DOI: http://dx.doi.org/10.12989/sem.2007.26.6.673

On the use of the Lagrange Multiplier Technique for the unilateral local buckling of point-restrained plates, with application to side-plated concrete beams in structural retrofit

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(Received June 30, 2004, Accepted May 17, 2007)

Abstract. Reinforced concrete beams can be strengthened in a structural retrofit process by attaching steel plates to their sides by bolting. Whilst bolting produces a confident degree of shear connection under conditions of either static or seismic overload, the plates are susceptible to local buckling. The aim of this paper is to investigate the local buckling of unilaterally-restrained plates with point supports in a generic fashion, but with particular emphasis on the provision of the restraints by bolts, and on the geometric configuration of these bolts on the buckling loads. A numerical procedure, which is based on the Rayleigh-Ritz method in conjunction with the technique of Lagrange multipliers, is developed to study the unilateral local buckling of rectangular plates bolted to the concrete with various arrangements of the pattern of bolting. A sufficient number of separable polynomials are used to define the flexural buckling displacements, while the restraint condition is modelled as a tensionless foundation using a penalty function approach to this form of mathematical contact problem. The additional constraint provided by the bolts is also modelled using Lagrange multipliers, providing an efficacious method of numerical analysis. Local buckling coefficients are determined for a range of bolting configurations, and these are compared with those developed elsewhere with simplifying assumptions. The interaction of the actions in bolted plates during buckling is also considered.

Keywords: bolting; contact problem; lagrange multipliers; local buckling; side-plates; unilateral.

1. Introduction

Steel plates may be bolted to the sides of a concrete beam in order to design against seismic

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loading, as well as to strengthen the beam in a structural retrofit procedure (Oehlers 1990). The process of bolting provides a unique but reliable form of shear connection (Nguyen *et al.* 2001), and which allows for the transfer of shear, axial and bending actions into the plates, for which they need to be designed in order to utilise their strengthening and ductility properties. These actions, however, may precipitate local buckling of the steel plate. The local buckling that occurs is of a unilateral nature, in that the buckles may only form away from the concrete to which the steel plate is juxtaposed. This is a special case of a mathematical contact problem that arises in engineering practice, and has been studied extensively by Smith and his colleagues (Smith *et al.* 1999a,b,c, Bradford *et al.* 2000).

The solution of contact problems by the technique of mathematical programming, and a general mathematical procedure for obtaining the best distribution of pressure over the contacting region, were presented by Conry (1970) and Conry and Seireg (1971). Chand *et al.* (1976) developed a compatible solution technique with the finite element method to handle contact problems with complex boundary configurations, although the efficiency of finite element approaches to the problem has been questioned recently by Smith *et al.* (1999d). A numerical procedure for large transverse deflections and a buckling analysis of clamped and simply supported plates constrained by the presence of a rigid plane, based on a finite element approximation, was proposed by Ohtake *et al.* (1980a,b). Fischer and Melosh (1987) introduced a new variant of the well-known Simplex algorithm to distinguish between contact and non-contact points for elastic contact problems.

The effect of large deformations on the behaviour of circular plates on unilateral Winkler-type foundations using the finite difference method was investigated by Khathlan (1994). Shahwan and Wass (1994) presented a mathematical model for the buckling of unilaterally constrained, finite, rectangular plates. In their analysis, the influence of different boundary conditions, material orthotropy and the distribution of transverse load were investigated, and the weak form of the governing differential equation was solved using the Galerkin method. A perturbation technique was presented to determine the interactive buckling loads and post-buckling equilibrium paths of composite laminated plates on two-parametric elastic foundations (Shen and Williams 1997).

Oehlers *et al.* (1994) developed simple design procedures to prevent the local buckling of the steel decking in a composite profiled trough girder before the ultimate strength of the beam is reached, while Uy (2001) developed a counterpart rule for the design of concrete-filled steel tubes against unilateral buckling of the steel plating. These studies showed that since the steel plate is constrained to buckle only outward from the concrete, to which it is juxtaposed, this mode of the buckling permits increases in the width of the plate of up to 70%. Wright (1995) also studied the local buckling of plates with various boundary conditions, including the effect of their being juxtaposed with a rigid medium such as concrete. The derivation of limiting width to thickness ratios for this class of plates was presented in Wright's study, and it was shown that these limiting ratios were much higher than their counterparts for plates not in contact with a rigid medium. Recent work by Bradford *et al.* (2000) has quantified the section classification limits for various classes of compactness for plates that may buckle only in a unilateral mode. The problem of finding the buckling loads of unilaterally constrained plates was considered by Shahwan and Wass (1998). In their analysis, the condition of contact at buckling was resolved by modelling the constrained plate as having two distinct regions, viz. a contacted and an uncontacted region. They showed that owing to the constraint on the deformation being 'one-sided', an increase in the local buckling stress of approximately 30% over the unconstrained situation was obtained.

More recently, Smith *et al.* (1999a,b) studied the buckling of side-plated reinforced concrete beams, both experimentally and numerically. In their numerical study, they presented a Rayleigh-Ritz formulation for the local buckling analysis of rectangular unilaterally restrained plates in compression, bending and shear. In their experiments, a series of tests was used to investigate the local buckling of mild steel plates bolted to the sides of reinforced concrete beams, and to validate the results of their numerical treatment.

Although not quantified hitherto, the number and configuration of the bolts on a plate bolted to the sides of a concrete beam will affect the boundary conditions that have been assumed in all previous studies as of an idealised type, and so will affect the buckling response of the plate. In order to circumvent the idealisations that need to be made to this type of unilateral buckling, this paper uses the Lagrange Multiplier technique, which has enjoyed considerable success in the treatment of constrained minimisation problems, to consider the constraint provided by the bolts on the local buckling of side plates. This is done in a generic fashion for elastic unilateral plate buckling with discrete restraint points, but with the application to side plating for retrofit being borne in mind. Both the unilateral and bilateral local buckling coefficients of side plates with different types of bolting configurations are determined. The practical application of the theory is that it forms a basis for allowing a more accurate choice, in lieu of the approximate guidelines developed with simplifying assumptions for the boundary conditions, for a suitable and optimal bolt spacing and arrangement in order to avoid premature buckling of the plate.

2. Theory

2.1 General

The well-known Rayleigh-Ritz method for solving elastic unilateral local buckling problems for plates under combined loading has been set out by Smith *et al.* (1999c). In this section, the relevant augmentation of this method to include the discrete point constraints imposed by the bolts is developed. Fig. 1 shows the geometry of a unilaterally restrained steel plate that is bolted to the side of a concrete beam. In this figure, the basic state of stress at the onset of local buckling is shown, and consists of a uniform shearing load N_{xy} and a longitudinal compressive load of average value N_x that varies linearly across the width of the plate from $(N_x - N_o)$ at the edge $y = 0$ to $(N_x + N_o)$ at the edge $y = b$. The axial force $\pm N_o$ is therefore that caused by the longitudinal in-plane bending action, and the prebuckling actions can be determined using the partial interaction modelling of Nguyen *et al.* (2001).

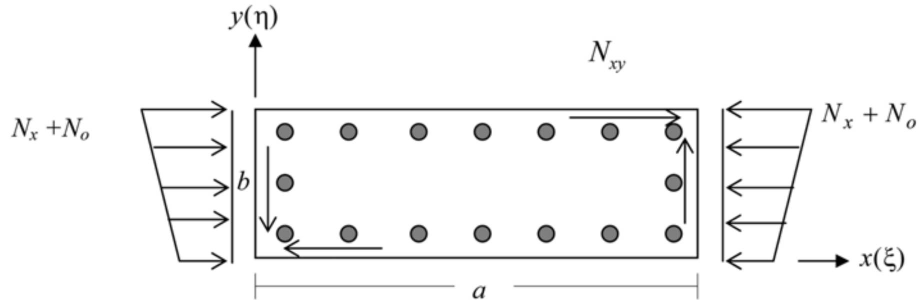

Fig. 1 Side-plated and bolt layout

2.2 Buckling deformations

In the Rayleigh-Ritz formulation, the out-of-plane displacement field w can be assumed to be of separable form, where shape functions in $x(\xi)$ are multiplied by those in $y(\eta)$, where $\xi = x/a$ and $\eta = y/b$, with $\xi \in [0, 1]$ and $\eta \in [0, 1]$. The flexural buckling displacements $w(\xi, \eta)$ are chosen to be polynomial-based, consisting of complete two-dimensional polynomials. This formulation has been previously used with success for modelling bilateral buckling and plate vibration problems (Wang and Liew 1994, Liew and Wang 1995), and has the form

$$w(\xi, \eta) = \sum_{i=0}^{m} \sum_{j=0}^{n} \alpha_{ij} \xi^i \eta^j \tag{1}$$

where α_{ij} are Ritz coefficients with the dimensions of length. Eq. (1) may be written conveniently in matrix form as

$$w(\xi, \eta) = \langle 1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \ldots \rangle \{\alpha\} = \vec{P}^T \vec{\alpha} \tag{2}$$

In the present study for the buckling analysis, instead of using the usual boundary conditions such as free, clamped and simply supported at the plate edges as described by Smith *et al.* (1999c), the restraints that are provided by a particular configuration of bolting are considered. This is achieved by use of the technique of Lagrange Multipliers, as is developed subsequently.

Strain energy stored during buckling

The elastic strain energy of a plate during buckling U_P is that due to flexural buckling in the normal direction, and is given by Timoshenko and Gere (1961)

$$U_P = \frac{D}{2} \int_A \{ (w_{,xx} + w_{,yy})^2 - 2(1 - \nu)(w_{,xx} w_{,yy} - w_{,xy}^2) \} \mathrm{d}A \tag{3}$$

where A is the area of the plate, commas denote partial differentiation, and the plate flexural rigidity is

$$D = \frac{Et^3}{12(1 - \nu^2)} \tag{4}$$

in which E is Young's modulus, t is the plate thickness and ν is Poisson's ratio.

In order to account for the condition of unilateral restraint imposed on the flexural buckling deformations of the plate because of the concrete beam, a nonlinear elastic foundation model that exhibits a sign-dependent relationship is used (Shawan and Wass 1994, Smith *et al.* 1998, 1999a). The numerical attributes of this method for solving this form of contact problem in terms of convergence and accuracy have been discussed previously by Smith *et al.* (1999d). The continuous (infinitesimal) pressure distribution imposed by the concrete surface is approximated by a set of forces acting at discrete points, defined *a priori* by discretising the plate into a sufficiently fine grid of 'elements' of dimension $\Delta x \times \Delta y$. To achieve this, a sufficiently large spring stiffness k_f is adopted to simulate the rigid concrete surface. By invoking a linear elastic relationship for the spring, since both the plate and the highly stiff juxtaposed restraining medium obey the laws of linear elasticity, the additional change in the strain energy of the system due to the deformation of the springs U_S

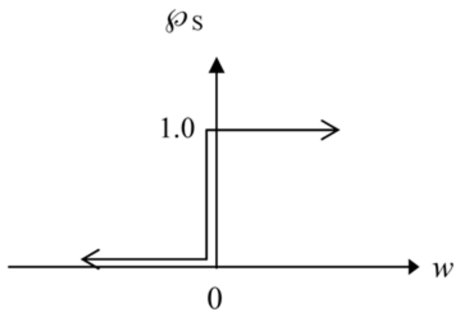

Fig. 2 Contact status or penalty function $\wp_S$

may be written as

$$U_S = \frac{1}{2}\sum_{i=1}^{p} \wp_S \bar{k}[w_i(\xi_g, \eta_g)]^2 \tag{5}$$

where $\bar{k} = k_f \Delta x \times \Delta y$, $w_i(\xi_g, \eta_g)$ represents a grid point (of which there are p in total) at which a spring is defined, and $\wp_S$ is a penalty function that represents the contact status, and is chosen to define the unilateral constraint of prohibiting the foundation to have tensile normal stresses, and of prohibiting the penetration of any part of the plate into the concrete medium. The relationship between the flexural displacement w and the contact status $\wp_S$ with respect to a rigid support to satisfy compatibility is depicted in Fig. 2. Its mathematical representation is simply

$$\wp_S = \begin{cases} 0 & w \in [0, \infty) \\ 1 & w \in (-\infty, 0) \end{cases} \tag{6}$$

which has the physical interpretation for the present problem (Baniotopoulos *et al.* 1994, Smith *et al.* 1999a,c)

$$\wp_S = \begin{cases} 0 & \text{(separation)} \\ 1 & \text{(contact)} \end{cases} \tag{7}$$

2.3 Change of total potential energy

The unilaterally restrained steel plate that is subjected to the initial in-plane forces that consist of longitudinal and shear forces (Fig. 1) experiences a decrease in the total potential energy of these forces during buckling. This effect is documented elsewhere (Azhari and Bradford 1993), and the potential decrease V_P may be written as

$$V_P = \frac{1}{2}\int_A \{[N_x + (2\eta - 1)N_o]w_{,x}^2 + 2N_{xy}w_{,x}w_{,y}\}\,\mathrm{d}A \tag{8}$$

Using Eqs. (3), (5) and (8), the total change in potential of the system during buckling in the absence of any constraints provided by the bolts can be expressed as

$$\Pi = U_P + U_S - V_P \tag{9}$$

Note that in formulating Eq. (9) for various conditions of idealised edge support, Smith *et al.* (1999a,c) made use of boundary polynomials in what is referred to as the *pb*-2 Rayleigh-Ritz treatment of the problem. Instead of the use of boundary polynomials, the Lagrange multiplier formulation for restraints at the discrete bolt positions will be used, as is developed in the following.

2.4 Lagrange multiplier technique

A convenient device for modelling the effect of the discrete point restraints that are provided by the bolts is the Lagrange Multiplier technique (Timoshenko and Gere 1961). Herein, the n_B bolt restraints are assumed to provide restraint conditions of the form

$$w(\xi_i, \eta_i) = 0 \qquad i = 1, 2, \ldots, n_B \tag{10}$$

where $w(\xi_i, \eta_i)$ are the buckling displacements at the bolt positions that are assumed to vanish. These n_B conditions may be expressed in terms of the Ritz coefficients $\vec{\alpha}$ in the form

$$\Gamma_k = \vec{P}_k^T \vec{\alpha} \qquad k = 1, 2, \ldots, n_B \tag{11}$$

in which $\vec{P}_k$ is the vectorial representation of the two-dimensional separable polynomial $\xi^i \eta^j$ at the location of the k-th discrete bolt restraint.

In accordance with the Lagrange Multiplier technique, a new functional Π^β, developed by multiplying each of the (null) constraints in Eqs. (10) and (11) by an appropriate Lagrange multiplier β_k, is used to augment the total change in potential Π in Eq. (9) caused by the elastic and geometric strain energies. This general form of the new functional may be stated as

$$\Pi^\beta = \sum_{k=1}^{n_B} \beta_k \Gamma_k \tag{12}$$

where the term on the right hand side of Eq. (12) is the sum of the product of the constraints and their associated Lagrange multipliers, and n_B is the number of bolts. Of course, in Lagrange multiplier methodology, Eq. (12) vanishes, and it is worth noting further that the physical significance of the Lagrange multipliers is that they represent the (infinitesimal) forces induced in the bolts during buckling.

An augmented functional, Π^a, may be constructed by the addition of the original energy functional Π and the augmenting functional Π^β, so that

$$\Pi^a = \Pi + \sum_{k=1}^{n_B} \beta_k \Gamma_k \tag{13}$$

By substituting Eq. (1) into Eqs. (3), (5) and (8), and using Eqs. (9) and (13), the augmented energy functional Π^a can be expressed in terms of the arbitrary Ritz coefficients $\vec{\alpha}$ and the Lagrange multipliers β_k. In formulating the condition of bifurcation of equilibrium in a variational form (since w represents the buckled shape and not the pre-buckling primary equilibrium path), a necessary and sufficient condition for attaining neutral equilibrium (and thus determining the buckling load) is that the functional Π^a is stationary. This condition of the stationary energy functional must be with respect to any perturbation of either $\vec{\alpha}$ or β_k, whose mathematical statement is

$$(N_{xy}, N_x, N_o)_{\text{at buckling}} \Leftrightarrow (\delta\Pi^a = 0 \;\forall\; \delta\vec{\alpha}, \delta\beta_k) \tag{14}$$

This produces i equations of the type

$$\left[\frac{\partial\Pi}{\partial\alpha_i} + \sum_k (\beta_k \vec{P}_k)_i\right]\delta\alpha_i = 0 \tag{15}$$

and a further n_B equations of the type

$$(\Gamma_k)\delta\beta_k = 0 \tag{16}$$

By noting that the variations $\delta\alpha_i$ and $\delta\beta_k$ are arbitrary, differentiating with respect to $\vec{\alpha}$ and manipulating the expression algebraically, Eqs. (15) and (16) may be written compactly in matrix form as

$$(\overline{K}_P + \overline{K}_S - \overline{K}_G)\begin{Bmatrix} \vec{\alpha} \\ \vec{\beta} \end{Bmatrix} = \vec{0} \tag{17}$$

in which $[K_P]$, $[K_S]$ and $[K_G]$ are the stiffness matrix of the plate, the stiffness matrix of the rigid foundation (and which is nonlinear in the buckling deformations owing to the nonlinearity of the contact penalty function $\wp$), and the stability matrix of the plate, respectively. It should be noted that for consistency of the dimension of the stiffness matrices $(p + n_B) \times (p + n_B)$, the matrices $[K_P]$ and $[K_S]$ should have null sub-matrices in the n_B rows and columns corresponding to the Lagrange multipliers, while the matrix $[K_P]$ is null, except for the n_B rows and columns corresponding to the Lagrange multipliers β_k. Details of the entries in the stiffness matrices in Eq. (17) are given in Hedayati (2001). The buckling load factor, as well as the buckling mode shape $\vec{\alpha}$ and normalised bolt forces $\vec{\beta}$, may be extracted from Eq. (17) using a standard eigensolver (such as the power method based routine of Swartz and O'Neill 1995), in conjunction with the iterative procedure outlined by Smith *et al.* (1999a,c) that makes recourse to Aitken's Δ^2 process to hasten the convergence.

3. Numerical results

3.1 General

The numerical procedure based on the Rayleigh-Ritz method and the Lagrange Multiplier technique described in the previous section has been implemented in FORTRAN computer code, and applied to study the elastic buckling of side plates of various geometrical configurations. The side-plated reinforced concrete beam and bolting layout with the applied loading is shown in Fig. 1. In order to present the results in a dimensionless format, they are given as local buckling coefficients that are defined for axial action, bending and shear respectively as (Allen and Bulson 1980)

$$k_a = \frac{N_x b}{\pi^2 D}; \quad k_b = \frac{N_0 b}{\pi^2 D}; \quad k_s = \frac{N_{xy} b}{\pi^2 D} \tag{18}$$

The sides of the plate are assumed to be free. In the following, the numbers and arrangement of the bolts that are needed to simulate the plate boundary conditions as being either simply supported or clamped, in the normal structural idealisation of the boundary conditions, are considered.

3.2 Conventional boundary conditions for side-plated beams

Smith *et al.* (1999c) studied the local buckling of side plates with the idealised boundary conditions of SSSS, CSCF and CFCF, in which F denotes free, S denotes simply supported and C denotes clamped. In order to demonstrate the validity and accuracy of the numerical procedure, in addition to the boundary conditions used by Smith *et al.* (1999c), the local buckling coefficients for plates with different boundary conditions have been determined.

Table 1 shows a summary of the axial compressive (k_a), pure bending (k_b) and pure shear (k_s) local buckling coefficients for each plate type that are required to cause bifurcation, for both the bilateral case (subscripted b) and the unilateral case (subscripted u). It should be noted that the term bilateral indicates that the local buckling is free to occur in both directions, in deference to the unilateral mode for which the plate may only buckle away from the rigid restraint. Over the range of plate aspect ratios $\gamma = b/a$ for plates with SSSS, CSCF and CFCF boundary conditions, the results computed by the method herein agree almost exactly with those reported by Smith *et al.* (1999c).

Table 1 Bilateral and unilateral buckling coefficients for pure compression, bending, and shear

Boundary conditions	$\gamma = a/b$	Axial compression		Bending		Shear	
		$k_{a,b}$	$k_{a,u}$	$k_{b,b}$	$k_{b,u}$	$k_{s,b}$	$k_{s,u}$
SSSS	1.0	4.00	4.00	25.53	26.53	9.32	9.33
	2.0	4.00	4.50	23.88	26.27	6.54	6.68
	3.0	4.00	4.50	24.13	26.59	5.84	6.68
CCCC	1.0	10.08	10.08	47.75	47.78	14.64	14.78
	2.0	7.97	9.78	41.95	42.16	10.25	11.85
	3.0	7.43	9.70	41.70	42.50	9.54	11.85
CSCS	1.0	6.74	6.74	32.00	32.00	12.56	12.70
	2.0	4.85	5.32	26.09	26.20	6.71	6.83
	3.0	4.41	5.28	25.01	25.13	5.93	6.62
CCCF	1.0	4.58	4.58	7.26	7.26	8.50	8.66
	2.0	1.92	1.92	3.19	3.19	2.76	2.80
	3.0	1.68	1.86	2.81	2.90	2.07	2.11
CSCF	1.0	4.37	4.37	7.24	7.24	8.43	8.65
	2.0	1.41	1.41	2.69	2.69	2.35	2.36
	3.0	0.86	0.86	1.70	1.70	1.35	1.36
CFCF	1.0	3.92	3.92	7.24	7.24	7.49	8.70
	2.0	0.97	0.97	2.68	2.68	1.77	1.78
	3.0	0.43	0.43	1.62	1.62	0.69	0.70

3.3 Discrete restraint points for bolted side plates

The advantage of the method developed herein over that presented by Smith *et al.* (1999a,c) is that it can handle the constraints imposed by the bolts, at discrete point locations, on the boundary conditions for the plate, which in turn influence the local buckling coefficients. The local buckling of plates with free edge boundary conditions, but with bolts consisting of several configurations in both arrangement and bolt number have been investigated. Table 2 shows the local buckling coefficients of such plates with different aspect ratios subjected to axial compression, pure bending and pure shear. The results shown in Table 2 that have been obtained from the Lagrange Multiplier technique are discussed summarily in the following, with reference to the various configurations illustrated in Fig. 3.

It was reported by Smith *et al.* (1999c) that the local buckling coefficient for a simply supported square side plate under axial compression, for both the bilateral and unilateral constraint cases, is 4.0. For a square side plate whose edge conditions are FFFF, the use of 8 bolts (Fig. 3a) gives local buckling coefficients of $(k_a)_b$ = $(k_a)_u$ = 3.52, while the use of 12 bolts (Fig. 3b) or 16 bolts (Fig. 3c) increases these values to $(k_a)_b$ = $(k_a)_u$ = 3.95 and $(k_a)_b$ = $(k_a)_u$ = 3.99 respectively. This means that using 8, 12 and 16 bolts on the side plate replicates a condition of simple support to an accuracy

Table 2 Local buckling coefficients of plates bolted to the sides of beams

Bolting arrangement	$\gamma = a/b$	Axial compression		Bending		Shear	
		$k_{a,b}$	$k_{a,u}$	$k_{b,b}$	$k_{b,u}$	$k_{s,b}$	$k_{s,u}$
Fig. 3(a)	1.0	3.52	3.52	6.91	6.92	4.00	4.79
	2.0	0.95	1.428	2.55	2.60	1.87	2.12
	3.0	0.41	0.68	1.55	1.73	1.28	1.42
Fig. 3(b)	1.0	3.95	3.95	13.48	13.48	7.01	7.09
	2.0	2.17	3.89	4.53	4.54	3.16	3.29
	3.0	0.95	2.64	2.59	2.78	1.78	2.32
Fig. 3(c)	1.0	3.99	3.99	22.14	22.14	8.51	8.52
	2.0	3.75	4.58	7.04	7.06	5.08	5.45
	3.0	1.70	2.30	3.85	3.88	2.64	3.99
Fig. 3(d)	1.0	2.96	3.06	5.61	5.84	5.41	7.43
	2.0	0.98	0.98	2.68	2.68	1.91	1.94
	3.0	0.46	0.46	1.70	1.70	0.78	0.78
Fig. 3(e)	1.0	3.90	3.91	6.83	6.89	8.15	8.69
	2.0	1.08	1.08	2.86	2.86	2.02	2.04
	3.0	0.48	0.48	1.75	1.75	0.81	0.81
Fig. 3(f)	1.0	6.44	8.64	10.59	10.60	6.97	9.46
	2.0	2.02	2.02	4.34	4.35	4.20	5.89
	3.0	0.95	1.81	2.62	4.07	2.67	3.20
Fig. 3(g)	1.0	9.998	10.47	18.97	29.02	14.27	15.14
	2.0	3.59	3.72	6.65	8.21	6.01	10.57
	3.0	1.67	3.37	3.73	3.85	3.76	5.98

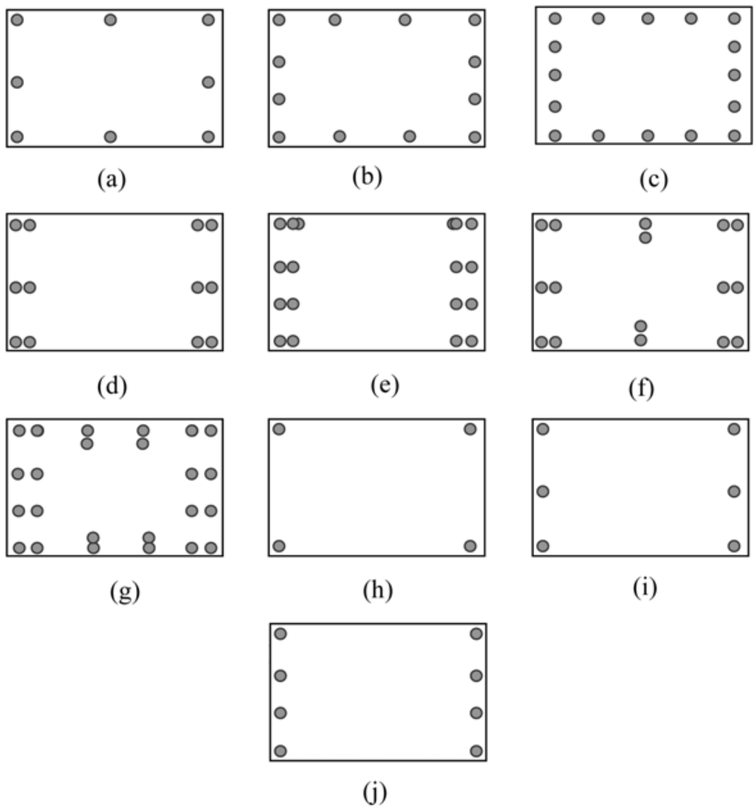

Fig. 3 Bolting configurations

(on the unconservative side) of 12%, 1.25% and 0.25% respectively.

If groups of paired bolts are used, as in Figs. 3(d) to 3(g), the free boundary condition may tend towards a clamped boundary condition. When the side plate is bolted with the configuration shown in Figs. 3(d) and 3(e), the boundary conditions of the plate move from a condition of FFFF to CFCF. It can be seen from Table 1 that for a plate with idealised restraints of CFCF at its edges, then $(k_a)_u$ = 3.92, $(k_b)_u$ = 7.24 and $(k_s)_u$ = 8.70, which are the same as the values quoted by Smith *et al.* (1999c). From Table 2, the use of 12 bolts in the configuration depicted in Fig. 3(d) results in local buckling coefficients of $(k_a)_u$ = 3.06, $(k_b)_u$ = 5.84 and $(k_s)_u$ = 7.43, while using 16 bolts in the configuration shown in Fig. 3(e) results in local buckling coefficients of $(k_a)_u$ = 3.91, $(k_b)_u$ = 6.89 and $(k_s)_u$ = 8.69 which are very close to the idealised CFCF local buckling coefficients.

The use of the bolting configurations shown in Figs. 3(f) and 3(g) may be suitable to replicate a condition of CCCC for the plate edges. The local buckling coefficients of a square plate with four edges free and that is bolted with 24 bolts in the configuration of Fig. 3(g) are $(k_a)_b$ = 9.998, $(k_a)_u$ = 10.47, $(k_b)_b$ = 18.97, $(k_b)_u$ = 29.02, $(k_s)_b$ = 14.27 and $(k_s)_u$ = 15.14. Allen and Bulson (1980) quote the local buckling coefficient for a square plate under compression without the restraint of a rigid medium (bilateral buckling) as 10.08, to which the value obtained with 24 bolts is within an accuracy of 0.8%, while Bradford (1998) has derived the local buckling coefficient for the same plate restrained by a rigid medium (unilateral buckling) to be 10.67, with which the 24 bolt solution agrees to within an accuracy of 1.9%. It is important to note that the arrangement and number of bolts in the configuration that may be required to further restrain the plate's boundary and so increase the local buckling coefficients is dependent on the aspect ratio of the side plate.

3.4 Buckling interaction analysis

In the deployment of side plates in retrofit, the shear connection may transfer axial, bending and shear actions simultaneously, and so their interaction on the values at buckling is important. The unilateral local buckling behaviour with different bolting configurations under combined shear and compression and under combined shear and bending has been investigated. Figs. 4 and 5 show interaction curves for the unilateral local buckling coefficients for square plates under combined shear and compression, while Figs. 6 and 7 show these counterparts for various bolting

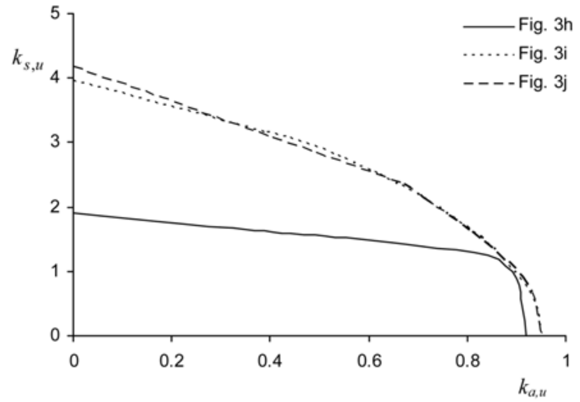

Fig. 4 Interaction curves for unilateral buckling of square side plates in shear and compression for bolting configurations shown in Figs. 3(h), 3(i) and 3(j)

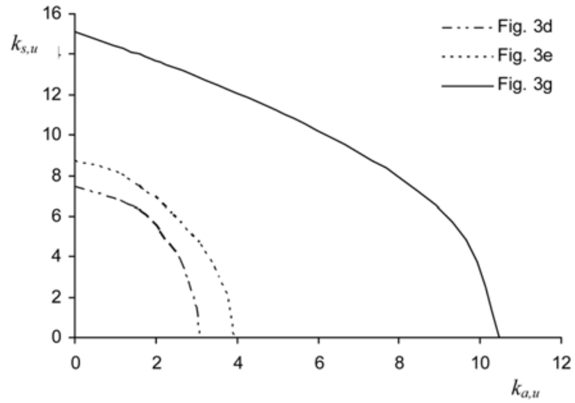

Fig. 5 Interaction curves for unilateral buckling of square side plates in shear and compression for bolting configurations shown in Figs. 3(d), 3(e) and 3(g)

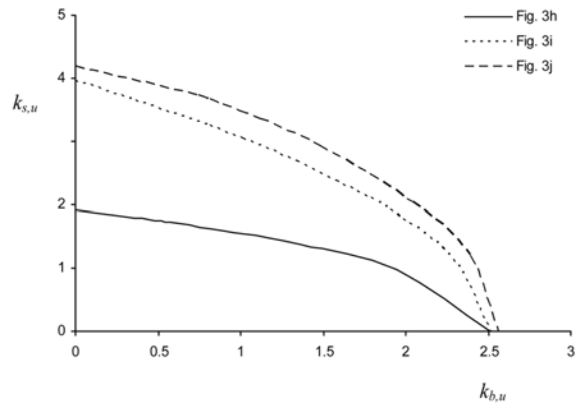

Fig. 6 Interaction curves for unilateral buckling of square side plates in shear and bending for bolting configurations shown in Figs. 3(h), 3(i) and 3(j)

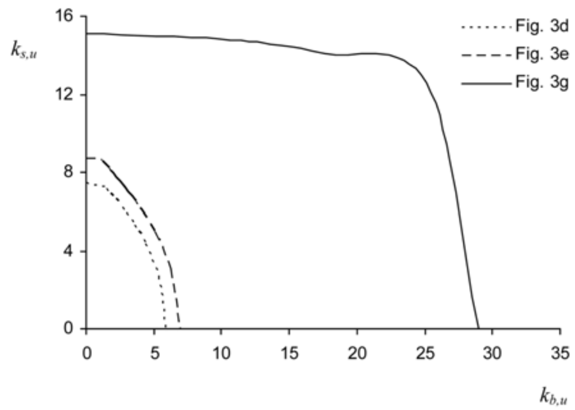

Fig. 7 Interaction curves for unilateral buckling of square side plates in shear and bending for bolting configurations shown in Figs. 3(d), 3(e) and 3(g)

configurations. The interaction curves were determined by fixing the ratio between the compression and shear actions (in Figs. 4 and 5) and between the bending and shear (in Figs. 6 and 7), and factoring these monotonically in a proportional loading regime by the buckling load factor in the numerical analysis. The solution procedure requires that the axial and bending stability matrices ($[K_G]$ in Eq. (17)) to be subtracted from the stiffness matrix ($[K_S]$ in eqn. (17)), and the solution to be obtained based on the shear stability matrix only. It can be seen from Figs. 4 to 7 that the interaction curves vary between being parabolic and circular, and in most cases are close to being parabolic.

4. Conclusions

A method of solution for the local instability of plates juxtaposed with a rigid medium and with discrete point supports has been developed, based on the Rayleigh-Ritz method in conjunction with a Lagrange Multiplier technique. A framework has been developed within this generic modelling whereby the local buckling coefficients for plates that are bolted to the sides of beams in a strength or seismic retrofit can be determined. The method is computationally efficient, and agrees with independent solutions for specific restraint cases. The technique has been used to determine the unilateral and bilateral local buckling coefficients for plates under shear, axial and bending actions for a variety of bolting configurations, as well as their interaction at buckling. The local buckling coefficients are strongly dependent on the regime of the point restraints, and indeed several bolts may be needed to replicate the simplified boundary conditions for which the local buckling coefficients have been reported by other researchers. Nevertheless, the paper sheds quantitative light on the effect of the configuration of bolts for side plates on the elastic local buckling coefficients.

References

Allen, H.B. and Bulson, P.S. (1980), *Background to Buckling*, McGraw-Hill, London.

Azhari, M. and Bradford, M.A. (1993), "Inelastic local buckling of plates with and without residual stresses", *Eng. Struct.*, **15**(1), 31-39.

Baniotopoulos, C.C., Abdalla, K.M. and Panagiotopoulos, P.D. (1994), "A variational inequality and duadratic programming approach to the separation problem of steel bolted bracket", *Comput. Struct.*, **53**(4), 983-991.

Bradford, M.A. (1998), "Service-load deformations of concrete-filled RHS columns", *Eighth Int. Symposium on Tubular Structures*, Singapore (Y.S. Choo *et al.* eds.), Balkema, 425-423.

Bradford, M.A., Smith, S.T. and Oehlers, D.J. (2000), "Semi-compact steel plates with unilateral restraint subjected to bending, compression and shear", *J. Constr. Steel Res.*, **56**(1), 47-67.

Chand, R., Huang, E.J. and Rimm, K. (1976), "Analysis of unbonded contact problems by means of quadratic programming", *J. Optimiz. Theory App.,* **20**, 171-189.

Conry, T.F. (1970), "The use of mathematical programming in design of uniform distribution in nonlinear elastic system", *PhD Thesis*, The University of Wisconsin, WI, USA.

Conry, T.F. and Seireg, A. (1971), "A mathematical programming method for design of elastic bodies in contact", *J. Appl. Mech.*, **2**, 387-392.

Fischer, U. and Melosh, R.J. (1987), "Solving discretized contact problems using linear programming", *Comput. Struct.*, **25**, 661-664.

Hedayati, P. (2001), Unilateral Local Buckling of Steel Bolted Plates, MSc Thesis, Department of Civil Engineering, Isfahan University of Technology, Isfahan, Iran.

Khathlan, A.A. (1994), "Large deformation analysis of plates on unilateral elastic foundation", *J. Eng. Mech., Am. Soc. Civil Eng.*, **120**(8), 1820-1827.

Liew, K.M. and Wang, C.M. (1995), "Elastic buckling of regular polygonal plates", *Thin Wall. Struct.,* **21**, 163-173.

Nguyen, N.T., Oehlers, D.J. and Bradford, M.A. (2001), "An analytical model for reinforced concrete beams with bolted side plates accounting for longitudinal and transverse partial interaction", *Int. J. Solids Struct.,* **38**(38-39), 6997-7015.

Oehlers, D.J. (1990), "Strengthening reinforced concrete beams by bonding steel plates to their soffits", *Struct. Eng. Conf.*, Adelaide, I.E. Australia, 346-350.

Oehlers, D.J., Wright, H.D. and Burnet, M.J. (1994), "Flexural strength of profiled beams", *J. Struct. Eng., Am. Soc. Civil Eng.*, **120**(2), 378-393.

Ohtake, K., Oden, J.T. and Kikuchi, N. (1980a), "Analysis of certain unilateral problems in von Karman plate theory by a penalty method. Part 1: A variational principle with penalty", *Comput. Meth. Appl. Mech. Eng.*, **24**, 187-213.

Ohtake, K., Oden, J.T. and Kikuchi, N. (1980b), "Analysis of certain unilateral problems in von Karman plate theory by a penalty method. Part 2: Approximation and numerical analysis", *Comput. Meth. Appl. Mech. Eng.*, **24**, 317-337.

Shahwan, K.W. and Wass, A.M. (1994), "A mechanical model for the buckling of unilaterally constrained rectangular plates", *Int. J. Solids Struct.*, **31**(1), 75-78.

Shahwan, K.W. and Wass, A.M. (1998), "Buckling of unilaterally constrained infinite plates", *J. Eng. Mech., Am. Soc. Civil Eng.*, **124**(2), 127-136.

Shen, H. and Williams, F.W. (1997), "Biaxial buckling and post-buckling of composite laminated plates on two-parameter elastic foundations", *Comput. Struct.*, **63**(6), 1177-1185.

Smith, S.T., Bradford, M.A. and Oehlers, D.J. (1999a), "Elastic buckling of unilaterally constrained plates in pure shear", *Eng. Struct.*, **21**, 443-453.

Smith, S.T., Bradford, M.A. and Oehlers, D.J. (1999b), "Local buckling of side-plated reinforced concrete beams. II: Experimental study", *J. Struct. Eng., Am. Soc. Civil Eng.*, **125**(6), 635-643.

Smith, S.T., Oehlers, D.J. and Bradford, M.A. (1999c), "Local buckling of side-plated reinforced concrete beams. I: Theoretical study", *J. Struct. Eng., Am. Soc. Civil Eng.,* **125**(6), 622-634.

Smith, S.T., Bradford, M.A. and Oehlers, D.J. (1999d), "Numerical convergence of simple and orthogonal polynomials for the unilateral plate buckling problem using the Rayleigh-Ritz method", *Int. J. Numer. Meth. Eng.*, **44**, 1685-1707.

Swartz, S.E. and O'Neill, R.J. (1995), "Linear elastic buckling of plates subjected to combined loads", *Thin Wall. Struct.*, **21**, 1-15.

Timoshenko, S.P. and Gere, J.M. (1961), *Theory of Elastic Stability*, McGraw-Hill, New York.

Uy, B. (2001), "Local and post local buckling of fabricated steel and composite cross-sections", *J. Struct. Eng., Am. Soc. Civil Eng.*, **127**(6), 666-677.

Wang, C.M. and Liew, K.M. (1994), "Buckling of triangular plates under uniform compression", *Eng. Struct.*, **16**(1), 43-50.

Wright, H.D. (1995), "Local stability of filled and encased steel sections", *J. Struct. Eng., Am. Soc. Civil Eng.*, **121**(10), 1382-1388.