be expressed as

$$T = \frac{1}{2}\int_A \left( \sum_{k=1}^{N} \int_{Z_{k-1}}^{Z_k} \rho^{(k)}\{\dot{u}\}^T\{\dot{u}\}dz \right) dA \tag{14}$$

where $\rho^{(k)}$ is the density of the $k$th layer of the laminate.

### *2.7 Finite element model*

#### *2.7.1 Strain energy analysis*

For an isoparametric element, the displacement vector and the element geometry are represented by same interpolation functions.

$$\{\Lambda\} = \sum_{i=1}^{NN} \varphi_i\{\Lambda\}_i; \quad x = \sum_{i=1}^{NN} \varphi_i x_i; \quad y = \sum_{i=1}^{NN} \varphi_i y_i \tag{15}$$

where $\varphi_i$ is the interpolation function for the $i$th node, $\{\Lambda\}_i$ is the vector of unknown displacements for the $i$th node, $NN$ is the number of nodes per element and $x_i$ and $y_i$ are Cartesian coordinates of the $i$th node.

Using Eq. (6b), the strain vector given in Eq. (10) can be written as

$$\{\bar{\varepsilon}\} = [L]\{\Lambda\} \tag{16}$$

where $[L]$ is a differential operator (Appendix).

The functional is computed for each element and then summed over all the elements in the domain to get total functional for the domain. Following this, Eq. (9) can be written as

$$U = \sum_{e=1}^{NE} U^{(e)} = \sum_{e=1}^{NE} \frac{1}{2}\int_{A^{(e)}} \{\bar{\varepsilon}\}^T[D]\{\bar{\varepsilon}\}dA \tag{17}$$

where, $NE$ is the number of elements.

From Eqs. (15-17), we get

$$U^{(e)} = \{\Lambda\}^{T(e)}[K]^{(e)}\{\Lambda\}^{(e)} \tag{18}$$

Here $\{\Lambda\}^{(e)}$ is the displacements vector of the $e$th element and $[K]^{(e)}$ is the bending stiffness matrix of the $e$th element, which is expressed as

$$[K]^{(e)} = \frac{1}{2}\int_{A^{(e)}}[BB]^T[D][BB]dA \tag{19}$$

where, $[BB] = [[BB_1]\ [BB_2]\ \ldots\ [BB_{NN}]]$, $[BB_i] = [L]\varphi_i$
where $[BB_i]$ is the strain-displacement matrix for the $i$th node.

Adopting numerical integration, the element bending stiffness matrix can be obtained from Eq. (19), by transforming expression in $x$, $y$ coordinate system to natural coordinate system $\xi$, $\eta$.

### *2.7.2 Foundation analysis*

Using finite element notation given in Eq. (15), Eq. (12) may be written as

$$U_f = \sum_{e=1}^{NE} U_f^{(e)} = \sum_{e=1}^{NE}\int_{A^{(e)}}\begin{Bmatrix} w \\ \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y}\end{Bmatrix}^T \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_2\end{bmatrix}\begin{Bmatrix} w \\ \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y}\end{Bmatrix} dA \tag{20}$$

We have,

$$\begin{Bmatrix} w \\ \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y}\end{Bmatrix} = [L_g]\{\Lambda\} \tag{21a}$$

where, $[L_g]$ is a differential operator due to the foundation (Appendix).

Hence Eq. (21a) may be written as

$$\begin{Bmatrix} w \\ \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y}\end{Bmatrix} = [L_g]\sum_{i=1}^{NN}\varphi_i[\Lambda_i] = [BB_g]\{\Lambda\}^{(e)} \tag{21b}$$

Here

$$[BB_g] = [[BB_{g1}]\ [BB_{g2}]\ \ldots\ [BB_{gNN}]] \text{ with } [BB_{gi}] = [L_g]\varphi_i$$

where $[BB_{gi}]$ is the strain-displacement matrix due to the foundation for the $i$th node.

Applying similar steps as given in sub-section 2.7.1, Eq. (20) may be written as

$$U_f^{(e)} = \{\Lambda\}^{T(e)}[K_f]^{(e)}\{\Lambda\}^{(e)} \tag{22}$$

Here $[K_f]^{(e)}$ is the stiffness matrix of $e$th element due to the foundation and written as

$$[K_f]^{(e)} = \frac{1}{2}\int_{A^{(e)}}[BB_g]^T[D_f][BB_g]dA \tag{23}$$

Here
$$[D_f] = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_2 \end{bmatrix}$$

Adopting numerical integration in natural coordinate system, the stiffness matrix due to the foundation can be obtained using Gaussian quadrature.

### *2.7.3 Kinetic energy analysis*

The displacement field model given by Eq. (4) may be represented as

$$\{u\} = [N]\{\Lambda\} \tag{24}$$

where, $[N]$ is given in Appendix.

Using Eqs. (13), (14) and (24), the following is obtained

$$T = \frac{1}{2}\int_A \{\dot{\Lambda}\}^T[m]\{\dot{\Lambda}\}dA \tag{25a}$$

where $[m]$ is the inertia matrix and may be written as

$$[m] = \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}\rho[N]^T[N]dz \tag{25b}$$

Using Eq. (15), the kinetic energy of the $e$th element is obtained

$$T^{(e)} = \frac{1}{2}\int_{A^{(e)}}[\dot{\Delta}]^{T(e)}[M]^{(e)}[\dot{\Delta}]^{(e)}dA \tag{26a}$$

where

$$[M]^e = \frac{1}{2}\int_{A^{(e)}}[\varphi]^T[m][\varphi]dA \tag{26b}$$

Summing over total number of element, the kinetic energy of the vibrating plate is obtained

$$T = \sum_{e=1}^{NE}T^{(e)} \tag{27}$$

## *2.8 Governing equations of motion*

The governing equations of the motion is obtained using Hamilton's variational principal

$$\delta\int_{t_1}^{t_2}(U + U_f - T)dt = 0 \tag{28}$$

Substituting Eqs. (17), (18), (22) and (27) in Eq. (28), ones obtain as

$$[M]\{\ddot{q}\} + [K + K_f]\{q\} = 0 \tag{29}$$

Assuming the system vibrates in principal mode in free vibration case, the Eq. (29) can be written as

$$\{[K_s] - \lambda[M]\}\{q\} = 0 \quad \text{with} \quad [K_s] = [K] + [K_f] \tag{30}$$

where, $\{q\} = \sum_{e=1}^{NE}\{\Lambda\}^{(e)}$, $[K] = \sum_{e=1}^{NE}[K]^{(e)}$, $[K_f] = \sum_{e=1}^{NE}[K_f]^{(e)}$, $[M] = \sum_{e=1}^{NE}[M]^{(e)}$ and $\lambda = \omega^2$. Also $\{q\}$, $[K]$, $[K_f]$, and $[M]$ and $\lambda$ are defined as a global displacement vector, global bending stiffness matrix, global foundation stiffness matrix, global mass matrix and eigenvalues, respectively, and $\omega$ is the frequency of natural vibration.

Eq. (30) is random in nature, being dependent on the system properties. Consequently, the natural frequencies and mode shapes are random in nature. A mean centered first order perturbation technique in conjunction with $C^0$ finite element method has been used to obtain the solution of the governing random equations.

## 3. Solutions-perturbation technique

We consider a class of problems where the random variation is very small as compared to the mean part of random material properties. Further it is quite logical to assume that the dispersions in the derived quantities like $[K]$, $\lambda$, etc. are also small with respect to their mean values. In the present analysis, the elastic constants (Young's modulus, shear modulus, Poisson's ratio, etc.) of each constituent material are treated as independent random variables. Since the foundation stiffness parameters are totally dependent on the material properties of the supporting elastic medium which also possess random fluctuations, the randomness in both $k_1$ and $k_2$ is taken into consideration. Consequently, $\lambda$, $[K_s]$, and $\{q\}$ in Eq. (30) are random. In general, a random variable can be represented as the sum of the mean value and a zero mean random variable, denoted by superscripts '$d$' and '$r$', respectively (Singh *et al.* 2001)

$$[K_s] = [K_s^d] + [K_s^r]; \quad \lambda_i = \lambda_i^d + \lambda_i^r \;\text{ and }\; \{q_i\} = \{q_i^d\} + \{q_i^r\} \tag{31}$$

where

$$\lambda_i^d = \omega_i^d, \;\; \lambda_i^r = 2\omega_i^d\omega_i^r + \omega_i^{r^2} \qquad i = 1, 2, \ldots, p \tag{32}$$

The parameter $p$ indicates the size of eigen problem. Substituting Eq. (31) in Eq. (30) and collecting same order of the magnitude term and keeping only up to the first order terms, we obtain

$$[K_s^d]\{q_i^d\} = \lambda_i^d\{q_i^d\} \tag{33}$$

$$[K_s^d]\{q_i^r\} + [K_s^r]\{q_i^d\} = \lambda_i^r\{q_i^d\} + \lambda_i^d\{q_i^r\} \tag{34}$$

Because Eq. (33) is the deterministic equation relating to the mean values, the mean eigenvalues and corresponding mean eigenvectors can be determined by conventional eigen solution procedures. According to the orthogonality properties, the normalized eigenvector meet the following conditions

$$\{q_i^d\}^T[M]\{q_i^d\} = \delta_{ij}$$

$$\{q_i^d\}^T[K_s^d]\{q_i^d\} = \delta_{ij}\lambda_i^d, \quad (i,j) = 1,2,\ldots,p \tag{35}$$

where $\delta_{ij}$ is the Kronecker delta.

The eigenvectors, after being properly normalized, form a complete orthonormal set and any vector in the space can be expressed as a linear combination of these eigenvectors. Hence, the *i*th random part of the eigenvectors can be expressed as

$$\{q_i^r\} = \sum_{j=1}^{p} C_{ij}^r\{q_i^d\}, \quad i \neq j, \quad C_{ii}^r = 0, \quad i = 1,2,\ldots,p \tag{36}$$

where $C_{ij}^r$'s are small random coefficients to be determined.

Substituting Eq. (36) in Eq. (34), premultiplying the first by $\{q_i^d\}^T$ and the second by $\{q_i^d\}^T$ $(j \neq i)$, respectively and making use of orthogonality Eq. (35), we have

$$\lambda_i^r = \{q_i^d\}^T[K_s^r]\{q_i^d\} \tag{37}$$

$$C_{ij}^r = \{q_j^d\}[K_s^r]\{q_i^d\}/(\lambda_i^d - \lambda_j^d), \quad j \neq i \tag{38}$$

Substituting Eq. (38) into Eq. (36), we obtain

$$\{q_i^r\} = \sum_{j=1}^{p}\{q_i^d\}\frac{\{q_j^d\}[K_s^r]\{q_i^d\}}{\lambda_i^d - \lambda_j^d}, \quad j \neq i \tag{39}$$

For the present case, as discussed earlier, the derived quantities are random because of the system properties. Let $b_1$, $b_2$, …., $b_n$ denote random system properties. Following Eq. (31), $b_i$ can be expressed as

$$b_i = b_i^d + b_i^r \tag{40}$$

The FEM in conjunction with FOPT has been found to be accurate and efficient (e.g., Vanmarke and Grigoriu 1983, Kareem and Sun 1990, Kleiber and Hein 1992, Yamin *et al.* 1996, Lin and Kam 2000). According to this method, the random variables are expressed by Taylor's series. Keeping the first-order terms and neglecting the second- and higher-order terms, Eq. (31) can be written as follows because, the first order is sufficient to yield results with desired accuracy for problems with low variability.

$$\lambda_i^r = \sum_{i=1}^{p}\frac{\partial\lambda_i^d}{\partial b_i}b_i^r; \quad \{q_i^r\} = \sum_{i=1}^{p}\frac{\partial\{q_i^d\}}{\partial b_i}b_i^r; \quad [K_s^r] = \sum_{i=1}^{p}\frac{\partial[K_s^d]}{\partial b_i}b_i^r \tag{41}$$

Substituting Eq. (41) into Eqs. (37) and (39), we obtain

$$\frac{\partial \lambda_i^d}{\partial b_i} = \{q_i^d\}^T \frac{\partial [K_s^d]}{\partial b_i}\{q_i^d\} \tag{42}$$

$$\frac{\partial q_i^d}{\partial b_i} = \sum_{\substack{s=1 \\ s \neq i}}^{p} \{q_s^d\} \frac{\{q_s^d\}^T \frac{\partial [K_s^d]}{\partial b_i}\{q_i^d\}}{\lambda_i^d - \lambda_s^d} \tag{43}$$

The variances of the eigenvalues and the eigenvectors can now be expressed as

$$Var(\lambda_i) = \sum_{j=1}^{p}\sum_{k=1}^{p} \frac{\partial \lambda_i^d}{\partial b_j}\frac{\partial \lambda_i^d}{\partial b_k} Cov(b_j^r, b_k^r) \tag{44}$$

$$Var\{\{q_i\}\{q_i^*\}^T\} = \sum_{j=1}^{p}\sum_{k=1}^{p} \frac{\partial \{q_i^d\}}{\partial b_j}\frac{\partial \{q_i^d\}^T}{\partial b_k} Cov(b_j^r, b_k^r) \tag{45}$$

where $Cov(b_j^r, b_k^r)$ is the cross variance between $b_j^r$ and $b_k^r$. The standard deviation (SD) is obtained by the square root of the variance (Nigam and Narayanan 1994).

## 4. Numerical results and discussion

The approach outlined for the free vibration analysis of the composite plates resting on elastic foundation with random system properties is illustrated through a number of examples. The technique has been validated by comparing the results. A nine noded Lagrange isoparametric element, with 63 degrees of freedom (DOFs) for the present HSDT model has been used for discretizing the laminate. Based on convergence study conducted for the fundamental frequency, a (5 × 5) mesh has been used throughout the study. All the results reported in this paper have been obtained by employing the full (3 × 3) integration rule for thick plate and reduced (2 × 2) integration rule for thin plates. The following dimensionless mean fundamental frequency and foundation stiffness parameters $k_1$ and $k_2$ have been used in this study as $\varpi = (\omega^d a^2 \sqrt{\rho/E_{22}^d})/h$, $k_1 = k_1 b^4/E_{22}^d h^3$ and $k_2 = k_2 b^2/E_{22}^d h^3$, where $\overline{\omega}$, $k_1$ and $k_2$ are dimensional mean natural frequency, dimensionless Winkler foundation stiffness parameter and dimensionless Pasternak foundation parameter, respectively.

In the present study various combination of edge support conditions namely clamped (C), free (F) and simply supported (S) have been used for the investigation. For example, CFCF means clamped edges at $x = 0$, a and free edges at $y = 0$ and $b$. The boundary conditions for the plate are

*Simply supported edges*:

$$v = w = \theta_y = \psi_y = 0 \text{ at } x = 0,\ a; \quad u = w = \theta_x = \psi_x = 0 \text{ at } y = 0,\ b$$

*Clamped edges:*

$$u = v = w = \theta_y = \theta_x = \psi_y = \psi_x = 0 \text{ at } x = 0,\ a \text{ and } y = 0,\ b$$

*Free edges*:

$$u \neq v \neq w \neq \theta_y \neq \theta_x \neq \psi_y \neq \psi_x \neq 0, \text{ at } x = 0, a \text{ and } y = 0, b$$

The second order statistics of dimensionless fundamental frequency of graphite-epoxy plate resting on Winkler and Pasternak foundations with various boundary conditions have been presented for a standard deviation (SD) of system properties varying from 0 to 25 percent. Uncertain variations of system properties are incorporated into the prediction of the dimensionless fundamental frequency of laminated composite plate resting on elastic foundations. The lamina material properties and foundation stiffness parameters modeled as independent RVs are longitudinal and transverse elastic moduli $E_{11}, E_{22}$ in plane shear modulus $G_{12}$ out of plane shear moduli $G_{13}, G_{23}$, Poisson ratio $\nu_{12}$ and the elastic foundation stiffness parameters $k_1$ and $k_2$. These RVs $(b_i)$ are sequenced as $b_1 = E_{11}, b_2 = E_{22}, b_3 = G_{12}, b_4 = E_{13}, b_5 = G_{23}, b_6 = \nu_{12}, b_7 = k_1$ and $b_8 = k_2$. The following numerical values and relationship between the mean values of the material properties for graphite/epoxy composite have been used in the present investigation:

$$E_{11}^d = 40E_{22}^d, G_{12}^d = G_{13}^d = 0.6E_{22}^d, G_{23}^d = 0.5E_{22}^d, \nu_{12}^d = 0.25$$

The plate geometry used is characterized by aspect ratios (*a/b*) = 1 and 2, side to thickness ratios (*a/h*) = 10 and 100.

### 4.1 Validation study

#### 4.1.1 Mean value

The proposed outlined approach is validated by comparing the present obtained mean fundamental frequency with those available in the literature. The dimensionless mean fundamental frequency for

Table 1 Comparison of dimensionless mean fundamental frequency, $\varpi = (\omega^d a^2 \sqrt{\rho/E_{22}^d})/h$ for cross-ply laminated composite square plates with all edges simply supported

| *a/h* | Present | | Singh *et al.* (2001) | |
|---|---|---|---|---|
| | [0°/90°] | [0°/90°/90°/0°] | [0°/90°] | [0°/90°/90°/0°] |
| 10 | 10.5684 | 15.0794 | 10.56565 | 15.10799 |
| 100 | 11.5261 | 19.1406 | 11.9049 | 19.13079 |

Table 2 Comparison of dimensionless mean fundamental frequency, $\varpi = (\omega^d a^2 \sqrt{\rho/E_{22}^d})/h$ for a $[0/90]_s$ all edges simply supported laminated square plate with various side to thickness ratios (*a/h*)

| *a/h* | Dimensionless mean fundamental frequency | | |
|---|---|---|---|
| | Present | Shen *et al.* (2003) | Handian and Nayfeh (1993) |
| 5 | 10.6786 | 10.263 | 10.263 |
| 10 | 15.0794 | 14.702 | 14.702 |
| 20 | 17.6786 | 17.483 | 17.483 |
| 25 | 18.1163 | 17.950 | - |
| 50 | 18.7724 | 18.641 | 18.681 |
| 100 | 19.1406 | 18.828 | 18.828 |

Table 3 Comparison of dimensionless mean fundamental frequency, $\varpi = (\omega^d a^2 \sqrt{\rho/E_{22}^d})/h$ for all edges simply supported composite square plates resting on elastic foundation

| Lay-up | *a/h* | Dimensionless mean fundamental frequency | | | | | |
|---|---|---|---|---|---|---|---|
| | | Present | Shen *et al.* (2003) | Present | Shen *et al.* (2003) | Present | Shen *et al.* (2003) |
| | | $(k_1, k_2)$ = (0, 0) | $(k_1, k_2)$ = (0, 0) | $(k_1, k_2)$ = (100, 0) | $(k_1, k_2)$ = (100, 0) | $(k_1, k_2)$ = (100, 10) | $(k_1, k_2)$ = (100, 10) |
| 0/90/0 | 50 | 18.7695 | 18.689 | 21.2656 | 21.152 | 25.4874 | 25.390 |
| | 20 | 17.5231 | 17.483 | 20.1561 | 20.132 | 24.5577 | 24.536 |
| | 10 | 14.7106 | 14.702 | 17.7559 | 17.753 | 22.7414 | 22.596 |
| | 5 | 10.3745 | 10.263 | 14.362 | 14.244 | 19.9434 | 19.879 |
| $(\pm 45)_{2T}$ | 50 | 24.0159 | 23.225 | 26.0135 | 25.285 | 29.5619 | 28.924 |
| | 20 | 22.4154 | 21.812 | 24.5376 | 23.989 | 28.2538 | 27.789 |
| | 10 | 18.6916 | 18.333 | 21.1833 | 20.868 | 25.3679 | 25.132 |
| | 5 | 12.7037 | 12.544 | 16.1502 | 16.022 | 21.2923 | 21.278 |

all edges simply supported anti-symmetric and symmetric cross-ply square plate with various side to thickness ratios has been obtained. The results are presented in Tables 1-3 and compared with those available in the literature (e.g., Handian and Nayfeh 1993, Singh *et al.* 2001, Shen *et al.* 2003). It is observed that the results are in good agreement.

### *4.1.2 Standard deviation*

The proposed outlined probabilistic approach has also been validated by comparing the present standard deviation results of the fundamental frequency with independent Monte Carlo simulation which is considered to be exact method in probabilistic analysis and those available in the literature (Singh *et al.* 2001). Figs. 2(a) and (b) present a comparison between results obtained by the present

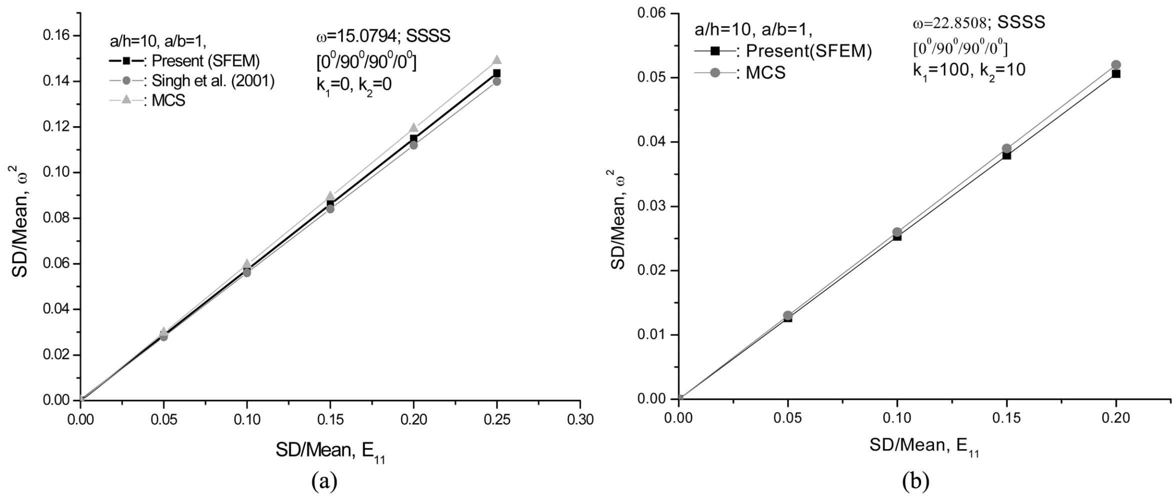



Fig. 2 Validation of SD/mean of the plate fundamental frequency from MCS and Singh *et al*. (2001) with the present SFEM for all edges simply supported [0°/90°/90°/0°] laminated composite square plate with $a/h = 10$ (a) no foundation ($k_1 = 0$, $k_2 = 0$), and (b) Pasternak foundation ($k_1 = 100$, $k_2 = 10$)

approach, the MCS and the FOPT based on close form solution by Singh *et al.* (2001) for $[0^o/90^o/90^o/0^o]$ symmetric cross-ply square plate, $b/h$ = 10, without foundation and between the present approach and the MCS for $[0^o/90^o/90^o/0^o]$ square plate resting on Pasternak foundation ($k_1$ = 100, $k_2$ = 10), respectively. All edges are simply supported. Only one material property $E_{11}$ has been considered random, others deterministic. For the MCS approach, the samples are generated using MatLab to fit the desired mean and SD. These samples are used in Eq. (30), which is solved repeatedly, adopting conventional eigen value procedure, to generate a sample of the fundamental frequency. The number of samples used for MCS approach is 10,000 based on satisfactory convergence of the results. The normal distribution has been assumed for random number generations in MCS. However, the present perturbation approach used in the study does not put any limitation as regard to probability distribution of the system property. This is an advantage over the MCS. It is observed that the results are in overall good agreement.

### *4.2 Numerical results: Mean and standard deviation*

#### *4.2.1 Mean fundamental frequency*

Table 4 presents dimensionless mean fundamental frequency with $a/b$ = 1 and 2, ($k_1$ = 0, $k_2$ = 0), ($k_1$ = 100, $k_2$ = 0), and ($k_1$ = 100, $k_2$ = 10) and $a/h$ = 10 and 100 for stacking sequence of [0º/90º/90º/0º] and [0º/90º/0º/90º] graphite-epoxy plates with SSSS boundary condition. It is observed that the fundamental frequency changes significantly with $a/h$ ratio. It is also observed that the fundamental frequency of Pasternak model is higher than that obtained by Winkler model. The changes between three cases of the foundation are very small for moderately thick rectangular plates ($a/h$ = 10) and large for thin rectangular plates ($a/h$ = 100). However, the changes are almost same order of magnitude for moderately thick and thin square laminates. The effect of nature of lay-up, i.e., symmetry and anti-symmetry of cross-ply laminate has significant role on the fundamental frequency. The aspect ratio of the plate also plays important roles. The fundamental frequency in general increases as the aspect ratio changes from 1 to 2 for both the foundation models.

Table 5 presents the dimensionless mean fundamental frequency with $a/b$ = 1, ($k_1$ = 0, $k_2$ = 0), ($k_1$ = 100, $k_2$ = 0), and ($k_1$ =100, $k_2$ = 10) and $a/h$ = 10 and 100 for an anti-symmetric angle ply [45º/−45º/45º/−45º] laminated composite plates with SSSS, CCCC and CFCF boundary conditions.

Table 4 Dimensionless mean fundamental frequency, $\varpi = (\omega^d a^2 \sqrt{\rho/E_{22}^d})/h$ for all edges simply supported cross-ply symmetric and anti-symmetric composite plates resting on elastic foundation

| Lay-up | $a/h$ | $a/b$ | Dimensionless mean fundamental frequency | | |
|---|---|---|---|---|---|
| | | | ($k_1$ = 0, $k_2$ = 0) | ($k_1$ = 100, $k_2$ = 0) | ($k_1$ = 100, $k_2$ = 10) |
| [0º/90º/90º/0º] | 10 | 1 | 15.5261 | 18.0955 | 22.8767 |
| | | 2 | 24.1116 | 24.3181 | 24.3181 |
| | 100 | 1 | 19.1404 | 21.8977 | 26.024 |
| | | 2 | 34.2855 | 52.4057 | 68.5534 |
| [0º/90º/0º/90º] | 10 | 1 | 14.899 | 17.9131 | 22.7243 |
| | | 2 | 24.6260 | 24.6414 | 24.6389 |
| | 100 | 1 | 17.5999 | 20.2420 | 24.6415 |
| | | 2 | 49.6917 | 63.7369 | 77.9896 |

Table 5 Dimensionless mean fundamental frequency, $\varpi = (\omega^d a^2 \sqrt{\rho/E_{22}^d})/h$ for an anti-symmetric angle-ply composite square plate resting on elastic foundation with various support conditions

| Lay-up | a/h | BCs | Dimensionless mean fundamental frequency | | |
|---|---|---|---|---|---|
| | | | ($k_1 = 0$, $k_2 = 0$) | ($k_1 = 100$, $k_2 = 0$) | ($k_1 = 100$, $k_2 = 10$) |
| [45°/–45°/45°/–45°] | 10 | SSSS | 18.7272 | 21.2147 | 25.4234 |
| | | CCCC | 22.8167 | 24.9020 | 28.7305 |
| | | CFCF | 6.7448 | 11.9827 | 13.9787 |
| | 100 | SSSS | 24.6154 | 26.5688 | 30.0581 |
| | | CCCC | 41.1869 | 42.3832 | 45.4118 |
| | | CFCF | 7.7769 | 12.5094 | 15.3158 |

There are significant changes in the fundamental frequency between the three cases of the foundations. The fundamental frequency of the plate on Pasternak foundation is the largest, while it is the lowest for the plate with no foundation. It is noticed that the fundamental frequency changes significantly with *a/h* ratios. The fundamental frequency for the CCCC boundary condition is the largest as compared to any other support conditions, while it is the smallest for CFCF for both the *a/h* ratios and the three elastic foundation cases.

### *4.2.2 Standard deviation of fundamental frequency*

Figs. 3(a)-(f) show the variation of SD/Mean of dimensionless fundamental frequency with random changes in only one material property at a time, keeping the others deterministic for [0°/90°/90°/0°] laminated square plate with SSSS boundary condition for three elastic foundation cases ($k_1$ = 0, $k_2$ = 0), ($k_1$ =100, $k_2$ = 0), ($k_1$ = 100, $k_2$ = 10), with $a/h = 10$. Among the three foundation cases considered, $(k_1, k_2) = (100, 10)$ and $(k_1, k_2) = (100, 0)$ correspond to the Pasternak type and the Winkler type foundation, respectively, whereas $(k_1, k_2) = (0, 0)$ corresponds to no elastic foundation, the scattering in the fundamental frequency is the lowest in case of the plate resting on two-parameter Pasternak foundation model, while it is the highest in case of no elastic foundation. In general, the plate on Winkler model shows approximately 27-32 percent less scattering, while the plate on Pasternak foundation shows approximately 55-58 percent less scattering as compared to the plate with no foundation. The effect of $E_{11}$ on scattering of fundamental frequency is the highest, while it is the lowest for $\nu_{12}$. It is seen that foundation stiffness has a significant effect on the frequency response of the plate. With reference to Pasternak foundation, the dispersion in the fundamental frequency of the plate decreases more sensibly as compared to Winkler foundation and no elastic foundation and these are about 30 and 38 percent, respectively.

From application point of view, it is appropriate to consider the case where all the material properties vary simultaneously. Fig. 4 shows the variation of the SD of the fundamental frequency with the SD of basic RVs ($b_i$, $i$ = 1, 2, ..., 6) changing simultaneously and with the same value of the ratio of its SD to mean. It has been analyzed 4-layers symmetric [0°/90°/90°/0°], square laminate with $a/h$ = 10 and 100 for the three foundation cases, as explained before. It is observed that the dispersion in the frequency increases if side to thickness ratio increases from $a/h$ = 10 to 100. For both the values of a/h, no elastic foundation plate is more sensitive as compared to the plate resting on elastic foundation. It is interesting to note that the Pasternak model shows about 56 percent for $a/h$ = 10 and 43.75 percent for $a/h$ = 100 and the Winkler model shows about 30 percent for $a/h$ = 10 and 21 percent for $a/h$ = 100 less scattering as compared to the plate with no elastic foundation.

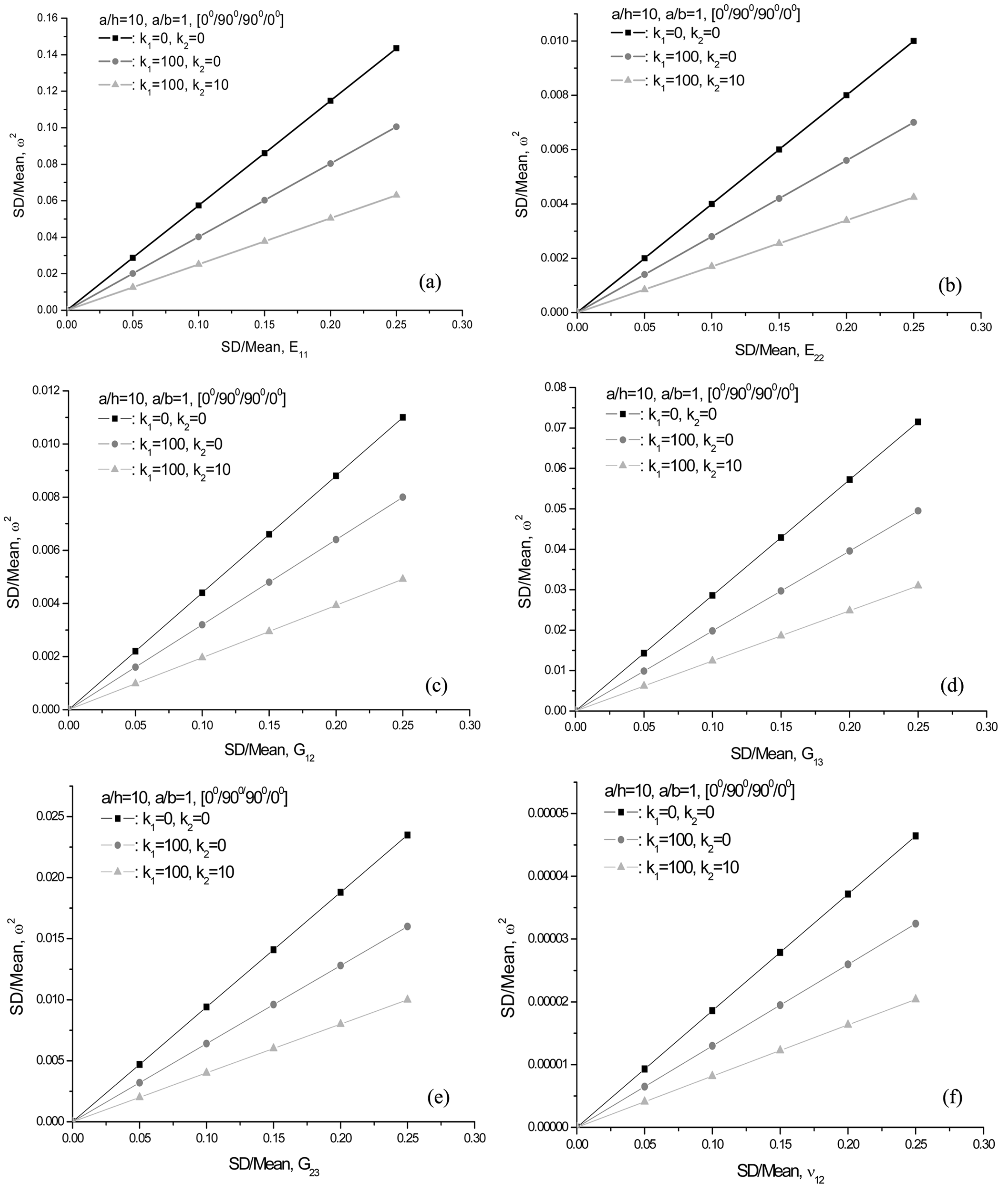
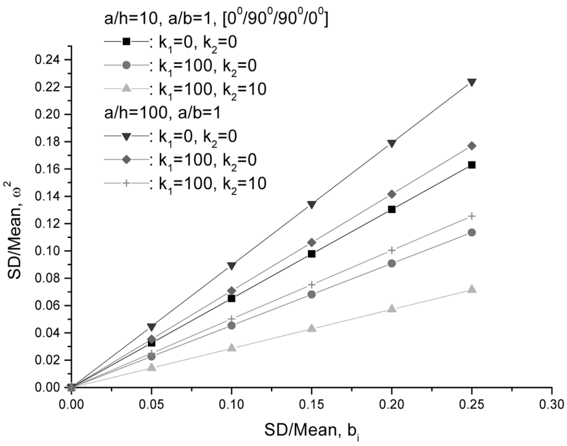
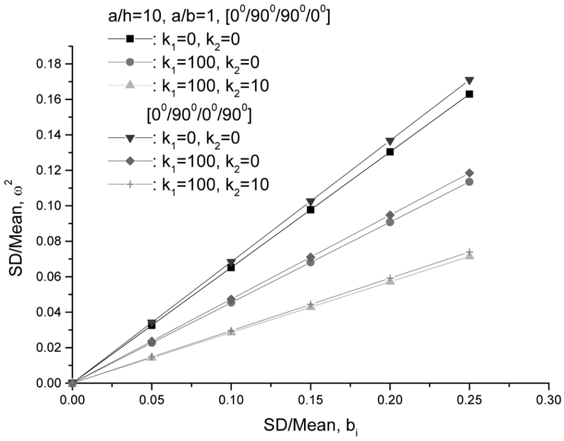



Fig. 3 Dispersion of the fundamental frequency square ($\omega^2$) of a simply supported square plate with $a/h$ = 10 resting on elastic foundations: (a) with respect to $E_{11}$ (b) with respect to $E_{22}$ (c) with respect to $G_{12}$ (d) with respect to $G_{13}$ (e) with respect to $G_{23}$ (f) with respect to $\nu_{12}$




( $i$ = 1, 2, 3, ..........., 6 )

Fig. 4 Effect of side-to-thickness ratio ($a/h$) on the dispersion of the fundamental frequency square ($\omega^2$) of a simply supported square plate resting on elastic foundations with all random material variables changing simultaneously



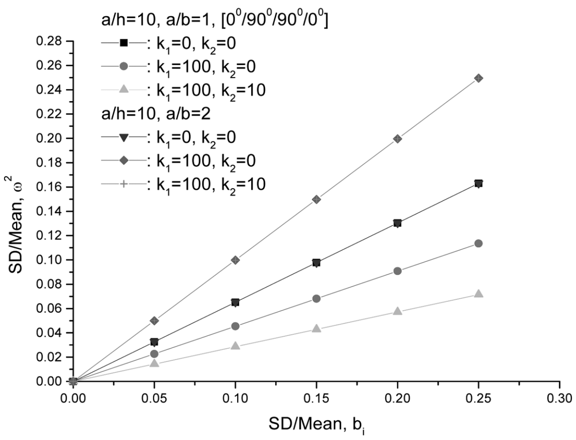

( $i$ = 1, 2, 3, ..........., 6 )

Fig. 5 Comparison of variation of SD/Mean of fundamental frequency square ($\omega^2$) with SD of material properties, for symmetric and anti-symmetric cross-ply laminated composite plate resting on elastic foundation with all basic material properties changing simultaneously

The comparison of variation of the SD of the fundamental frequency with the SD of basic random material properties changing simultaneously, as explained before, for 4 layers symmetric [0°/90°/90°/0°], and 4 layers anti-symmetric [0°/90°/0°/90°], square laminates with $a/h = 10$ is shown in Fig. 5 for three foundation cases, as explained before. It is observed that the anti-symmetric cross ply laminate is more sensitive than symmetric cross-ply laminate. The anti-symmetric plate shows 4.9, 4.8 and 4 percent more dispersion for the plate with no foundation, Winkler foundation and Pasternak foundation, respectively as compared to the symmetric plate with the respective cases of foundation. In general, it can been seen that the scattering in the frequency is significantly affected by simultaneous change in the all random material properties considered.

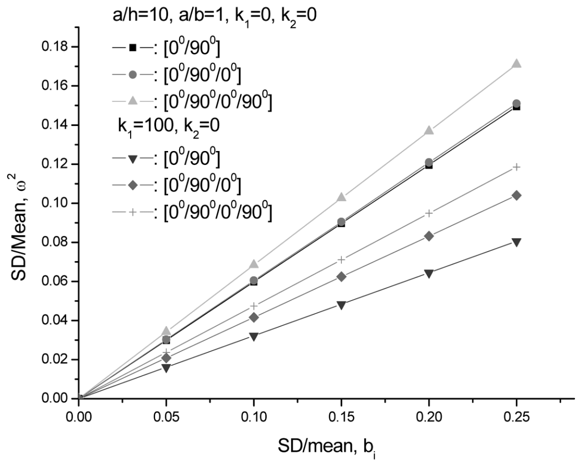
Fig. 6 shows the comparison of variation of SD of the fundamental frequency with the SD of basic RVs ($b_i$, $i$ = 1, 2, ..., 6) changing simultaneously each assuming the same value for the ratio of its SD to mean for 4 layers symmetric [0°/90°/90°/0°] cross-ply laminate with $a/b$ = 1 and 2 for $a/h = 10$. It is observed that the dispersion in the frequency is higher if aspect ratio increases from to $a/b$ = 1 to 2, i.e., the scattering in rectangular plate is greater than that of square plate for all three foundation cases considered, thus indicating that randomness in basic variables has more effect on the sensitivity of the fundamental frequency as the plate aspect ratio increases. In case of Pasternak and Winkler models, the rectangular plate shows relatively very large value of dispersion, while the rectangular plate with no foundation shows almost same sensitivity if compared with the square plate.

The comparison of variation of the SD of the fundamental frequency with all basic material properties simultaneously each assuming the same value for the ratio of its SD to mean for 2-layers anti-symmetric [0°/90°], 3-layers symmetric [0°/90°/0°], and 4-layers anti-symmetric [0°/90°/0°/90°] square laminates with $a/h = 10$ and 100 for no elastic foundation and Winkler foundation case is




( $i = 1, 2, 3, \ldots\ldots\ldots, 6$ )

Fig. 6 Effect of plate aspect ratio (*a/b*) on the dispersion of the fundamental frequency square ($\omega^2$) of simply supported square plates resting on elastic foundations with all random material variables changing simultaneously




( $i = 1, 2, 3, \ldots\ldots\ldots, 6$ )

Fig. 7 Effect of stacking sequence on the dispersion of the fundamental frequency square ($\omega^2$) of a simply supported square plate resting on elastic foundations with all random material variables changing simulaneously

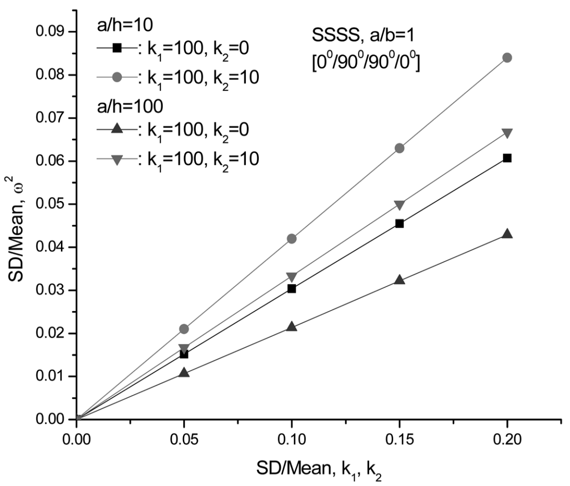



Fig. 8 Effect of randomness of the foundation stiffness parameters on the dispersion of the fundamental frequency square ($\omega^2$) of simply supported square plates, $a/h = 10$ and 100, resting on elastic foundation

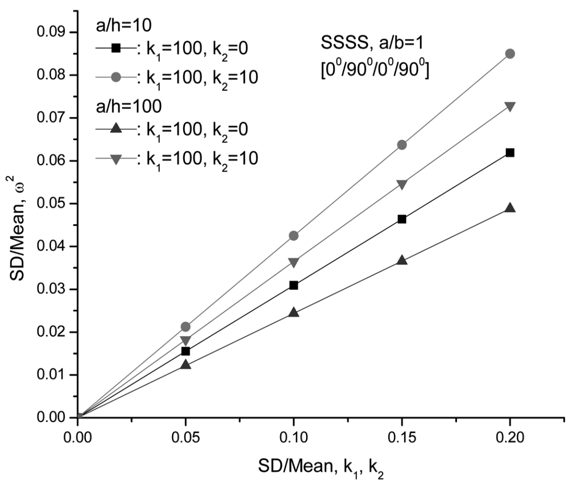



Fig. 9 Effect of randomness of the foundation stiffness parameters on the dispersion of the fundamental frequency square ($\omega^2$) of simply supported square plates, *a/h* 10, $a/b = 1$ and 2, resting on elastic foundation

shown in Fig. 7. It is observed that the dispersion in the frequency increases with number of layer of lay-ups. With reference to 2 layers anti-symmetric case, the [0º/90º/0º] and [0º/90º/0º/90º] laminates give 1 percent and 12.6 percent higher scattering in case of no foundation and 22.5 percent and 32.3 percent higher scattering in case of Winkler foundation.

Fig. 8 examines the influence of scattering in the foundation stiffness parameters $k_1$ and $k_2$ on the fundamental frequency for 4-layer symmetric [0°/90°/90°/0°] cross-ply square laminate with *a/h* = 10 and 100 for two cases of foundation, ($k_1$ = 100, $k_2$ = 0) and ($k_1$ = 100, $k_2$ = 10). The lamina material properties of the constituent materials are kept deterministic. It is observed that the frequency of moderately thick plate is more sensitive as compared to thin plate for each case of the foundation. Out of all, the Pasternak model for *a/h* = 10 is the most sensitive, while Winkler model for *a/h* = 100 is the least sensitive. The scattering in the plate of *a/h* = 10 on Winkler and the plate of *a/h* = 100 on Pasternak is close to each other in comparison to other combinations of the plate and foundation. The Pasternak model shows 38.3 percent and 55.4 percent more dispersion as compared to Winkler model for *a/h* = 10 and 100, respectively.

Fig. 9 examines the influence of scattering in the foundation stiffness parameters, $k_1$ and $k_2$ for 4-layer anti-symmetric [0°/90°/0°/90°] cross-ply square laminate with two cases of foundation, ($k_1$ = 100, $k_2$ = 0) and ($k_1$ = 100, $k_2$ = 10) for *a/h* = 10 and 100. The lamina material properties of the constituent materials are kept deterministic. It is noticed from the figure that the fundamental frequency is more sensitive for the plate of *a/h* = 10 as compared to the plate of *a/h* = 100. The trends are similar to the Fig. 8. However, the Pasternak model shows 37.3 percent and 49.2 percent more dispersion for *a/h* 01 and 100, respectively as compared to Winkler model.

Figs. 10(a) and (b) present the variation of the scattering in the fundamental frequency with simultaneous changes in the foundation stiffness parameters for 4-layer symmetric [0°/90°/90°/0°] and anti-symmetric [0°/90°/0°/90°] cross-ply rectangular (*a/b* = 2) laminates, respectively with two cases of foundation, as explained in Fig. 9 for *a/h* = 10 and 100. It is interesting to note that the moderately thick plate is almost insensitive to the random changes in the foundation stiffness parameters. Similar observations have been noted for the plate having side to thickness ratio less than 10. However, the results for the plate having a/h ratio less than 10 are not presented here. The symmetric cross-ply plate is more sensitive as compared to anti-symmetric cross-ply plate. The sensitivity of the thin plate is the largest, while the sensitivity is the smallest for thick plates and as

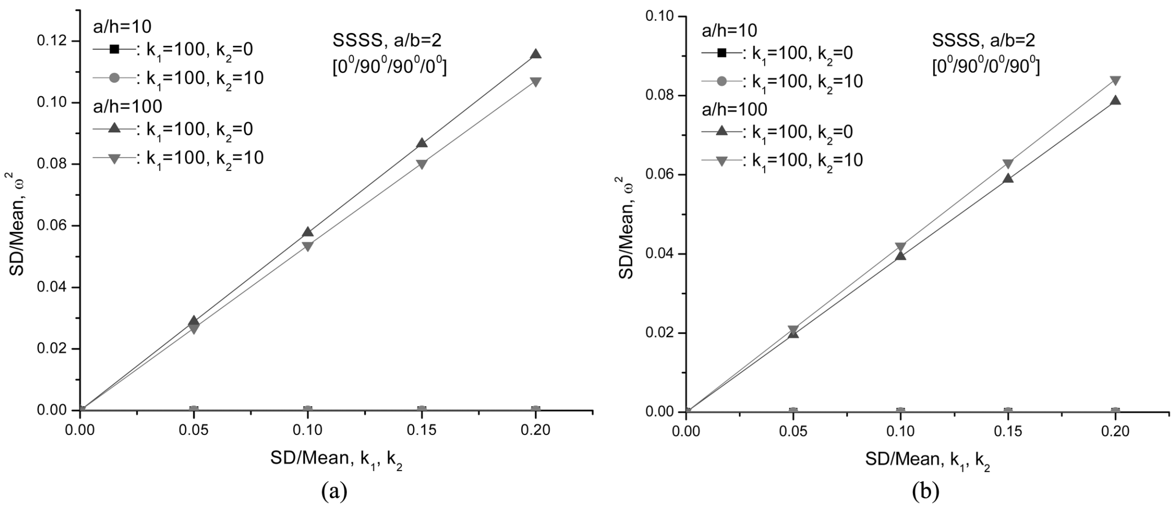



Fig. 10 Effect of randomness of the foundation stiffness parameters on the dispersion of the fundamental frequency square ($\omega^2$) of simply supported rectangular (*a/b* = 2), *a/h* = 10 and 100, resting on elastic foundation (a) [00°/90°/90°/0°], and (b) [0°/90°/0°/90°]

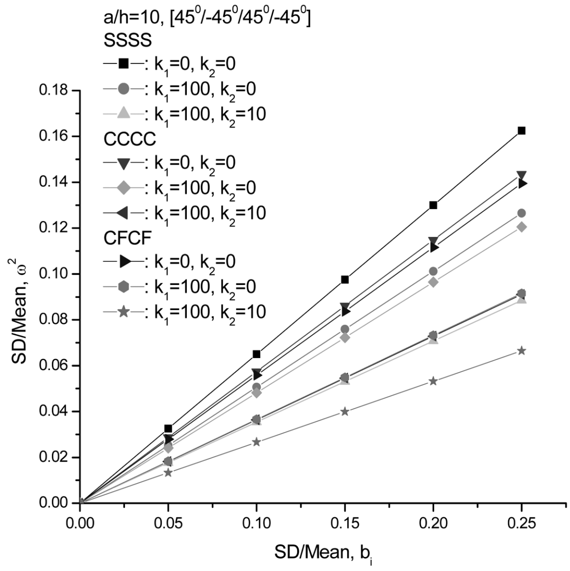



Fig. 11 Effect of support conditions on the dispersion of the fundamental frequency square ($\omega^2$) of laminated composite square plates resting on elastic foundations with all system properties changing simultaneously for $a/h$ = 10

stated earlier it almost negligible.

Fig. 11 shows the effect of CCCC, SSSS and CFCF support conditions on the dispersion of the dimensionless fundamental frequency for lay-up of [45°/–45°/45°/–45°] angle-ply square laminate with all system properties changing simultaneously for $a/h$ = 10. All the inputs RVs ($b_i$, $i$ = 1, 2, ..., 8) are assumed to have same SD to mean ratio. It is observed that the scatter in dimensionless fundamental frequency is of equal order of magnitude for CCCC ($k_1$ = 100, $k_2$ =10) and CFCF (($k_1$ = 0, $k_2$ = 0); CFCF ($k_1$ = 100, $k_2$ = 0) and SSSS ($k_1$ = 100, $k_2$ = 10); CCCC ($k_1$ = 100, $k_2$ = 0) and SSSS ($k_1$ = 100, $k_2$ = 0). The scatter in the fundamental frequency is strongest in case of plate with SSSS ($k_1$ = 0, $k_2$ = 0), while it is lowest for CFCF ($k_1$ = 100, $k_2$ = 10). The plate with Winkler and Pasternak foundation shows 22 percent and 30 percent for SSSS boundary condition, 15.9 percent and 36 percent for CCCC boundary condition and 34.4 percent and about 50 percent for CFCF boundary condition more dispersion as compared to the plate with no elastic foundation.

## 5. Conclusions

A $C^0$ finite element method in conjunction with FOPT has been outlined to obtain the second order statistics of dimensionless fundamental frequency of laminated composite plates that are resting on elastic foundation. A higher order shear deformation theory has been used and different boundary conditions analyzed. The following conclusions can be drawn from this limited study:

(1) The SD of the fundamental frequency shows different sensitivity to different system properties. The sensitivity changes with the lay-up sequence, the plate side to thickness ratio, the plate aspect ratio, the boundary condition, the material properties and the foundation stiffness parameters.

(2) Among the different system properties studied, $E_{11}$ causes the highest scatter in the fundamental frequency, while $\nu_{12}$ the lowest scatter.
(3) Among the different stacking sequences studied, 4 layers anti-symmetric and symmetric cross-ply plates show the highest dispersion in the fundamental frequency for random material properties and foundation stiffness parameters, respectively.
(4) Thick plates show almost negligible sensitivity with random foundation stiffness parameters.
(5) The effect of different foundation cases on the scattering of the dimensionless fundamental frequency is quite significant and comparable with the material properties. The uncertainty in these parameters cannot be ignored in design.

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## Appendix

$$[L] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1\frac{\partial}{\partial x} \\ 0 & 0 & 0 & 0 & 0 & c_1\frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_1\frac{\partial}{\partial x} & c_1\frac{\partial}{\partial y} \\ 0 & 0 & 0 & 0 & -c_4\frac{\partial}{\partial x} & 0 & -c_2\frac{\partial}{\partial x} \\ 0 & 0 & 0 & -c_4\frac{\partial}{\partial y} & 0 & -c_2\frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & -c_4\frac{\partial}{\partial x} & -c_4\frac{\partial}{\partial y} & -c_2\frac{\partial}{\partial x} & -c_2\frac{\partial}{\partial y} \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 & c_1 & 0 \\ 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & -3c_4 & 0 & -3c_2 & 0 \\ 0 & 0 & 0 & 0 & -3c_4 & 0 & -3c_2 \end{bmatrix} \tag{A-1}$$

$$[L_g] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A-2}$$

$$[N] = \begin{bmatrix} 1 & 0 & 0 & 0 & f_2(z) & 0 & f_1(z) \\ 0 & 1 & 0 & f_2(z) & 0 & f_1(z) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A-3}$$

## Notation

| | |
|---|---|
| $A_{ij}$, $B_{ij}$, etc | : Laminate stiffnesses |
| $BB$ | : Strain-displacement matrix |
| $BB_g$ | : Strain-displacement matrix due to foundation |
| $a$, $b$ | : Plate length and breadth |
| $b_i$ | : Basic random system properties |
| $E_{11}$, $E_{22}$ | : Longitudinal and Transverse elastic moduli |
| $G_{12}$, $G_{13}$, $G_{23}$ | : Shear moduli |
| $h$ | : Thickness of the plate |
| $K$ | : Bending stiffness matrix |
| $K_1$, $k_1$ | : Winkler elastic foundation stiffness (normal) and its dimensionless form |
| $K_2$, $k_2$ | : Pasternak elastic foundation stiffness (shear) and its dimensionless form |
| $M$, $m$ | : Mass and inertia matrices |
| $NE$, $N$ | : Number of elements, number of layers in the laminated plate |
| $NN$ | : Number of nodes per element |
| $\varphi_i$ | : Shape function of $i$th node |
| $\overline{Q}_{ij}$ | : Reduced elastic material constants |
| $\Lambda$, $\{\Lambda\}^{(e)}$ | : Vector of unknown displacements, displacement vector of $e$th element |
| $U$, $U_f$ | : Strain energy due to bending and foundation, respectively |
| $u$, $v$, $w$ | : Displacements of a point on the mid plane of plate |
| $\overline{u}$, $\overline{v}$, $\overline{w}$ | : Displacement of a point ($x$, $y$, $z$) |
| $\{\sigma\}$, $\{\varepsilon\}$ | : Stress vector, Strain vector |
| $\psi_x$, $\psi_y$ | : Rotations of normal to mid plane about the $x$ and $y$ axis respectively |
| $\theta_x$, $\theta_y$, $\theta_k$ | : Two slopes and angle of fiber orientation wrt $x$-axis for $k$th layer |
| $x$, $y$, $z$ | : Cartesian coordinates |
| $\rho$, $\lambda$, Var(.) | : Mass density, eigenvalue, variance |
| $\omega$, $\varpi$ | : Natural frequency and its dimensionless form |
| $RVs$ | : Random variables |