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Bilinear plate bending element for thin and moderately thick plates using Integrated Force Method

H. R. Dhananjaya†

Department of Civil Engineering, Manipal Institute of Technology, Manipal - 576 104, India

J. Nagabhushanam‡

Department of Aerospace Engineering, Indian Institute of Science, Bangalore - 560 012, India

P. C. Pandey‡†

Department of Civil Engineering, Indian Institute of Science, Bangalore - 560 012, India

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Abstract. Using the Mindlin-Reissner plate theory, many quadrilateral plate bending elements have been developed so far to analyze thin and moderately thick plate problems via displacement based finite element method. Here new formulation has been made to analyze thin and moderately thick plate problems using force based finite element method called Integrated Force Method (IFM). The IFM is a novel matrix formulation developed in recent years for analyzing civil, mechanical and aerospace engineering structures. In this method all independent/internal forces are treated as unknown variables which are calculated by simultaneously imposing equations of equilibrium and compatibility conditions. In this paper the force based new bilinear quadrilateral plate bending element (MQP4) is proposed to analyze the thin and moderately thick plate bending problems using Integrated Force Method. The Mindlin-Reissner plate theory has been used in the formulation of this element which accounts the effect of shear deformation. Standard plate bending benchmark problems are analyzed using the proposed element MQP4 via Integrated Force Method to study its performance with respect to accuracy and convergence, and results are compared with those of displacement based 4-node quadrilateral plate bending finite elements available in the literature. The results are also compared with the exact solutions. The proposed element MQP4 is free from shear locking and works satisfactorily in both thin and moderately thick plate bending situations.

Keywords: stress-resultant fields; flexibility matrix; equilibrium matrix; displacement fields; Mindlin-Reissner theory; Integrated Force Method.

1. Introduction

The investigations on the Mindlin-Reissner quadrilateral plate bending elements is perhaps the

† Professor, Ph.D.,, Corresponding author, E-mail: djaya_hr@yahoo.com
‡ Professor, Ph.D., Emeritus Fellow, E-mail: naga@aero.iisc.ernet.in
‡† Professor, Ph.D., E-mail: pcpandey@civil.iisc.ernet.in

most interesting problem in finite element analysis. For the Mindlin-Reissner plate elements, only C^0 continuity is required and therefore the difficulties of C^1 continuity requirement for thin plate element are solved easily. Moreover, both thin (Kirchhoff) and moderately thick (Mindlin-Reissner) plate analysis can be integrated into single element model. In this paper the force-based approach called the Integrated Force Method has been extended to analyze thin and moderately thick plate using the Mindlin-Reissner theory.

During the pre-computer era, the force method was the most popular analyzing tool for civil, mechanical and aerospace engineering structures. This popularity can be attributed to its ability to determine forces/stresses in the structures with reasonable accuracy. During the formulation period of structural analysis by matrix methods, earnest research was directed to automate the force method, which includes computer-assisted generation of compatibility conditions. This effort, by and large, was partially successful (Robinson and Haggenmacher 1971, Kaneko *et al.* 1983) because redundant analysis in continuum structures was not possible using the conventional force method. This was the main cause of the failure of the force method. It acted as dominating road block in the path of the automation of force method.

The Integrated Force Method (IFM) is the new matrix formulation (Patnaik 1973) of the classical force method of analysis. The IFM is independent of redundant forces and the basis determinate structure and hence it is as flexible as that of displaced based finite element method as far as automation of the method to computers is concerned. And further it retains the potential of the force method i.e., primary importance will be on force/stress computations which is well accepted by all design engineers.

In any method of analysis of structural mechanics problems, in general, equilibrium equations, compatibility conditions have to be satisfied in addition to the constitutive relations which describe the material behavior. The IFM integrates the system equilibrium equations and the global compatibility conditions in a fashion paralleling approaches in continuum mechanics (e.g., the Beltrami - Michel formulation of elasticity (Love 1944)). The IFM provides a natural way of integrating the equilibrium equations and the compatibility conditions while performing structural analysis. The Integrated Forces Method is based on variational principles (Patnaik 1986) and its stationary condition of the functional yields the equilibrium equations, compatibility and natural boundary conditions. The compatibility conditions in sparse and banded form for IFM has been given in the Nagabhushanam and Patnaik (1990). The application of the IFM on certain kind of discrete and continuum structures for their static and dynamic behavior are cited in (Nagabhushanam and Srinivas 1991, Patnaik and Yadagiri 1976, Krishnam Raju and Nagabhushanam 2000). The IFM exhibits potential to eliminate certain shortcomings of the displacement method: indirect stress calculations, shear locking, circuitous treatment of the initial deformations and stress inaccuracy in stress concentration zones and to simplify design sensitivity analysis (Patnaik *et al.* 1996).

In this paper the IFM has been extended to analyze the Mindlin-Reissner plates and a new four-node bilinear quadrilateral plate bending element (MQP4) is proposed for analyzing the plate bending problems using IFM. The Mindlin-Reissner theory has been employed in the formulation of this element which accounts for the shear deformation. This element considers three degrees of freedom namely a transverse displacement w and two rotations θ_x, θ_y at each node. Suitable displacement and stress-resultants fields are chosen over the element and the corresponding element equilibrium and flexibility matrices are developed. The shear correction factor (Reissner 1945) has been considered in the formulation. Standard plate bending benchmark problems are analyzed to study the accuracy and convergence of the proposed element (MQP4). The results obtained by the

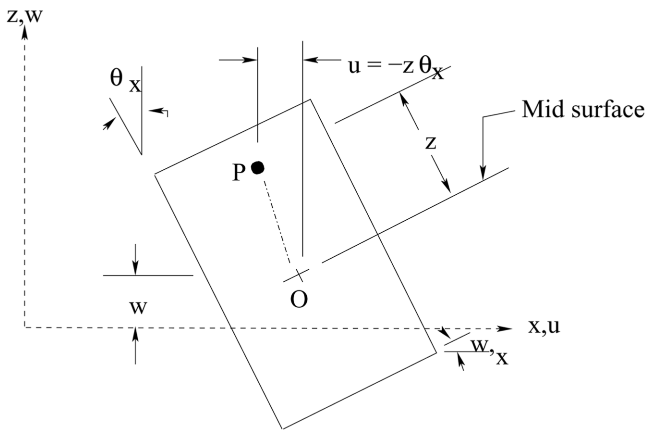


Fig. 1 Differential plate element after deformation (as per Mindlin-Reissner theory)

proposed element are compared with those of displacement based four node quadrilateral elements available in the literature. Results are also compared with exact solutions. The proposed element MQP4 works satisfactorily in both thin and moderately thick plate bending situations.

2. Formulation of element equilibrium and flexibility matrices

In this section formulation of equilibrium and flexibility matrices for the plate bending element is discussed. The Mindlin-Reissner plate theory has been employed in the formulation which accounts for the shear deformation. In the Mindlin-Reissner plate theory where a section that is straight and normal to mid-surface of the un-deformed plate remain straight but not necessarily normal to the mid-surface of the deformed plate (Fig. 1). This leads to the following definition of the displacement components u, v, w in the x, y, z Cartesian coordinates system

$$u = -z\theta_x(x, y); \quad v = -z\theta_y(x, y); \quad w = w(x, y) \tag{1}$$

where

x, y are coordinates in the reference mid-surface
z is the coordinate through the thickness t with $-t/2 <= z <= t/2$
w is the transverse (lateral) displacement
θ_x, θ_y represent the rotations of the normal in x-z and y-z planes respectively

Engineering strains for the Mindlin-Reissner theory can be written as

$$\{\varepsilon\} = -z\{k_1\} \tag{2}$$

where

$$\{\varepsilon\} = [\varepsilon_x \ \ \varepsilon_y \ \ \gamma_{xy} \ \ \gamma_{yz} \ \ \gamma_{zx}]^T$$

$$\{k_1\} = \left[\frac{\partial\theta_x}{\partial x} \;\; \frac{\partial\theta_y}{\partial y} \;\; \frac{\partial\theta_x}{\partial y}+\frac{\partial\theta_y}{\partial x} \;\; \frac{\theta_y-\frac{\partial w}{\partial y}}{z} \;\; \frac{\theta_x-\frac{\partial w}{\partial x}}{z}\right]^T$$

The stress-strain relations for an isotropic two-dimensional plate material is given by

$$\{\sigma\} = [C_{con}]\{\varepsilon\} \tag{3}$$

where $\{\sigma\} = [\sigma_x \;\; \sigma_y \;\; \tau_{xy} \;\; \tau_{yz} \;\; \tau_{xz}]^T$ = Vector of stress components

$\{\varepsilon\} = [\varepsilon_x \;\; \varepsilon_y \;\; \gamma_{xy} \;\; \gamma_{yz} \;\; \gamma_{xz}]^T$ = Vector of strain components

$$[C_{con}] = \text{constitutive matrix} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}$$

E = Young's modulus; ν = Poisson's ratio

The stress-resultants $\{M\}$ can be written as

$$\{M\} = \int_{-t/2}^{t/2} z\{\sigma_r\}\,dz \tag{4}$$

where $\{M\} = [M_x \;\; M_y \;\; M_{xy} \;\; Q_y \;\; Q_x]^T$ = Vector of stress-resultants

$$\{\sigma_r\} = \left[\sigma_x \;\; \sigma_y \;\; \tau_{xy} \;\; \frac{\tau_{yz}}{z} \;\; \frac{\tau_{xz}}{z}\right]^T$$

Eqs. (2), (3) and (4) yield the moment-curvature relations as

$$\{M\} = [C_1]\{k\} \tag{5}$$

Where $\{k\}$ = Vector of Curvatures

$$= \left[\frac{\partial\theta_x}{\partial x} \;\; \frac{\partial\theta_y}{\partial y} \;\; \frac{\partial\theta_x}{\partial y}+\frac{\partial\theta_y}{\partial x} \;\; \theta_y-\frac{\partial w}{\partial y} \;\; \theta_y-\frac{\partial w}{\partial y}\right]^T$$

$[C_1]$ = matrix relating stress-resultants to curvatures

From the Eq. (5), the curvature-moment relation becomes

$$\{k\} = [C_1]^{-1}\{M\} = [H]\{M\} \tag{6}$$

where $[H] = [C_1]^{-1}$
= matrix relating curvatures to stress-resultants

For the Mindlin-Reissner plate, the Matrix $[H]$ can be written with Reissner's shear correction factor of 5/6 as

$$[H] = \frac{1}{D_1}\begin{bmatrix} 1 & -\nu & 0 & 0 & 0 \\ -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & \dfrac{t^2(1+\nu)}{5} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{t^2(1+\nu)}{5} \end{bmatrix} \tag{7}$$

where

$D_1 = Et^3/12$

t = thickness of the plate; ν = Poisson's ratio

The Strain energy U_p of the elastic plate in bending and shear is given by

$$U_p = \iint \frac{1}{2}\left[M_x\frac{\partial \theta_x}{\partial x} + M_y\frac{\partial \theta_y}{\partial y} + M_{xy}\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) + Q_y\left(\theta_y - \frac{\partial w}{\partial y}\right) + Q_x\left(\theta_x - \frac{\partial w}{\partial x}\right)\right]dxdy \tag{8}$$

For a discrete plate bending element the stress-resultants $\{M\}$ and curvatures $\{k\}$ can be expressed in the matrix notation in terms of assumed stress-resultant fields and displacement fields as

$$\{M\} = [\psi]\{F_e\} \tag{9}$$

$$\{k\} = [D_{op}][\phi_1]\{\alpha\} = [D_{op}][\phi]\{X_e\} \tag{10}$$

where

$[\psi]$ = matrix of polynomial terms for stress-resultant fields

$\{F_e\}$ = vector of force components of the discrete element

$[\phi_1]$ = matrix of polynomial terms for displacement fields

$[\phi] = [\phi_1][A]^{-1}$

$[A]$ = matrix formed by substituting the coordinates of the element nodes into the polynomial of displacement fields

$\{\alpha\}$ = coefficients of the displacement field polynomial

$\{X_e\}$ = vector of displacements of the discrete element

$$[D_{op}] = \text{Differential operator matrix} = \begin{bmatrix} 0 & \dfrac{\partial}{\partial x} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} & 0 & 1 \\ \dfrac{\partial}{\partial x} & 1 & 0 \end{bmatrix}$$

Substituting Eqs. (9) and (10) into the Eq. (8), the strain energy for the discrete element can be expressed as

$$U_p = \frac{1}{2}\{X_e\}^T[B_e]\{F_e\} \tag{11}$$

where $[B_e]$ represents the element equilibrium matrix and is given by

$$[B_e] = \iint[\phi]^T[D_{op}]^T[\psi]dxdy \tag{12}$$

The complementary strain energy for the elastic plate in bending and shear is written as

$$U_p = \iint\frac{1}{2}\frac{1}{D_1}\left[M_x^2 + M_y^2 - 2\nu M_x M_y + 2M_{xy}^2 + Q_y^2\frac{2t^2(1+\nu)}{5} + Q_x^2\frac{2t^2(1+\nu)}{5}\right]dxdy$$

where $D_1 = Et^3/12$

E = young's modulus, t = thickness of the plate; ν = Poisson's ratio

For the discrete plate element, complementary strain energy can be expressed as

$$U_c = \frac{1}{2}\{F_e\}^T[G_e]\{F_e\} \tag{13}$$

where $[G_e]$ represents the element flexibility matrix and is given by

$$[G_e] = \iint[\psi]^T[H][\psi]dxdy \tag{14}$$

The Eqs. (12) and (14) can be used to obtain element equilibrium and flexibility matrices $[B_e]$ and $[G_e]$ respectively. These element equilibrium matrix $[B_e]$ and element flexibility matrix $[G_e]$ of all elements are assembled to obtain the global equilibrium matrix $[B]$ and global flexibility matrix $[G]$ of the structure and they are used to setup the IFM governing equation to analyze the structure by IFM.

2.1 Displacement and stress-resultant fields

The geometry of the proposed typical 4-node bilinear quadrilateral plate bending element MQP4 is shown in the Fig. 2. Three degrees of freedom namely a transverse nodal displacement w and two rotations θ_x, θ_y are considered at each node of the element. The independent description of the assumed displacement and stress resultant fields required for the development of element equilibrium and flexibility matrices. The assumed polynomials for displacement fields should satisfy the convergence requirements. Assumed displacement fields for transverse nodal displacement (w) and two rotations (θ_x, θ_y) are given in the Eq. (15). The stress-resultant fields in terms of generalized force parameters should satisfy the homogeneous equilibrium equations. The assumed stress-resultant fields for this element are shown in the Eq. (16).

The displacement fields for w, θ_x and θ_y are assumed in terms of generalized coordinates α_1, α_2, …, α_{12} as

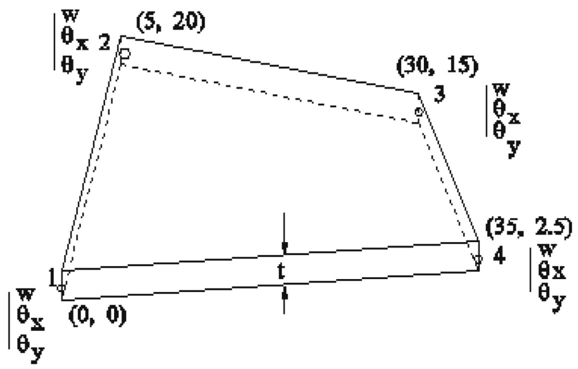


Fig. 2 Typical four node quadrilateral element

$$
\begin{aligned}
w &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy \\
\theta_x &= \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy \\
\theta_y &= \alpha_9 + \alpha_{10} x + \alpha_{11} y + \alpha_{12} xy
\end{aligned} \tag{15}
$$

and the assumed stress-resultant fields in terms of polynomial terms with independent generalized force parameters $F_1, F_2, \ldots, F_9$ are given as

$$
\begin{aligned}
M_x &= F_1 + F_2 x + F_3 y + F_4 xy \\
M_y &= F_5 + F_6 x + F_7 y + F_8 xy \\
M_{xy} &= F_9 \\
Q_y &= F_7 + F_8 x \\
Q_x &= F_2 + F_4 y
\end{aligned} \tag{16}
$$

The above displacement and stress-resultant fields can be written in matrix notation as

$$\{u\} = [\phi_2]\{\alpha\} \tag{17}$$

Where

$$\{u\} = [w \;\; \theta_x \;\; \theta_y]^T$$

$$[\phi_2] = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix}$$

$$\{\alpha\} = [\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; \ldots \;\; \alpha_{12}]^T$$

The curvatures for the discrete element can be expressed in matrix notation as

$$\{k\} = [D_{op}][\phi_1]\{\alpha\} \tag{18}$$

$$[D_{op}] = \begin{bmatrix} 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & 0 & 1 \\ \frac{\partial}{\partial x} & 1 & 0 \end{bmatrix} \quad [\phi_1] = \begin{bmatrix} -1 & -x & -y & -xy & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix}$$

And stress-resultants can be expressed in matrix notation as

$$\{M\} = [\psi]\{F\} \tag{19}$$

Where

$$\{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_y \\ Q_x \end{Bmatrix}$$

$$[\psi] = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & 0 \\ 0 & 1 & 0 & y & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\{F\} = [F_1F_2 \ldots F_9]^T$$

Substituting the Eqs. (17), (18), (19) and (7) into the Eqs. (12) and (14), the element equilibrium and flexibility matrices are obtained.

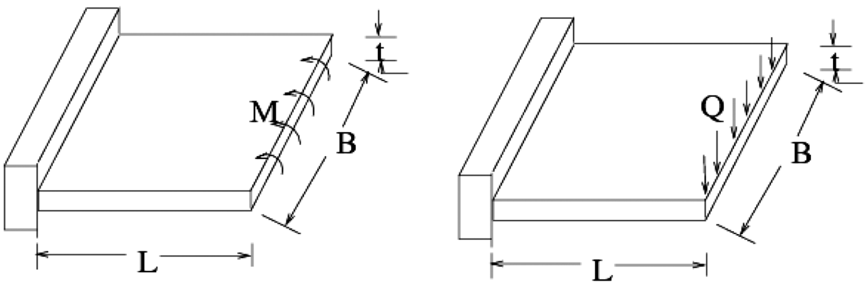
3. Numerical results and discussions

The patch tests and convergence tests are considered to study the performance of the proposed element MQP4.

3.1 Patch tests

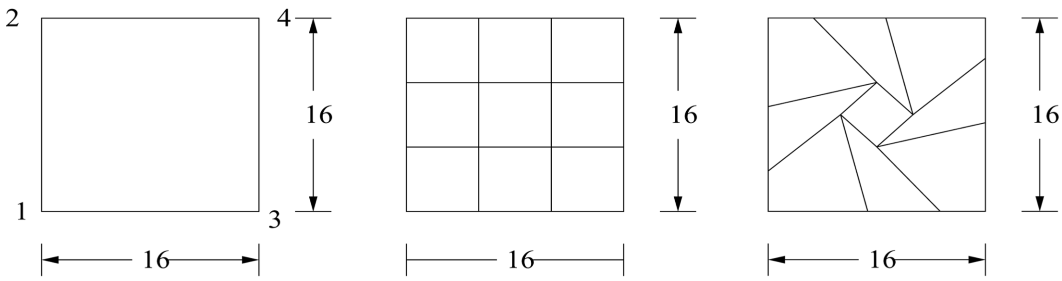
Constant curvature (pure bending) test and constant shear test are carried out on single element mesh and multiple elements patches (Mallikarjuna Rao and Srinivas 2001) (Figs. 3 and 4) using the proposed element MQP4 via Integrated Force Method. Parameters of the constant curvature and constant shear test problems are: $L = B = 16$, $t = 1$, $E = 1e+06$, $\nu = 0.25$, end moment = 100, vertical end shear = 10.

Boundary and loading conditions for constant curvature (pure bending) test: Distributed constant



(a) Plate with end moment (b) Plate with vertical end shear

Fig. 3 Constant curvature (pure bending) test and constant shear test



A B C

Fig. 4 Various meshes(regular A. 1 × 1, B. 3 × 3, C. irregular 3 × 3 mesh) in the plate for patch tests

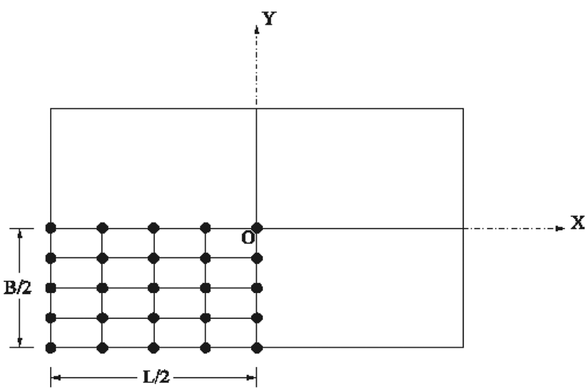


Fig. 5 A typical (4 × 4) mesh in one quadrant of the rectangular plate

edge moments along one edge, the opposite edge is clamped and all lateral boundary tangential rotations are fixed.

Boundary and loading conditions for constant shear test: Distributed constant edge forces on one edge, the opposite edge is clamped and all rotations fixed in order to prevent bending.

The proposed element has yielded the following results: In the constant curvature test problem the estimated moment M_x at all nodes of the plate is 6.25 (exact = 6.25), stress σ_x at all nodes is 37.5 (exact = 37.5). In the constant shear test problem, the estimated shear Q_x at all the nodes is 0.625 (exact = 0.625). The proposed element has also yielded the identical results for (3 × 3) regular mesh (Fig. 5) and satisfactory results for (3 × 3) irregular mesh. Hence the proposed element MQP4 has passed the patch tests.

3.2 Convergence tests

A square thin/thick plate with simply supported/clamped boundary conditions and cantilever plate subjected to point load/uniform load over the full plate are considered and they are analyzed for various mesh sizes using the proposed element MQP4 via the Integrated Force Method to estimate moments and deflections. The results of the proposed element MQP4 with respect to accuracy and convergence are compared with few 4-node displacement based quadrilateral plate bending elements available in the literature (Wanji and Cheung 2000, Kikuchi and Ando 1972) and commercial finite element software (NISA Version 9.3 and ANSYS Version 5.6). The results of the proposed element MQP4 are also compared with the exact solutions (Timoshenko and Krieger 1959, Jane *et al*. 2000). The example problems considered are

1. A square thin/thick plate with simply supported/clamped boundary conditions subjected to uniform load. The parameter of the problem are: size of the plate = 100 × 100, t = 1 or 20, E = 1092000, ν = 0.3, q = 1 (Wanji and Cheung 2000)
2. A square thin plate with simply supported/clamped boundary conditions subjected to central point load. The parameter of the problem are : size of the plate = 100 × 100, t = 1, E = 109200, ν = 0.3, P = 1 (Kikuchi and Ando 1972)
3. A long cantilever beam (strip plate) subjected to point load at the tip or uniform load over the whole plate. The parameter of the problem are: L = 1000, B = 30, t = 5, E = 2e + 05, ν = 0.0, P = 25, q = 0.01. Here the Poisson's ratio is considered as zero, to compare the results with the beam solution.

Because of the symmetry of the geometry, loading and boundary conditions of the plate in the example problems 1 and 2 above, one quadrant of the plate is considered for the analysis. The typical (4 × 4) mesh considered in one quadrant is as shown in the Fig. 5.

The results obtained by proposed element MQP4 using IFM for the above example problems 1, 2 and 3 are compared with results from displacement based 4-node quadrilateral plate bending elements given in the element groups I, II and III respectively. These element groups are formed by selecting the elements from literature (Wanji and Cheung 2000, Kikuchi and Ando 1972) and the commercial finite element software NISA and ANSYS and they are as given below.

Group I elements:

RDKQM the refined quadrilateral element based on Mindlin/Reissner plate theory by Wanji and Cheung (2000)

Q4-R	the reduced/selective integration quadrilateral element, which can not pass the patch test and has two extra energy modes (Malkus and Hughes 1978)
Q4	non-conforming quadrilateral element presented in the paper (Zienkiewicz *et al.* 1971)
DKQ	the discrete Kirchhoff quadrilateral element proposed by Stricklin *et al.* (1969), Dhatt (1969), Batoz and Tahar (1982)
MITC4	the quadrilateral element proposed by Bathe and Dvorkin (1985)

Group II elements: R-0, R-1, R2-3, R-4 = Quadrilateral elements from Kikuchi and Ando (1972)

Group III elements: 4-node quadrilateral elements from the commercial finite element software NISA and ANSYS

Exact displacements and moments for the plates with various boundary conditions and loadings are calculated from Timoshenko and Krieger (1959), Jane *et al*. (2000) for the example problems 1 and 2 and from the beam solutions for the example problem 3.

Tables 1-4 summarize central deflections and moments of the square thin plate (t/L = 0.01) for the example problem 1 with various mesh sizes and the corresponding convergence trends are shown in the Figs. 6-9. Central deflections and moments for square thick plate (t/L = 0.2) of the example problem 1 for various mesh sizes are given in the Tables 5-8. Figs. 10-13 show the corresponding converging trends.

Table 1 Central deflection for a simply supported square plate with uniform load (t/L = 0.01)

W_c ($10^{-5}qL^4/D$) (Example Problem 1)						
Elements	Q4	Q4-R	DKQ	MITC4	RDKQM	MQP4
1 × 1	1.1	319.2	378.5	319.1	378.7	496.5
2 × 2	4.5	397.1	404.6	397.1	404.8	440.9
4 × 4	17.3	404.4	406.0	404.4	406.2	415.5
6 × 6	36.9	405.5	406.1	405.5	406.3	409.9
8 × 8	61.3	405.9	406.2	405.9	406.3	408.8
10 × 10	88.3	406.1	406.2	406.1	406.4	407.9
Exact = 407						

Table 2 Central moment for a simply supported square plate with uniform load (t/L = 0.01)

M_c (Example Problem 1)						
Elements	Q4	Q4-R	DKQ	MITC4	RDKQM	MQP4
1 × 1	0.833	331.6	603.1	331.6	603.6	899.3
2 × 2	5.119	477.0	501.0	477.1	501.5	580.7
4 × 4	20.81	479.0	483.9	479.0	484.2	502.4
6 × 6	44.74	478.9	405.5	478.9	481.4	488.7
8 × 8	74.38	478.9	480.1	478.9	480.4	484.6
10 × 10	107.0	478.9	479.6	478.9	480.0	482.6
Exact = 479						

Table 3 Central deflection for a clamped square plate with uniform load (t/L = 0.01)

W_c (10^{-5}qL^4/D) (Example Problem 1)						
Elements	Q4	Q4-R	DKQ	MITC4	RDKQM	MQP4
1 × 1	0.3	0.4	156.2	0.3	156.5	182.0
2 × 2	1.0	121.4	146.1	121.3	146.3	158.3
4 × 4	3.7	125.3	131.9	125.3	132.2	135.3
6 × 6	7.8	126.1	129.0	126.1	129.2	130.5
8 × 8	13.2	126.4	127.9	126.4	128.1	129.0
10 × 10	19.5	126.6	127.4	126.5	127.6	128.2
Exact = 126.6						

Table 4 Central moment for a clamped square plate with uniform load (t/L = 0.01)

M_c (Example Problem 1)						
Elements	Q4	Q4-R	DKQ	MITC4	RDKQM	MQP4
1 × 1	0.0	0.0	487.5	0.0	487.6	566.8
2 × 2	1.57	251.4	287.3	251.7	287.7	324.6
4 × 4	6.90	232.9	243.3	233.1	243.6	255.7
6 × 6	15.44	230.7	235.4	230.9	235.6	240.7
8 × 8	25.62	230.0	232.6	230.1	232.9	235.7
10 × 10	37.71	229.7	231.3	229.7	231.6	233.3
Exact = 231						

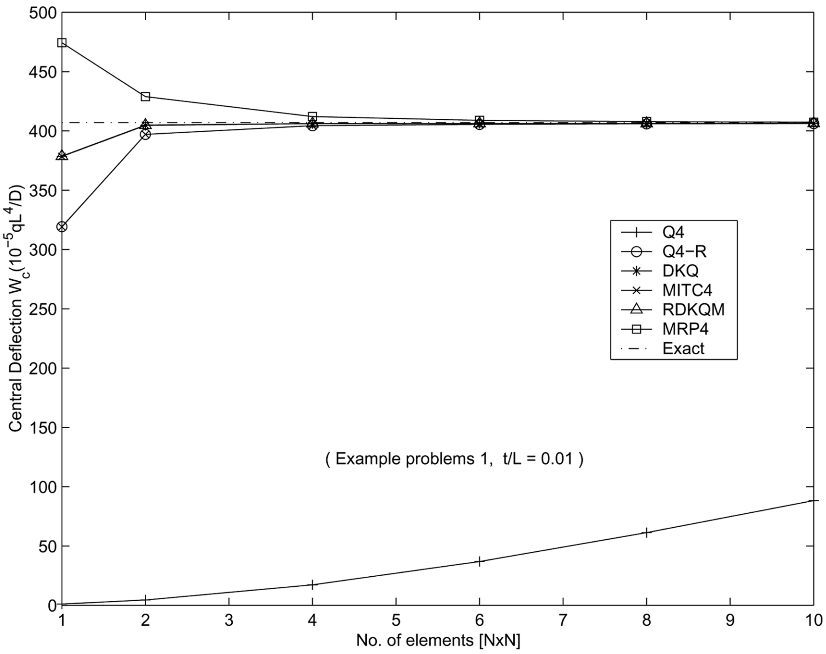


Fig. 6 Central deflection for a simply supported square thin plate with uniform load

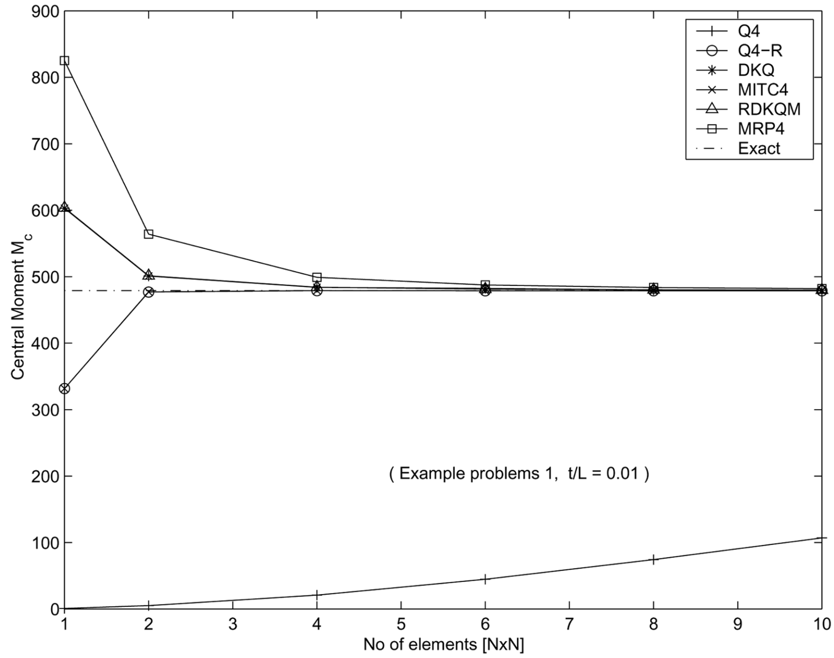


Fig 7 Central moment for a simply supported square thin plate with uniform load

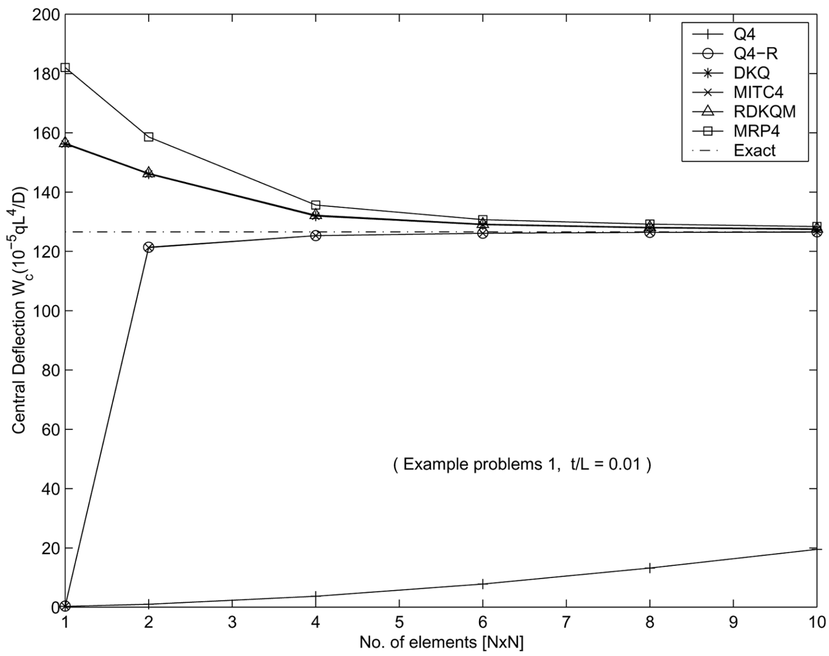


Fig 8 Central deflection for a clamped square thin plate with uniform load

Central deflections for a square thin plate ($t/L = 0.01$) with central point load of the example problem 2 for various mesh sizes are summarized in Tables 9 and 10 and the corresponding

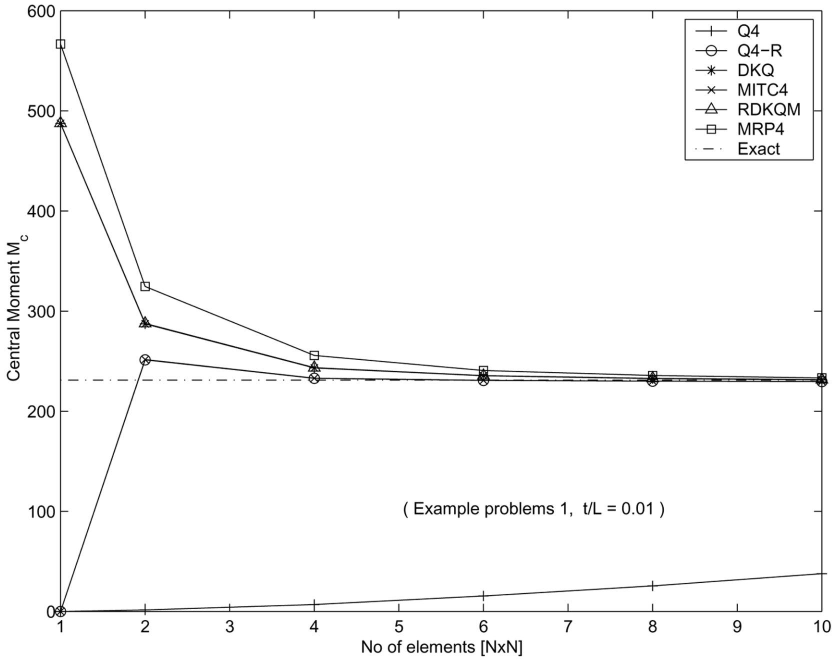


Fig 9 Central moment for a clamped square thin plate with uniform load

Table 5 Central deflection for a simply supported square plate with uniform load (t/L = 0.2)

W_c ($10^{-5}qL^4/D$) (Example Problem 1)						
Elements	Q4	Q4-R	DKQ	MITC4	RDKQM	MQP4
1 × 1	267	461.7	378.5	426.0	468.6	612.8
2 × 2	411	492.1	404.6	485.5	485.8	541.2
4 × 4	468	490.5	406.0	489.4	488.9	506.7
6 × 6	480	490.5	406.1	490.0	489.7	497.5
8 × 8	485	490.4	406.2	490.2	490.0	495.1
10 × 10	487	490.4	406.2	490.3	490.2	493.5
Exact = 490						

Table 6 Central moment for a simply supported square plate with uniform load (t/L = 0.2)

M_c (Example Problem 1)						
Elements	Q4	Q4-R	DKQ	MITC4	RDKQM	MQP4
1 × 1	166	331.6	603.1	331.6	685.5	884.7
2 × 2	386	471.1	501.0	475.0	543.0	586.9
4 × 4	454	479.0	483.9	478.9	497.3	507.2
6 × 6	468	478.9	481.1	478.9	487.3	491.6
8 × 8	473	478.9	480.1	478.9	483.7	486.4
10 × 10	475	478.9	479.6	478.9	482.0	483.7
Exact = 479						

Table 7 Central deflection for a clamped square plate with uniform load (t/L = 0.2)

			W_c ($10^{-5}qL^4/D$) (Example Problem 1)			
Elements	Q4	Q4-R	DKQ	MITC4	RDKQM	MQP4
1 × 1	107.1	147.2	156.2	107.1	260.4	289.9
2 × 2	175.7	217.7	146.1	209.4	235.0	247.5
4 × 4	204.7	217.4	131.9	215.7	221.7	224.4
6 × 6	211.4	217.3	129.0	216.6	219.2	220.4
8 × 8	213.9	217.2	127.9	216.9	218.3	219.1
10 × 10	215.1	217.2	127.4	217.0	217.9	218.4
			Exact = 217			

Table 8 Central moment for a clamped square plate with uniform load (t/L = 0.2)

			M_c (Example Problem 1)			
Elements	Q4	Q4-R	DKQ	MITC4	RDKQM	MQP4
1 × 1	0.00	0.00	487.5	0.00	524.7	555.8
2 × 2	158.2	213.8	287.3	220.1	316.7	326.7
4 × 4	215.9	235.4	243.3	235.1	256.5	258.0
6 × 6	226.8	235.7	235.4	235.5	245.0	245.3
8 × 8	230.7	235.7	232.6	235.6	241.0	241.2
10 × 10	235.7	235.7	231.3	235.7	239.1	239.3
			Exact = 231			

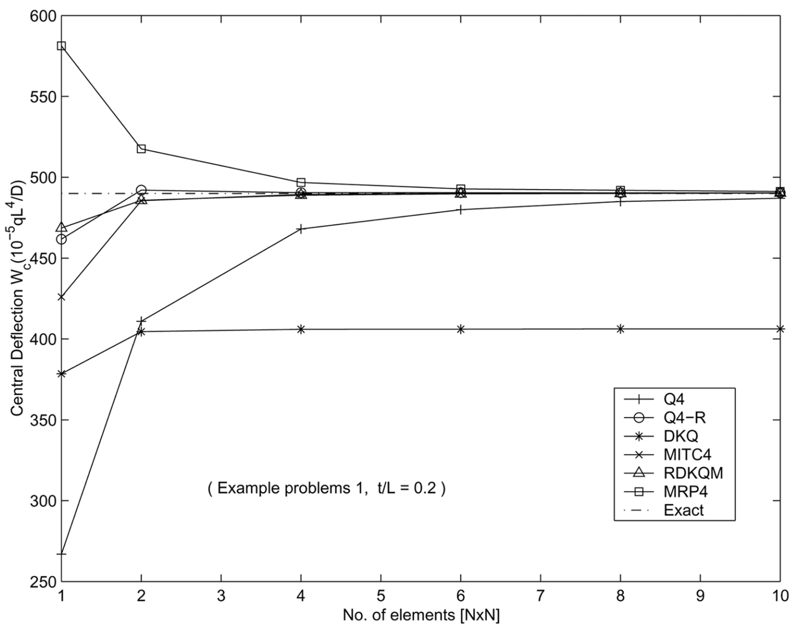


Fig. 10 Central deflection for a simply supported square thick plate with uniform load

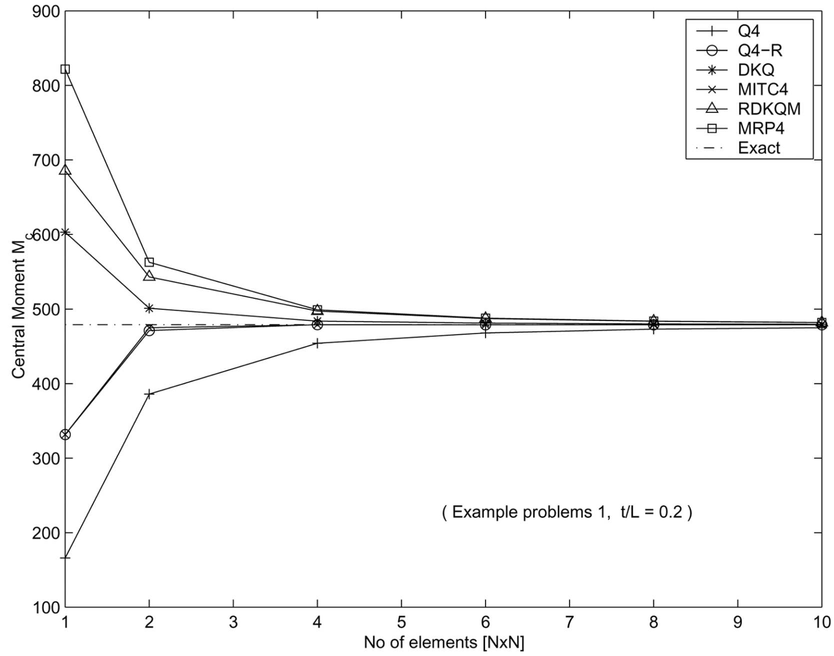


Fig. 11 Central moment for a simply supported square thick plate with uniform load

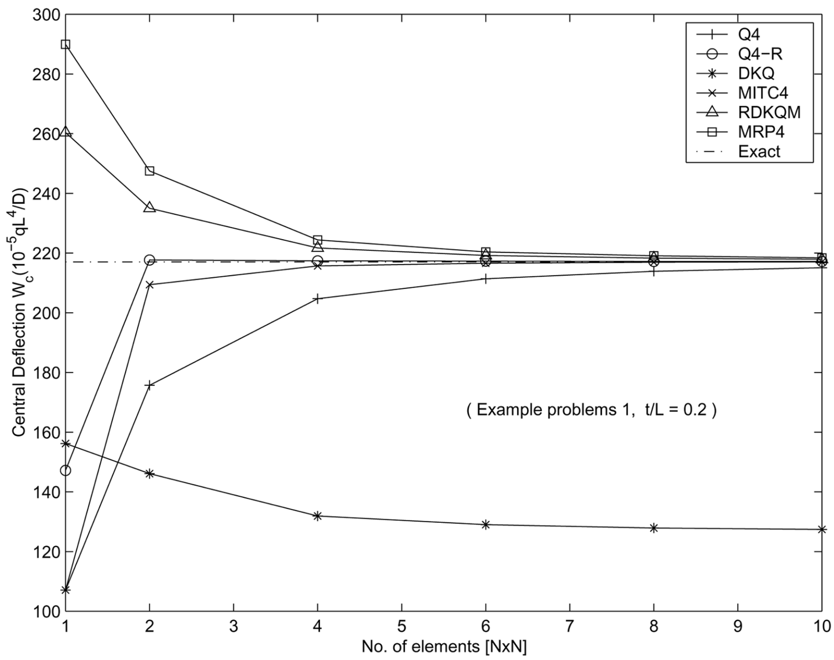


Fig. 12 Central deflection for a clamped square thick plate with uniform load

convergence trends are shown in the Figs. 14 and 15.

In the Tables 1-10, the central deflections and moments for both thin and thick square plates with

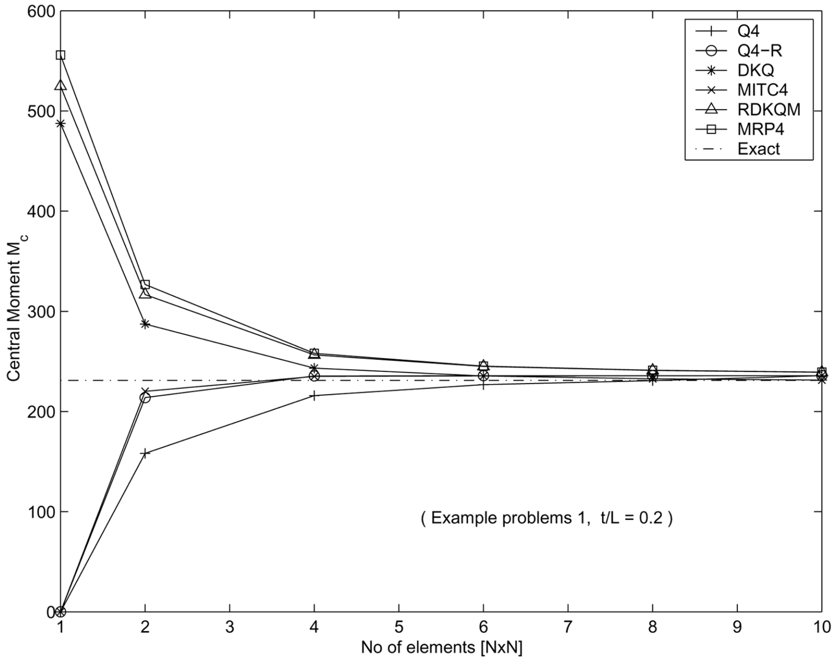


Fig. 13 Central moment for a clamped square thick plate with uniform load

Table 9 Central deflection for a simply supported square plate with central point load (*t/L* = 0.01)

$W_c\,(10^{-2})$ (Example Problem 2)					
Elements	R-0	R-1	R2-3	R-4	MQP4
1 × 1	1.378	1.045	1.119	1.945	1.986
2 × 2	1.233	1.138	1.167	1.374	1.393
4 × 4	1.183	1.155	1.164	1.223	1.228
10 × 10	1.165	1.159	1.161	1.172	1.172

Table 10 Central deflection for a clamped square plate with central point load (*t/L* = 0.01)

$W_c\,(10^{-3})$ (Example Problem 2)					
Elements	R-0	R-1	R2-3	R-4	MQP4
1 × 1	5.919	5.348	5.388	8.446	7.278
2 × 2	6.134	5.350	5.530	7.261	7.079
4 × 4	5.803	5.550	5.625	6.154	6.133
10 × 10	5.653	5.603	5.619	5.721	5.722
Exact = 5.60					

simply supported and clamped boundary conditions computed by the proposed element MQP4 via Integrated Force Method are comparable with those of other elements considered. Further the results are fast approaching with monotonic convergence towards the exact solutions as shown in the Figs. 6-15.

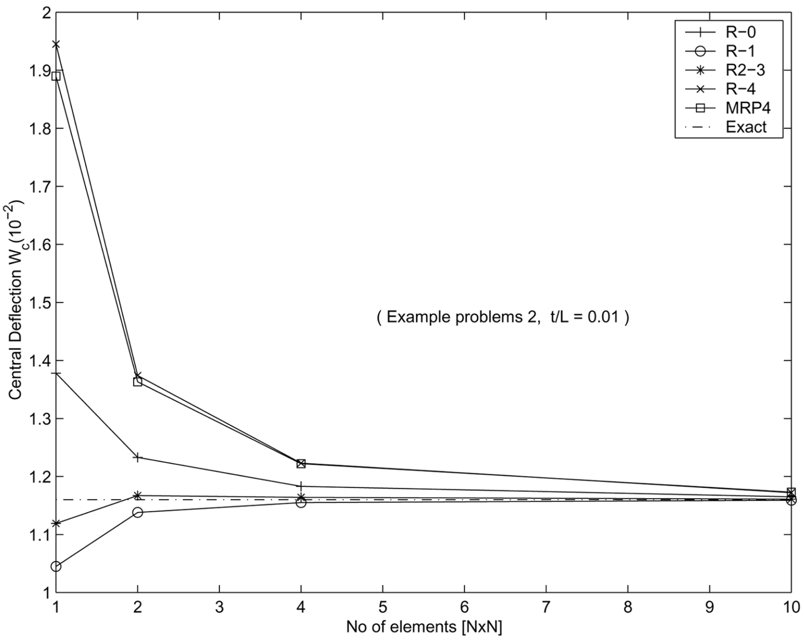


Fig. 14 Central deflection for a simply supported square thin plate with central point load

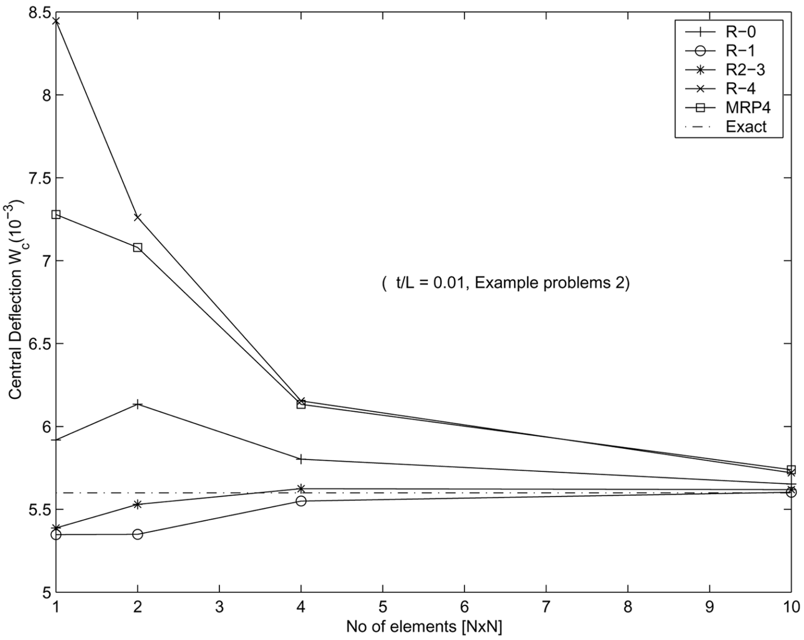


Fig. 15 Central deflection for a clamped square thin plate with the central point load

Tip deflections and bending moments at the clamped edge for long cantilever beam (strip plate) of the example problem 3 for various mesh sizes are given in the Tables 11-14 and the corresponding convergence trends are shown in the Figs. 16-19.

Table 11 Tip deflection for a cantilever strip plate with point load at the tip (t/L = 0.005)

W (Example Problem 3)			
Elements	NISA	ANSYS	MQP4
2 × 1	125.15	130.92	133.34
4 × 1	131.27	132.70	133.34
8 × 1	132.83	133.18	133.34
16 × 1	133.22	133.31	133.34
32 × 1	133.31	133.33	133.34
64 × 1	133.33	133.34	133.34
128 × 1	133.33	133.34	133.34
Exact = 133.33			

Table 12 Moment at the clamped edge for a cantilever strip plate with point load at the tip

M (Example Problem 3) (t/L = 0.005)			
Elements	NISA	ANSYS	MQP4
2 × 1	625.00	763.00	833.33
4 × 1	729.17	798.96	833.33
8 × 1	781.25	816.63	833.33
16 × 1	807.29	825.79	833.33
32 × 1	820.29	830.63	833.33
64 × 1	826.83	832.71	833.33
128 × 1	830.10	833.25	833.33
Exact = 833.33			

Table 13 Tip deflection for a cantilever strip plate with uniform load (t/L = 0.005)

W (Example Problem 3)			
Elements	NISA	ANSYS	MQP4
2 × 1	600.09	584.35	650.01
4 × 1	600.09	595.99	612.51
8 × 1	600.08	599.06	603.11
16 × 1	600.07	599.83	600.79
32 × 1	600.05	600.01	600.21
64 × 1	600.03	600.01	600.21
Exact = 600.00			

Table 14 Moments at the clamped edge for a cantilever strip plate with uniform load

M (Example Problem 3) (t/L = 0.005)			
Elements	NISA	ANSYS	MQP4
2 × 1	2500.00	4217.50	5000.00
4 × 1	3906.00	4599.17	5000.00
8 × 1	4414.04	4802.08	5000.00
16 × 1	4697.25	4910.42	5000.00
32 × 1	4846.25	4967.92	5000.00
64 × 1	4920.00	4992.50	5000.00
Exact = 5000.00			

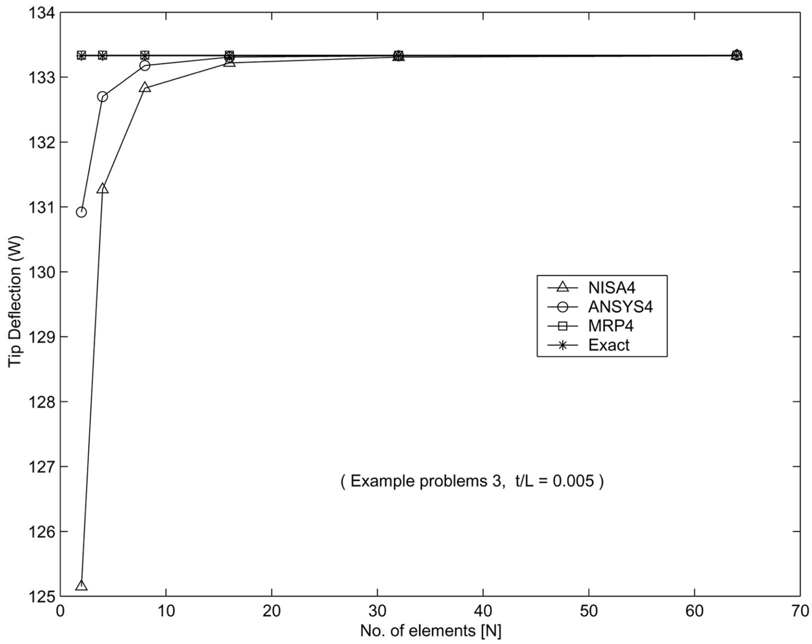


Fig 16 Tip deflection for a cantilever strip plate with the point load at tip

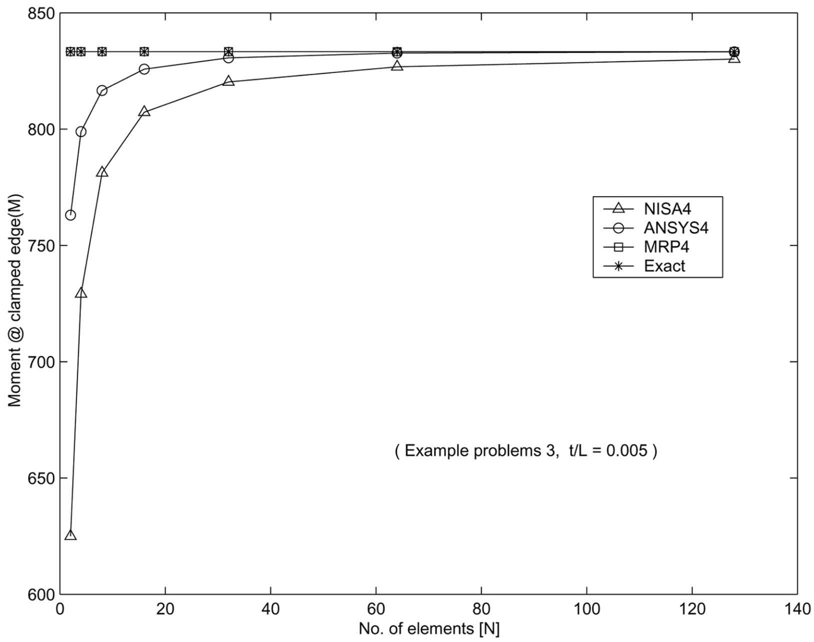


Fig. 17 Moment at clamped edge for a cantilever strip plate with the point load at tip

In the Tables 11-14, the tip deflections and moments at clamped edge of a cantilever beam (strip plate) with different loading conditions and for various mesh sizes computed by the proposed element MQP4 are superior when compared with the results of the other elements considered.

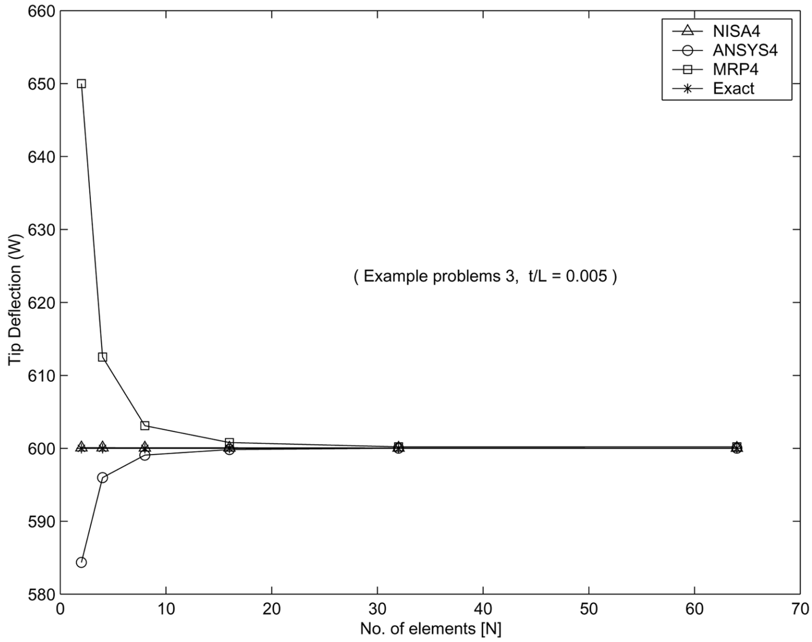


Fig. 18 Tip deflection for a cantilever strip plate with uniform load

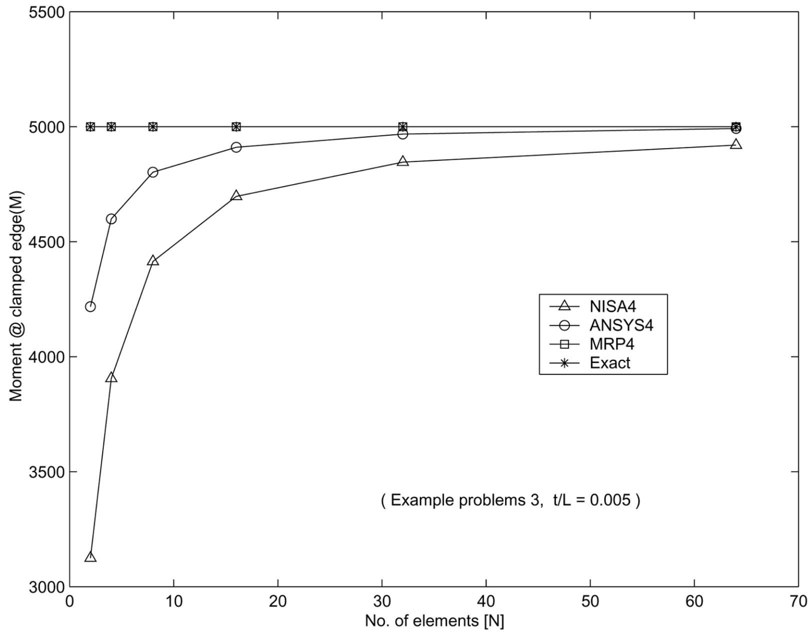


Fig. 19 Moment at clamped edge for a cantilever strip plate with uniform load

Further the results are fast approached, in general, to the exact solutions with even just for two elements as shown in the Figs. 16-19.

In order to study the shear locking behavior of the proposed element MQP4, a simply supported square plate subjected to uniform load is analyzed for various thickness-span ratios using the proposed element MQP4 via the Integrated Force Method for the mesh size 10 × 10 in one quadrant

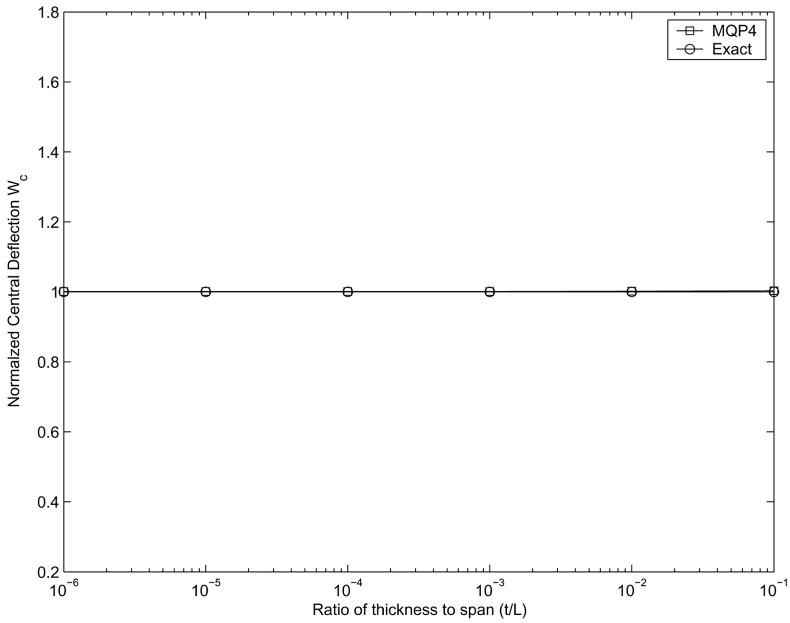


Fig. 20 Normalized central deflections for simply supported plate with uniform load (t/L = 0.1, 0.01, 0.001, 0.0001, 0.00001, 0.000001)

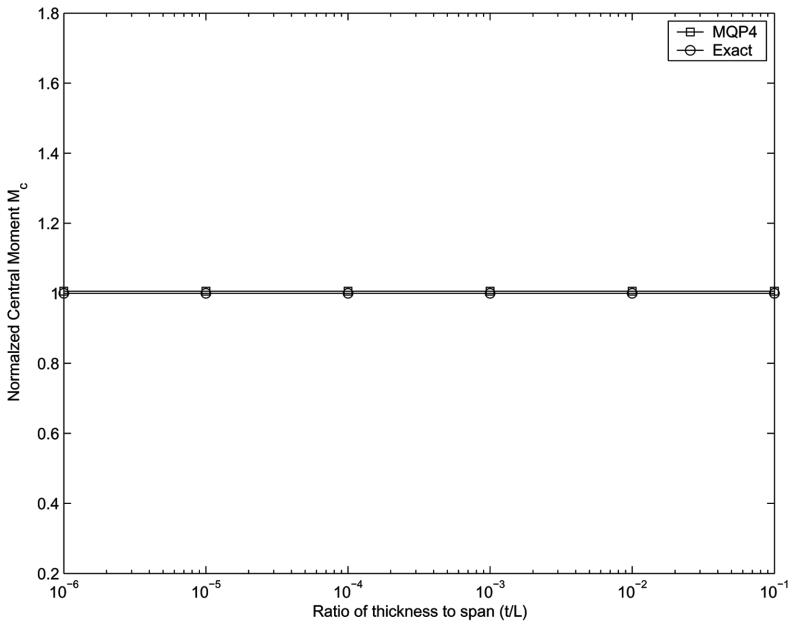


Fig. 21 Normalized central moments for simply supported plate with uniform load (t/L = 0.1, 0.01, 0.001, 0.0001, 0.00001, 0.000001)

of the plate to estimate the central deflections and moments. The parameters of the problem considered are: L = 50, B = 50, t = 5, 0.5, 0.05, 0.005, 0.0005 and 0.00005, E = 200000, n = 0.3, q = 1. The exact central displacements and moments are calculated from the Kirchhoff theory (Timoshenko and Krieger 1959) and the Mindlin plate theory (Jane *et al.* 2000) solutions for thin and moderately thick plate problems respectively. These results are shown in the Figs. 20 and 21. These Figs. indicate that the proposed element MQP4 works excellent for both thin and moderately thick plate bending problems.

4. Conclusions

The Integrated Force Method has been extended for the analysis of thin and moderately thick plates. The Mindlin-Reissner theory based new four-node bilinear quadrilateral plate bending element (MQP4) has been proposed for the analysis of plate bending problems using the Integrated Force Method. This element considers three degrees of freedom namely a transverse nodal displacement w and two rotations θ_x, θ_y at each node. The proposed element MQP4 is free from spurious energy modes. Further this proposed element (MQP4) is free from shear locking, i.e., this element does not lock under thin plate bending situations. Hence the same element can be used to analyze both thin and moderately thick plate bending problems. Various standard plate bending benchmark problems are analyzed using this proposed element (MQP4) via Integrated Force Method. This element, in general, works satisfactorily in all the example problems considered. Therefore this element becomes an alternative force based four-node quadrilateral element compared to displacement based similar four-node quadrilateral elements considered.

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Appendix: Basic theory of IFM

In the Integrated Force Method of analysis, a structure idealized by finite elements is designated as "structure (n, m)". Where (n, m) are force and displacement degrees of freedom of the discrete model, respectively. The structure (n, m) has m equilibrium equations and $r = (n - m)$ compatibility conditions. Equilibrium equations (EE) represent the vectorial summation of the internal forces $\{F\}$ to the external loads $\{P\}$ at the nodes of the finite element discritization. It can be written in symbolized matrix notation as

Equilibrium Equations[EE]: $$[B]\{F\} = \{P\} \tag{1}$$

Where $[B]$ = global equilibrium matrix
$\{F\}$ = Vector of internal forces of the structure
$\{P\}$ = vector of external loads on the structure

The Compatibility Conditions(CC) are constraints on strains, and for finite element models they are also constraints on member deformations.

In IFM, St. Venant's approach has been extended for discrete mechanics to develop the compatibility conditions. Development of CC is briefly explained below

The Deformation-Displacement Relationship (DDR) for discrete mechanics is equivalent to the strain-displacement relationship in elasticity. The DDR for discrete analysis was obtained during the development of the variational energy formulation for the IFM (Patnaik 1986)

According to work energy conservation theorem, the internal energy stored in the body in the structure is equal to the work done by the external load, that is

$$\frac{1}{2}\{F\}^T\{\beta\} = \frac{1}{2}\{P\}^T\{X\} \tag{2}$$

where $\{X\}$ represents nodal displacements. Eq. (2) can be rewritten by eliminating the load $\{P\}$ in favor of forces $\{F\}$, by using the Eq. (1) to obtain the following relation

$$\frac{1}{2}\{F\}^T[B]^T\{X\} = \frac{1}{2}\{F\}^T\{\beta\} \tag{3}$$

Eq. (3) can be simplified as

$$\frac{1}{2}\{F\}^T[[B]^T\{X\} - \{\beta\}] = 0 \tag{4}$$

Since $\{F\}$ is not a null vector, its coefficient must be equal to zero, which yields the DDR as

$$\{\beta\} = [B]^T\{X\} \tag{5}$$

Where $\{\beta\}$ are member deformations.

This equation represents the Deformation-Displacement Relations (DDR) for the discrete structure. The elimination of m displacements from n deformations displacement relations given by the above equation yields $r = (n - m)$ compatibility conditions and the associated matrix $[C]$. It can be symbolized in matrix notations as

$$[C]\{\beta\} = 0 \tag{6}$$

Substituting Eq. (5) into the Eq. (6), we obtain

$$[C]\{\beta\} = [C][B]^T\{X\} = 0$$

$$\{X\}^T([B][C]^T) = \{0\}$$

Since the displacement vector $\{X\}$ is not a null vector, we have

$$[B][C]^T = 0 \tag{7}$$

where $[C]$ is the $(r \times n)$ compatibility matrix. It is a kinematic relationship, and it is independent of design parameters, material properties and external loads. This matrix is rectangular and banded. The deformation $\{\beta\}$ in the compatibility conditions (CC) given by the Eq. (6) represents the total deformation consisting of an elastic component $\{\beta_e\}$ and the initial component $\{\beta_o\}$ as

$$\{\beta\} = \{\beta_e\} + \{\beta_0\} \tag{8}$$

The CC in terms of elastic deformation can be written as

$$[C]\{\beta\} = [C]\{\beta_e\} + [C]\{\beta_0\} = 0 \tag{9}$$

$$[C]\{\beta_e\} = \{\delta R\}$$

Where

$$\{\delta R\} = -[C]\{\beta_0\} \tag{10}$$

Using the flexibility characteristics, Eq. (6) with initial deformations can be rewritten as

$$[C][G]\{F\} = \{\delta R\} \tag{11}$$

Clubbing Eqs. (1) and (11), we lead to the IFM governing equation as

$$\begin{bmatrix} [B] \\ ([C][G]) \end{bmatrix}\{F\} = \begin{Bmatrix} P \\ \delta R \end{Bmatrix}$$

$$[S]\{F\} = \{P^*\} \tag{12}$$

Notation

$[A]$	: matrix relating nodal degrees of freedom and coefficients of the polynomial
$[B]$	: global equilibrium matrix ($m \times n$)
$[B_e]$	: element equilibrium matrix ($m_e \times n_e$)
$[C]$	: compatibility matrix ($r \times n$)
$[D_{op}]$	: differential operator matrix
E	: Young's modulus
$\{F\}$	: vector of internal forces of the structure ($n \times 1$)
$\{F_e\}$	: vector of internal forces of the discrete element ($n_e \times 1$)
$[G]$	: global flexibility matrix ($n \times n$)
$[G_e]$	: element flexibility matrix ($n_e \times n_e$)
$[H]$	: matrix relating the curvatures to stress resultants
$[J]$	: deformation coefficient matrix ($m \times n$)
L, B	: Length and breadth of the plate
M_c	: central moment of the plate
$\{M\}$	: vector of stress resultants
P	: point load at the center or tip of the plate
$\{P\}$	: vector of external loads ($m \times 1$)
q	: uniform load over the plate
$[S]$	: IFM governing matrix ($n \times n$)
W_c	: Central deflection of the plate
$\{X\}$	: vector of displacements of the structure ($m \times 1$)
$\{X_e\}$	: vector of displacements of the discrete element ($m_e \times 1$)
a, b	: length and breadth of the plate bending element
$\{k\}$	: vector of curvatures
n, m	: force and displacement degrees of freedom of the structures respectively
n_e, m_e	: element force and displacement degrees of freedom respectively
t	: thickness of the plate
$\{\alpha\}$	: generalized coordinates of the polynomial in the displacement field
$\{\beta\}$	: vector of elastic deformations
$\{\beta_o\}$	: vector of initial deformations
ν	: Poisson's ratio
$[\Phi_1]$	: matrix of polynomial terms for displacement fields
$[\psi]$	: matrix of polynomial terms for stress-resultants fields